Tunable IIR Digital Filters

- We have described earlier two 1st-order and two 2nd-order IIR digital transfer functions with tunable frequency response characteristics.
- We shall show now that these transfer functions can be realized easily using allpass structures providing independent tuning of the filter parameters.

Tunable Lowpass and Highpass Digital Filters

- We have shown earlier that the 1st-order lowpass transfer function
  \[ H_{LP}(z) = \frac{1 - \alpha}{2} \left( \frac{1 + z^{-1}}{1 - \alpha z^{-1}} \right) \]
  and the 1st-order highpass transfer function
  \[ H_{HP}(z) = \frac{1 + \alpha}{2} \left( \frac{1 - z^{-1}}{1 - \alpha z^{-1}} \right) \]
  are doubly-complementary pair.

Moreover, they can be expressed as

\[ H_{LP}(z) = \frac{1}{2} [1 + A(z)] \]
\[ H_{HP}(z) = \frac{1}{2} [1 - A(z)] \]

where
\[ A(z) = -\alpha + z^{-1} \]

is a 1st-order allpass transfer function.

A realization of \( H_{LP}(z) \) and \( H_{HP}(z) \) based on the allpass-based decomposition is shown below.

The 1st-order allpass filter can be realized using any one of the 4 single-multiplier allpass structures described earlier.

One such realization is shown below in which the 3-dB cutoff frequency of both lowpass and highpass filters can be varied simultaneously by changing the multiplier coefficient \( \alpha \).

Figure below shows the composite magnitude responses of the two filters for two different values of \( \alpha \).
Tunable Bandpass and Bandstop Digital Filters

• The 2nd-order bandpass transfer function

\[ H_{BP}(z) = \frac{1 - \alpha}{2} \left( \frac{1 - z^{-2}}{1 - \beta(1 + \alpha) z^{-1} + \alpha z^{-2}} \right) \]

and the 2nd-order bandstop transfer function

\[ H_{BS}(z) = \frac{1 + \alpha}{2} \left( \frac{1 - \beta z^{-1} + z^{-2}}{1 - \beta(1 + \alpha) z^{-1} + \alpha z^{-2}} \right) \]

also form a doubly-complementary pair.

Thus, they can be expressed in the form

\[ H_{BP}(z) = \frac{1}{2} [1 - A_2(z)] \]
\[ H_{BS}(z) = \frac{1}{2} [1 + A_2(z)] \]

where

\[ A_2(z) = \frac{\alpha - \beta(1 + \alpha) z^{-1} + \alpha z^{-2}}{1 - \beta(1 + \alpha) z^{-1} + \alpha z^{-2}} \]

is a 2nd-order allpass transfer function.

A realization of \( H_{BP}(z) \) and \( H_{BS}(z) \) based on the allpass-based decomposition is shown below.

The final structure is as shown below.

In the above structure, the multiplier \( \beta \) controls the center frequency and the multiplier \( \alpha \) controls the 3-dB bandwidth.

The parametric tuning property of the overall structure is illustrated below.

IIR Tapped Cascaded Lattice Structures

Realization of an All-pole IIR Transfer Function

• Consider the cascaded lattice structure derived earlier for the realization of an allpass transfer function.
IIR Tapped Cascaded Lattice Structures

• A typical lattice two-pair here is as shown below

\[ W_m(z) = W_{m+1}(z) - k_m z^{-1} S_m(z) \]

\[ S_{m+1}(z) = k_m W_m(z) + z^{-1} S_m(z) \]

Its input-output relations are given by

\[ \frac{W_m(z)}{S_m(z)} = \frac{1}{k_m} \]

From the input-output relations we derive the chain matrix description of the two-pair:

\[
\begin{bmatrix}
W_{m+1}(z) \\
S_{m+1}(z)
\end{bmatrix} =
\begin{bmatrix}
1 & k_m z^{-1} \\
k_m & 1
\end{bmatrix}
\begin{bmatrix}
W_m(z) \\
S_m(z)
\end{bmatrix}
\]

The chain matrix description of the cascaded lattice structure is therefore

\[
\begin{bmatrix}
X_1(z) \\
Y_1(z)
\end{bmatrix} =
\begin{bmatrix}
1 & k_3 z^{-1} & 1 & k_2 z^{-1} & 1 & k_1 z^{-1}
\end{bmatrix}
\begin{bmatrix}
W_f(z) \\
S_f(z)
\end{bmatrix}
\]

The transfer function \( W_f(z) \) is thus an all-pole function with the same denominator as that of the 3rd-order allpass function \( A_f(z) \):

\[
W_f(z) = \frac{1}{1 + d_1 z^{-1} + d_2 z^{-2} + d_3 z^{-3}}
\]

