Simple Digital Filters

• Later in the course we shall review various methods of designing frequency-selective filters satisfying prescribed specifications
• We now describe several low-order FIR and IIR digital filters with reasonable selective frequency responses that often are satisfactory in a number of applications

Simple FIR Digital Filters

• FIR digital filters considered here have integer-valued impulse response coefficients
• These filters are employed in a number of practical applications, primarily because of their simplicity, which makes them amenable to inexpensive hardware implementations

Simple FIR Digital Filters

Lowpass FIR Digital Filters

• The simplest lowpass FIR digital filter is the 2-point moving-average filter given by
  \[ H_0(z) = \frac{1}{2}(1 + z^{-1}) = \frac{z + 1}{2z} \]
• The above transfer function has a zero at \( z = -1 \) and a pole at \( z = 0 \)
• Note that here the pole vector has a unity magnitude for all values of \( \omega \)

• On the other hand, as \( \omega \) increases from 0 to \( \pi \), the magnitude of the zero vector decreases from a value of 2, the diameter of the unit circle, to 0
• Hence, the magnitude response \(|H_0(e^{j\omega})|\) is a monotonically decreasing function of \( \omega \) from \( \omega = 0 \) to \( \omega = \pi \)

Simple FIR Digital Filters

• The maximum value of the magnitude function is 1 at \( \omega = 0 \), and the minimum value is 0 at \( \omega = \pi \), i.e.,
  \[ |H_0(e^{j0})| = 1, \quad |H_0(e^{j\pi})| = 0 \]
• The frequency response of the above filter is given by
  \[ H_0(e^{j\omega}) = e^{-j\omega/2} \cos(\omega/2) \]

• The magnitude response \(|H_0(e^{j\omega})| = \cos(\omega/2)\) can be seen to be a monotonically decreasing function of \( \omega \)
Simple FIR Digital Filters

• The frequency \( \omega = \omega_c \) at which

\[
|H_0(e^{j\omega_c})| = \frac{1}{\sqrt{2}} |H_0(e^{j0})|
\]

is of practical interest since here the gain \( G(\omega_c) \) in dB is given by

\[
G(\omega_c) = 20 \log_{10} |H(e^{j\omega_c})| = 20 \log_{10} |H(e^{j0})| - 20 \log_{10} \sqrt{2} \cong -3 \text{ dB}
\]

since the dc gain \( G(0) = 20 \log_{10} |H(e^{j0})| = 0 \)

• Thus, the gain \( G(\omega) \) at \( \omega = \omega_c \) is approximately 3 dB less than the gain at \( \omega = 0 \)

• As a result, \( \omega_c \) is called the 3-dB cutoff frequency

• To determine the value of \( \omega_c \) we set

\[
|H_0(e^{j\omega_c})|^2 = \cos^2 (\omega_c / 2) = \frac{1}{2}
\]

which yields \( \omega_c = \pi / 2 \)

Simple FIR Digital Filters

• The 3-dB cutoff frequency \( \omega_c \) can be considered as the passband edge frequency

• As a result, for the filter \( H_0(z) \) the passband width is approximately \( \pi / 2 \)

• The stopband is from \( \pi / 2 \) to \( \pi \)

• Note: \( H_0(z) \) has a zero at \( z = -1 \) or \( \omega = \pi \), which is in the stopband of the filter

• A cascade of the simple FIR filter

\[
H_0(z) = \frac{1}{2}(1 + z^{-1})
\]

results in an improved lowpass frequency response as illustrated below for a cascade of 3 sections

Simple FIR Digital Filters

• The 3-dB cutoff frequency of a cascade of \( M \) sections is given by

\[
\omega_c = 2 \cos^{-1} \left( \frac{1}{2} \right) = 0.302 \pi
\]

• For \( M = 3 \), the above yields \( \omega_c = 0.302 \pi \)

• Thus, the cascade of first-order sections yields a sharper magnitude response but at the expense of a decrease in the width of the passband

• A better approximation to the ideal lowpass filter is given by a higher-order moving-average filter

• Signals with rapid fluctuations in sample values are generally associated with high-frequency components

• These high-frequency components are essentially removed by an moving-average filter resulting in a smoother output waveform
Simple FIR Digital Filters

Highpass FIR Digital Filters
• The simplest highpass FIR filter is obtained from the simplest lowpass FIR filter by replacing $z$ with $-z$.
• This results in
  \[ H_1(z) = \frac{1}{2}(1 - z^{-1}) \]
• Corresponding frequency response is given by
  \[ H_1(e^{j\omega}) = j e^{-j\omega/2} \sin(\omega/2) \]
  whose magnitude response is plotted below.

