Linear-Phase FIR Transfer Functions

- It is nearly impossible to design a linear-phase IIR transfer function.
- It is always possible to design an FIR transfer function with an exact linear-phase response.
- Consider a causal FIR transfer function $H(z)$ of length $N+1$, i.e., of order $N$:
  $$H(z) = \sum_{n=0}^{N} h[n] z^{-n}$$

- The above transfer function has a linear phase, if its impulse response $h[n]$ is either symmetric, i.e.,
  $$h[n] = h[N-n], \quad 0 \leq n \leq N$$
  or is antisymmetric, i.e.,
  $$h[n] = -h[N-n], \quad 0 \leq n \leq N$$

Since the length of the impulse response can be either even or odd, we can define four types of linear-phase FIR transfer functions:

- For an antisymmetric FIR filter of odd length, i.e., $N$ even,
  $$h[N/2] = 0$$
  We examine next each of the 4 cases.

Type 1: Symmetric Impulse Response with Odd Length

- In this case, the degree $N$ is even
- Assume $N = 8$ for simplicity
- The transfer function $H(z)$ is given by

Type 2: $N=7$

Type 3: $N=8$

Type 4: $N=7$
Linear-Phase FIR Transfer Functions

- The corresponding frequency response is then given by
  \[ H(e^{j\omega}) = e^{-j\omega} [2h[0]\cos(4\omega) + 2h[1]\cos(3\omega) + 2h[2]\cos(2\omega) + 2h[3]\cos(\omega) + h[4]] \]

- The quantity inside the braces is a real function of \( \omega \), and can assume positive or negative values in the range \( 0 \leq |\omega| \leq \pi \)

Linear-Phase FIR Transfer Functions

- The phase function here is given by
  \[ \theta(\omega) = -4\omega + \beta \]
  where \( \beta \) is either 0 or \( \pi \) and hence, it is a linear function of \( \omega \) in the generalized sense

- The group delay is given by
  \[ \tau(\omega) = -\frac{d\theta(\omega)}{d\omega} = 4 \]
  indicating a constant group delay of 4 samples

Linear-Phase FIR Transfer Functions

- In the general case for Type 1 FIR filters, the frequency response is of the form
  \[ H(e^{j\omega}) = e^{-j\omega} \tilde{H}(\omega) \]
  where the amplitude response \( \tilde{H}(\omega) \), also called the zero-phase response, is of the form
  \[ \tilde{H}(\omega) = h[N/2] + 2 \sum_{n=1}^{N/2} h[N - n] \cos(\omega n) \]

Linear-Phase FIR Transfer Functions

- Example - Consider
  \[ H_0(z) = \frac{1}{6}(1 + z^{-1}) \frac{1}{6} (1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5} + \frac{1}{2} z^{-6}) \]
  which is seen to be a slightly modified version of a length-7 moving-average FIR filter

- The above transfer function has a symmetric impulse response and therefore a linear phase response

Linear-Phase FIR Transfer Functions

- A plot of the magnitude response of \( H_0(z) \) along with that of the 7-point moving-average filter is shown below

- Note the improved magnitude response obtained by simply changing the first and the last impulse response coefficients of a moving-average (MA) filter

- It can be shown that we can express
  \[ H_0(z) = \frac{1}{2}(1 + z^{-1}) \frac{1}{6} (1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}) \]
  which is seen to be a cascade of a 2-point MA filter with a 6-point MA filter

- Thus, \( H_0(z) \) has a double zero at \( z = -1 \) i.e.,
  \( (\omega = \pi) \)
Linear-Phase FIR Transfer Functions

Type 2: Symmetric Impulse Response with Even Length

- In this case, the degree \( N \) is odd
- Assume \( N = 7 \) for simplicity
- The transfer function is of the form

\[
\]

- Making use of the symmetry of the impulse response coefficients, the transfer function can be written as

\[
H(z) = h[0](1 + z^{-7}) + h[1](z^{-1} + z^{-6}) + h[2](z^{-2} + z^{-5}) + h[3](z^{-3} + z^{-4})
\]

\[
= z^{-7/2}[h[0](z^{7/2} + z^{-7/2}) + h[1](z^{5/2} + z^{-5/2}) + h[2](z^{3/2} + z^{-3/2}) + h[3](z^{1/2} + z^{-1/2})]
\]