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• From the above equation we arrive at

\[
X_1(z) = \left[ 1 + [k_1(1+k_2)+k_2k_3]z^{-1} + [k_2+k_1k_2(1+k_3)]z^{-2} + k_3z^{-3} \right] W_f(z)
\]

\[
= \left( 1 + d_1 z^{-1} + d_2 z^{-2} + d_3 z^{-3} \right) W_f(z)
\]

using the relation \( S_f(z) = W_f(z) \) and the relations

\[
k_1 = d_1', \quad k_2 = d_2', \quad k_3 = d_3
\]

IIR Tapped Cascaded Lattice Structures

Gray-Markel Method

• A two-step method to realize an \( M \)-th order arbitrary IIR transfer function

\[
H(z) = P_M(z) / D_M(z)
\]

Step 1: An intermediate allpass transfer function \( A_M(z) = z^{-M} D_M(z^{-1}) / D_M(z) \) is realized in the form of a cascaded lattice structure

\[
H(z) = \frac{P_3(z)}{D_3(z)} = \frac{p_0 + p_1 z^{-1} + p_2 z^{-2} + p_3 z^{-3}}{1 + d_1 z^{-1} + d_2 z^{-2} + d_3 z^{-3}}
\]
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- In the first step, we form a 3rd-order allpass transfer function
  \[ A_3(z) = Y_1(z)/X_1(z) = z^{-3}D_3(z^{-1})/D_3(z) \]
- Realization of \( A_3(z) \) has been illustrated earlier resulting in the structure shown below.

\[
\left( \begin{array}{c}
A_3(z) = Y_1(z)/X_1(z) \\
A_2(z) = Y_2(z)/X_2(z) \\
A_1(z) = Y_3(z)/X_3(z)
\end{array} \right)
\]

IIR Tapped Cascaded Lattice Structures

- Objective: Sum the independent signal variables \( Y_1, S_1, S_2, \) and \( S_3 \) with weights \( \{a_i\} \) as shown below to realize the desired numerator \( P_3(z) \)

\[
Y_1(z) + S_1(z) + S_2(z) + S_3(z) = P_3(z)
\]

IIR Tapped Cascaded Lattice Structures

- To this end, we first analyze the cascaded lattice structure realizing and determine the transfer functions \( S_1(z)/X_1(z) \), \( S_2(z)/X_1(z) \), and \( S_3(z)/X_1(z) \).
- We have already shown
  \[
  \frac{S_1(z)}{X_1(z)} = \frac{1}{D_3(z)}
  \]

IIR Tapped Cascaded Lattice Structures

- From the figure it follows that
  \[
  S_2(z) = (k_1 + z^{-1})S_1(z) = (d_1^* + z^{-1})S_1(z)
  \]
  and hence
  \[
  \frac{S_2(z)}{X_1(z)} = \frac{d_1^* + z^{-1}}{D_3(z)}
  \]

IIR Tapped Cascaded Lattice Structures

- In a similar manner it can be shown that
  \[
  S_3(z) = (d_2 + d_4z^{-1} + z^{-2})S_1(z)
  \]
- Thus,
  \[
  \frac{S_3(z)}{X_1(z)} = \frac{d_2 + d_4z^{-1} + z^{-2}}{D_3(z)}
  \]
- Note: The numerator of \( S_3(z)/X_1(z) \) is precisely the numerator of the allpass transfer function \( A_3(z) = S_1(z)/W_1(z) \)

\[
\left( \begin{array}{c}
Y_1(z) = \alpha_1 \frac{Y_1(z)}{X_1(z)} + \alpha_2 \frac{S_1(z)}{X_1(z)} + \alpha_3 \frac{S_1(z)}{X_1(z)} + \alpha_4 \frac{S_1(z)}{X_1(z)}
\end{array} \right)
\]
IIR Tapped Cascaded Lattice Structures

• Substituting the expressions for the various transfer functions in the above equation we arrive at

\[ Y_o(z) = \frac{\alpha_1(d_3 + d_2 z^{-1} + d_1 z^{-2} + z^{-3})}{X(z)} = \frac{\alpha_2(d_2^* + d_1 z^{-1} + z^{-2}) + \alpha_3(d_1^* + z^{-1}) + \alpha_4}{D_3(z)} \]

\[ \alpha_4 = p_0 - \alpha_3 d_3 - \alpha_2 d_2 - \alpha_1 d_1^* \]