First-order FIR highpass filter

Simple FIR Digital Filters
• The monotonically increasing behavior of the magnitude function can again be demonstrated by examining the pole-zero pattern of the transfer function $H_1(z)$.
• The highpass transfer function $H_1(z)$ has a zero at $z = 1$ or $\omega = 0$ which is in the stopband of the filter.

Improved highpass magnitude response can again be obtained by cascading several sections of the first-order highpass filter.

Simple FIR Digital Filters
• Alternately, a higher-order highpass filter of the form
  \[ H_1(z) = \frac{1}{M} \sum_{m=0}^{M-1} (-1)^n z^{-n} \]
  is obtained by replacing $z$ with $-z$ in the transfer function of a moving average filter.

Simple FIR Digital Filters
• An application of the FIR highpass filters is in moving-target-indicator (MTI) radars.
• In these radars, interfering signals, called clutters, are generated from fixed objects in the path of the radar beam.
• The clutter, generated mainly from ground echoes and weather returns, has frequency components near zero frequency (dc).

Simple FIR Digital Filters
• The clutter can be removed by filtering the radar return signal through a two-pulse canceler, which is the first-order FIR highpass filter $H_1(z) = \frac{1}{2}(1 - z^{-1})$.
• For a more effective removal it may be necessary to use a three-pulse canceler obtained by cascading two two-pulse cancelers.
Simple IIR Digital Filters

Lowpass IIR Digital Filters

- A first-order causal lowpass IIR digital filter has a transfer function given by
  \[ H_{LP}(z) = \frac{1 - \alpha}{2} \left( 1 + \frac{1 + \alpha^{-1}}{1 - \alpha^{-1}} \right) \]
  where \(|\alpha| < 1\) for stability
- The above transfer function has a zero at \(z = -1\) i.e., at \(\omega = \pi\) which is in the stopband

\[ |H_{LP}(z)| = \frac{|\alpha - 1 + \alpha^{-1}|}{|1 - \alpha - 1 + \alpha^{-1}|} \]

Hence, \( |H_{LP}(z)| \) has a real pole at \(z = \alpha\)

As \(\omega\) increases from 0 to \(\pi\), the magnitude of the zero vector decreases from a value of 2 to 0, whereas, for a positive value of \(\alpha\), the magnitude of the pole vector increases from a value of \(1 - \alpha\) to \(1 + \alpha\).

The maximum value of the magnitude function is 1 at \(\omega = 0\), and the minimum value is 0 at \(\omega = \pi\)

The squared magnitude function is given by
\[ |H_{LP}(e^{\jmath \omega})|^2 = \frac{(1 - \alpha)^2 (1 + \cos \omega)}{2(1 + \alpha^2 - 2\alpha \cos \omega)} \]

The derivative of \(|H_{LP}(e^{\jmath \omega})|^2\) with respect to \(\omega\) is given by
\[ \frac{d}{d\omega} |H_{LP}(e^{\jmath \omega})|^2 = -\frac{(1 - \alpha)^2 (1 + 2\alpha + \alpha^2) \sin \omega}{2(1 - 2\alpha \cos \omega + \alpha^2)^2} \]

\[ \frac{d}{d\omega} |H_{LP}(e^{\jmath \omega})|^2 = \frac{\omega c}{\omega c} \]

\[ \omega c = \frac{2\alpha}{1 + \alpha^2} \]

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The above quadratic equation can be solved for \(\alpha\) yielding two solutions
Simple IIR Digital Filters

• The solution resulting in a stable transfer function $H_{LP}(z)$ is given by
  \[ \alpha = \frac{1 - \sin \omega_c}{\cos \omega_c} \]

• It follows from
  \[ |H_{LP}(e^{j\omega})|^2 = \frac{(1 - \alpha)^2 (1 + \cos \omega)}{2(1 + \alpha^2 - 2 \alpha \cos \omega)} \]
  that $H_{LP}(z)$ is a BR function for $|\alpha| < 1$

Highpass IIR Digital Filters

• A first-order causal highpass IIR digital filter has a transfer function given by
  \[ H_{HP}(z) = \frac{1 + \alpha}{2} \left( 1 - \frac{1 - \alpha z^{-1}}{1 - \alpha z^{-1}} \right) \]
  where $|\alpha| < 1$ for stability

• The above transfer function has a zero at $z = 1$ i.e., at $\omega = 0$ which is in the stopband