IIR Tapped Cascaded Lattice Structures

• Comparing the numerator of \( Y_o(z)/X(z) \) with the desired numerator \( P_1(z) \) and equating like powers of \( z^{-1} \) we obtain

\[ \alpha_4 d_3 + \alpha_2 d_2 + \alpha_3 d_1 + \alpha_4 = p_0 \]
\[ \alpha_4 d_3 + \alpha_2 d_2 + \alpha_3 + \alpha_4 = p_1 \]
\[ \alpha_4 d_3 + \alpha_2 d_2 + \alpha_1 = p_2 \]
\[ \alpha_4 d_3 + \alpha_3 = p_3 \]

IIR Tapped Cascaded Lattice Structures

• Solving the above equations we arrive at

\[ \alpha_4 = p_0 - \alpha_3 d_3 - \alpha_2 d_2 - \alpha_1 d_1^* \]

\[ \alpha_4 = p_0 - \alpha_3 d_3 - \alpha_2 d_2 - \alpha_1 d_1^* \]

IIR Tapped Cascaded Lattice Structures

• Example - Consider

\[ H(z) = \frac{P_3(z)}{D_3(z)} = \frac{0.44 z^{-1} + 0.362 z^{-2} + 0.02 z^{-3}}{1 + 0.4 z^{-1} + 0.18 z^{-2} - 0.2 z^{-3}} \]

• The corresponding intermediate allpass transfer function is given by

\[ A_3(z) = \frac{z^{-3} D_3(z)}{D_3(z)} = \frac{-0.2 + 0.18 z^{-1} + 0.4 z^{-2} + z^{-3}}{1 + 0.4 z^{-1} + 0.18 z^{-2} - 0.2 z^{-3}} \]

IIR Tapped Cascaded Lattice Structures

• The allpass transfer function \( A_3(z) \) was realized earlier in the cascaded lattice form as shown below

• In the figure,

\[ k_3 = d_3 = -0.2, \quad k_2 = d_2 = 0.2708333 \]
\[ k_1 = d_1^* = 0.3573771 \]

IIR Tapped Cascaded Lattice Structures

• Other pertinent coefficients are:

\[ d_1 = 0.4, \quad d_2 = 0.18, \quad d_3 = -0.2, \quad d_1 = 0.4541667 \]
\[ p_0 = 0, \quad p_1 = 0.44, \quad p_2 = 0.36, \quad p_3 = 0.02 \]

• Substituting these coefficients in

\[ \alpha_1 = p_3 \]
\[ \alpha_2 = p_2 - \alpha_3 d_1 \]
\[ \alpha_3 = p_1 - \alpha_3 d_2 - \alpha_2 d_1^* \]
\[ \alpha_4 = p_0 - \alpha_3 d_3 - \alpha_3 d_2 - \alpha_4 d_1^* \]
IIR Tapped Cascaded Lattice Structures

\[ \alpha_1 = 0.02, \quad \alpha_2 = 0.352 \]
\[ \alpha_3 = 0.2765333, \quad \alpha_4 = -0.19016 \]

- The final realization is as shown below

\[ k_1 = 0.3573771, \quad k_2 = 0.2708333, \quad k_3 = -0.2 \]

Tapped Cascaded Lattice Realization Using MATLAB

- Both the pole-zero and the all-pole IIR cascaded lattice structures can be developed from their prescribed transfer functions using the M-file \texttt{tf2latc}
- To this end, Program 6.4 can be employed

FIR Cascaded Lattice Structures

- An arbitrary \( N \)-th order FIR transfer function of the form
  \[ H_N(z) = 1 + \sum_{m=1}^{N} p_m z^{-m} \]
  can be realized as a cascaded lattice structure as shown below

\[
\begin{align*}
X_m(z) & = X_{m-1}(z) + k_m z^{-1} Y_{m-1}(z) \\
Y_m(z) & = k_m X_{m-1}(z) + z^{-1} Y_{m-1}(z)
\end{align*}
\]

- In matrix form the above equations can be written as

\[
\begin{bmatrix}
X_m(z) \\
Y_m(z)
\end{bmatrix} =
\begin{bmatrix}
1 & k_m z^{-1} \\
k_m & z^{-1}
\end{bmatrix}
\begin{bmatrix}
X_{m-1}(z) \\
Y_{m-1}(z)
\end{bmatrix}
\]

where \( m = 1, 2, \ldots, N \)
FIR Cascaded Lattice Structures

- From the previous equation we observe
  \[ H_1(z) = 1 + k_1 z^{-1}, \quad G_1(z) = k_1 + z^{-1} \]
  where we have used the facts
  \[ H_0(z) = X_0(z)/X_0(z) = 1 \]
  \[ G_0(z) = Y_0(z)/X_0(z) = X_0(z)/X_0(z) = 1 \]
- It follows from the above that
  \[ G_1(z) = z^{-1}(z k_1 + 1) = z^{-1}H_1(z^{-1}) \]
- \( G_1(z) \) is the mirror-image of \( H_1(z) \)