Simple IIR Digital Filters

• Its 3-dB cutoff frequency $\omega_c$ is given by
  \[ \alpha = (1 - \sin \omega_c)/\cos \omega_c \]
  which is the same as that of $H_{LP}(z)$

• Magnitude and gain responses of $H_{HP}(z)$ are shown below

Bandpass IIR Digital Filters

• $H_{BP}(z)$ is a BR function for $|\alpha| < 1$

• Example - Design a first-order highpass digital filter with a 3-dB cutoff frequency of $0.8\pi$

  Now \( \sin(\omega_c) = \sin(0.8\pi) = 0.587785 \) and \( \cos(0.8\pi) = -0.80902 \)

  Therefore
  \[ \alpha = (1 - \sin \omega_c)/\cos \omega_c = -0.5095245 \]

• Therefore,
  \[ H_{BP}(z) = \frac{1 + \alpha}{2} \left( 1 - \frac{1 - \alpha z^{-1}}{1 - \alpha z^{-1}} \right) \]
  \[ = 0.245238 \left( 1 - \frac{1 - \alpha z^{-1}}{1 + 0.5095245 z^{-1}} \right) \]

• A 2nd-order bandpass digital transfer function is given by
  \[ H_{BP}(z) = \frac{1 - \alpha}{2} \left( 1 - \frac{1 - \alpha^2 z^{-2}}{1 - \beta(1 + \alpha) z^{-2} + \alpha^2 z^{-2}} \right) \]

• Its squared magnitude function is
  \[ |H_{BP}(e^{j\omega})|^2 = \frac{(1 - \alpha)^2 (1 - \cos 2\omega)}{2[1 + \beta^2(1 + \alpha)^2 + \alpha^2 + 2\beta(1 + \alpha)(1 + \alpha^2 \cos \omega + 2\alpha \cos 2\omega)]} \]
Simple IIR Digital Filters

• \(|H_{BP}(e^{j\omega})|^2\) goes to zero at \(\omega = 0\) and \(\omega = \pi\)
• It assumes a maximum value of 1 at \(\omega = \omega_c\), called the center frequency of the bandpass filter, where \(\omega_c = \cos^{-1} (\beta)\)
• The frequencies \(\omega_{1}\) and \(\omega_{2}\), where \(|H_{BP}(e^{j\omega})|^2\) becomes \(1/2\) are called the 3-dB cutoff frequencies

Simple IIR Digital Filters

• The difference between the two cutoff frequencies, assuming \(\omega_{2} > \omega_{1}\), is called the 3-dB bandwidth and is given by

\[
B_w = \omega_{2} - \omega_{1} = \cos^{-1}\left(\frac{2\alpha}{1 + \alpha^2}\right)
\]

• The transfer function \(H_{BP}(z)\) is a BR function if \(|\alpha| < 1\) and \(|\beta| < 1\)

Simple IIR Digital Filters

• Plots of \(|H_{BP}(e^{j\omega})|\) are shown below

Simple IIR Digital Filters

• Example - Design a 2nd order bandpass digital filter with center frequency at 0.4\(\pi\) and a 3-db bandwidth of 0.1\(\pi\)
• Here \(\beta = \cos(\omega_c) = \cos(0.4\pi) = 0.309017\) and

\[
2\alpha = \cos(B_w) = \cos(0.1\pi) = 0.9510565
\]

• The solution of the above equation yields: \(\alpha = 1.376382\) and \(\alpha = 0.7265423\)

Simple IIR Digital Filters

• The corresponding transfer functions are

\[
H_{BP}(z) = \frac{1 - z^{-2}}{1 - 0.18819 - 0.7343424z^{-1} + 1.37638z^{-2}}
\]

and

\[
H_{BP}(z) = \frac{1 - z^{-2}}{1 - 0.13673 - 0.533531z^{-1} + 0.72654253z^{-2}}
\]

• The poles of \(H_{BP}(z)\) are at \(z = 0.3671712 \pm j1.11425636\) and have a magnitude > 1

Simple IIR Digital Filters

• Thus, the poles of \(H_{BP}(z)\) are outside the unit circle making the transfer function unstable
• On the other hand, the poles of \(H_{BP}(z)\) are at \(z = 0.2667655 \pm j0.8095546\) and have a magnitude of 0.8523746
• Hence \(H_{BP}(z)\) is BIBO stable
• Later we outline a simpler stability test
Simple IIR Digital Filters

- Figures below show the plots of the magnitude function and the group delay of $H_{BP}(z)$