FIR Cascaded Lattice Structures

- From the input-output relations of the \( m \)-th two-pair we obtain for \( m = 2 \):
  \[ H_2(z) = H_1(z) + k_2 z^{-1}G_1(z) \]
  \[ G_2(z) = k_2 H_1(z) + z^{-2}H_1(z^{-1}) \]
  \[ H_2(z) = H_1(z) + k_2 z^{-1}G_1(z) \]
  \[ G_2(z) = k_2 H_1(z) + z^{-2}H_1(z^{-1}) \]
- Now we can write
  \[ G_2(z) = k_2 H_1(z) + z^{-2}H_1(z^{-1}) \]
  \[ = z^{-2}[k_2 z^{-2}H_1(z) + H_1(z^{-1})] = z^{-2}H_2(z^{-1}) \]
- \( G_2(z) \) is the mirror-image of \( H_2(z) \)

FIR Cascaded Lattice Structures

- Substituting \( G_1(z) = z^{-1}H_1(z^{-1}) \) in the two previous equations we get
  \[ H_2(z) = H_1(z) + k_2 z^{-1}H_1(z^{-1}) \]
  \[ G_2(z) = k_2 H_1(z) + z^{-2}H_1(z^{-1}) \]
- Now we can write
  \[ G_2(z) = k_2 H_1(z) + z^{-2}H_1(z^{-1}) \]
  \[ = z^{-2}[k_2 z^{-2}H_1(z) + H_1(z^{-1})] = z^{-2}H_2(z^{-1}) \]
- \( G_2(z) \) is the mirror-image of \( H_2(z) \)

FIR Cascaded Lattice Structures

- To develop the synthesis algorithm, we express \( H_{m-1}(z) \) and \( G_{m-1}(z) \) in terms of \( H_m(z) \) and \( G_m(z) \) for \( m = N, N-1, \ldots, 2, 1 \) arriving at
  \[ H_{N-1}(z) = \frac{1}{1-k_N z} \{ H_N(z) - k_N G_N(z) \} \]
  \[ G_{N-1}(z) = \frac{1}{1-k_N z} \{ -k_N H_N(z) + G_N(z) \} \]

FIR Cascaded Lattice Structures

- Substituting the expressions for \( H_N(z) = 1 + \sum_{n=1}^{N} p_n z^{-n} \) and
  \[ G_N(z) = z^{-N} H_N(z^{-1}) = \sum_{n=0}^{N-1} p_n z^{-n} + z^{-N} \]
  in the first equation we get
  \[ H_{N-1}(z) = \frac{1}{1-k_N z} \{ (1-k_N p_N) + \sum_{n=1}^{N-1} (p_n - k_N p_{N-n}) z^{-n} \}
  \[ + (p_N - k_N z^{-N}) \]
FIR Cascaded Lattice Structures

- If we choose $k_N = p_N$, then $H_{N-1}(z)$ reduces to an FIR transfer function of order $N - 1$ and can be written in the form
  
  $H_{N-1}(z) = 1 + \sum_{n=1}^{N-1} p_n z^{-n}$
  
  where $p_n = \frac{D_n-k_N p_{N-n}}{1-k_N^2}$, $1 \leq n \leq N - 1$

- Continuing the above recursion algorithm, all multiplier coefficients of the cascaded lattice structure can be computed.

FIR Cascaded Lattice Structures

- Example - Consider
  
  $H_4(z) = 1 + 1.2 z^{-1} + 1.1 z^{-2} + 0.12 z^{-3} - 0.08 z^{-4}$

  - From the above, we observe $k_4 = p_4 = -0.08$
  
  - Using
    
    $p_n = \frac{D_n-k_N p_{N-n}}{1-k_N^2}$, $1 \leq n \leq 3$

  we determine the coefficients of $H_3(z)$:
    
    $p_3 = 0.2173913$, $p_2 = 1.2173913$

    $p_1 = 1.2173913$

- As a result, $H_3(z) = 1 + z^{-1} + z^{-2}$

- From the above, we get $k_3 = p_3 = 0.2173913$

- The final recursion yields the last multiplier coefficient $k_1 = p_1/k(1+k_2) = 0.5$

- The complete realization is shown below

FIR Cascaded Lattice Realization Using MATLAB

- The M-file tf2latc can be used to compute the multiplier coefficients of the FIR cascaded lattice structure

- To this end Program 6_6 can be employed

- The multiplier coefficients can also be determined using the M-file poly2rc