Simple IIR Digital Filters

- Its magnitude response is plotted below

Simple IIR Digital Filters

- The frequencies $\omega_{c_2}$ and $\omega_{c_1}$ where $|H_{BS}(e^{j\omega})|^2$ becomes 1/2 are called the 3-dB cutoff frequencies
- The difference between the two cutoff frequencies, assuming $\omega_{c_2} > \omega_{c_1}$, is called the 3-dB notch bandwidth and is given by $B_n = \omega_{c_2} - \omega_{c_1} = \cos^{-1}\left(\frac{2\omega_{c_1}}{1 + \omega^2}ight)$

Bandstop IIR Digital Filters

- A 2nd-order bandstop digital filter has a transfer function given by
  $$H_{BS}(z) = \frac{1 + \alpha}{2} \left(\frac{1 - 2\beta z^{-1} + z^{-2}}{1 - \beta(1 + \alpha) z^{-1} + \alpha z^{-2}}\right)$$
- The transfer function $H_{BS}(z)$ is a BR function if $|\alpha| < 1$ and $|\beta| < 1$

Simple IIR Digital Filters

- Here, the magnitude function takes the maximum value of 1 at $\omega = 0$ and $\omega = \pi$
- It goes to 0 at $\omega = \omega_o$, where $\omega_o$, called the notch frequency, is given by $\omega_o = \cos^{-1}(\beta)$
- The digital transfer function $H_{BS}(z)$ is more commonly called a notch filter

Higher-Order IIR Digital Filters

- By cascading the simple digital filters discussed so far, we can implement digital filters with sharper magnitude responses
- Consider a cascade of $K$ first-order lowpass sections characterized by the transfer function
  $$H_{LP}(z) = \frac{1 - \alpha}{2} \left(\frac{1 + z^{-1}}{1 - \alpha z^{-1}}\right)$$
Simple IIR Digital Filters

- The overall structure has a transfer function given by
  \[ G_{LP}(z) = \left( \frac{1 - \alpha}{2} \right)^K \left( \frac{1 + \alpha^{-1}}{1 - \alpha^{-1}} \right) \]

- The corresponding squared-magnitude function is given by
  \[ |G_{LP}(e^{j\omega})|^2 = \left( \frac{1 - \alpha^2}{2(1 + \alpha^2 - 2\alpha \cos \omega)} \right)^K \]

\[ K \]

Simple IIR Digital Filters

where

\[ C = 2^{(K-1)/K} \]

- It should be noted that the expression for \( \alpha \) given earlier reduces to
  \[ \alpha = \frac{1 - \sin \omega_c}{\cos \omega_c} \]
  for \( K = 1 \)

Simple IIR Digital Filters

- To determine the relation between its 3-dB cutoff frequency \( \omega_c \) and the parameter \( \alpha \), we set
  \[ \frac{(1 - \omega^2)(1 + \cos \omega_c)}{2(1 + \alpha^2 - 2\alpha \cos \omega_c)} = \frac{1}{2} \]
  which when solved for \( \alpha \), yields for a stable \( G_{LP}(z) \)
  \[ \alpha = \frac{1 + (1 - C) \cos \omega_c - \sin \omega_c \sqrt{2C - C^2}}{1 - C + \cos \omega_c} \]

Simple IIR Digital Filters

- Example - Design a lowpass filter with a 3-dB cutoff frequency \( \omega_c = 0.4\pi \) using a single first-order section and a cascade of 4 first-order sections, and compare their gain responses
  - For the single first-order lowpass filter we have
    \[ \alpha = \frac{1 + \sin 0.4\pi}{\cos 0.4\pi} = \frac{1 + \sin 0.4\pi}{\cos 0.4\pi} = 0.1584 \]

Simple IIR Digital Filters

- For the cascade of 4 first-order sections, we substitute \( K = 4 \) and get
  \[ C = 2^{(K-1)/K} = 2^{(4-1)/4} = 1.6818 \]
  Next we compute
  \[ \alpha = \frac{1 + (1 - C) \cos \omega_c - \sin \omega_c \sqrt{2C - C^2}}{1 - C + \cos \omega_c} \]
  \[ = \frac{1 + (1 - 0.1584) \cos (0.4\pi) - \sin (0.4\pi) \sqrt{2(1.6818) - (1.6818)^2}}{1 - 1.6818 + \cos (0.4\pi)} \]
  \[ = -0.251 \]

Simple IIR Digital Filters

- The gain responses of the two filters are shown below
- As can be seen, cascading has resulted in a sharper roll-off in the gain response