- The corresponding frequency response is given by

\[
H(e^{j\omega}) = e^{-j\pi \omega /2} \{h[0]\cos(\frac{\pi \omega}{2}) + 2h[1]\cos(\frac{3\pi \omega}{2}) + 2h[2]\cos(\frac{5\pi \omega}{2}) + 2h[3]\cos(\frac{7\pi \omega}{2})\}
\]

- As before, the quantity inside the braces is a real function of \( \omega \) and can assume positive or negative values in the range \( 0 \leq |\omega| \leq \pi \)

- Here the phase function is given by

\[
\theta(\omega) = -\frac{\pi}{2} \omega + \beta
\]

where again \( \beta \) is either 0 or \( \pi \)

- As a result, the phase is also a linear function of \( \omega \) in the generalized sense

- The corresponding group delay is

\[
\tau(\omega) = \frac{7}{2}
\]

indicating a group delay of \( \frac{7}{2} \) samples

Linear-Phase FIR Transfer Functions

Type 3: Antiymmetric Impulse Response with Odd Length

- In this case, the degree \( N \) is even
- Assume \( N = 8 \) for simplicity
- Applying the symmetry condition we get

\[
H(z) = z^{-4}[h[0](z^{4} - z^{-4}) + h[1](z^{3} - z^{-3}) + h[2](z^{2} - z^{-2}) + h[3](z - z^{-1})]
\]
Linear-Phase FIR Transfer Functions

• The corresponding frequency response is given by
  \[ H(e^{j\theta}) = e^{-j4\theta}e^{-j\pi/2} \{ 2h[0]\sin(4\omega) + 2h[1]\sin(3\omega) + 2h[2]\sin(2\omega) + 2h[3]\sin(\omega) \} \]

• It also exhibits a generalized phase response given by
  \[ \theta(\omega) = -4\omega + \frac{\pi}{2} + \beta \]
  where \( \beta \) is either 0 or \( \pi \)

• The group delay is constant and is given by
  \[ \tau(\omega) = 4 \]
  indicating a constant group delay of 4 samples

• In the general case we have
  \[ H(e^{j\theta}) = e^{-j4\theta}e^{-j\pi/2} \{ 2h[0]\sin(2\omega) + 2h[1]\sin(\omega/2) + 2h[2]\sin(\omega) + 2h[3]\sin(3\omega) \} \]

• In each of the four types of linear-phase FIR filters, the frequency response is of the form
  \[ H(e^{j\theta}) = e^{-j4\theta}e^{-j\pi/2} \{ 2h[0]\sin(\omega/2) + 2h[1]\sin(\omega) + 2h[2]\sin(3\omega) + 2h[3]\sin(2\omega) \} \]

• The amplitude response becomes negative over certain frequency ranges, typically in the stopband

Linear-Phase FIR Transfer Functions

Type 4: Antsymmetric Impulse Response with Even Length

• In this case, the degree \( N \) is even
• Assume \( N = 7 \) for simplicity
• Applying the symmetry condition we get
  \[ H(z) = z^{-7/2} \{ h[0](z^{7/2} - z^{-7/2}) + h[1](z^{5/2} - z^{-5/2}) + h[2](z^{3/2} - z^{-3/2}) + h[3](z^{1/2} - z^{-1/2}) \} \]
  \[ = 2 \sum_{n=1}^{N/2} h[\frac{N+1}{2} - n] \sin(\omega(n - \frac{1}{2})) \]

Linear-Phase FIR Transfer Functions

General Form of Frequency Response

• In each of the four types of linear-phase FIR filters, the frequency response is of the form
  \[ H(e^{j\theta}) = e^{-j4\theta}e^{-j\pi/2} \{ 2h[0]\sin(\omega) + 2h[1]\sin(\omega/2) + 2h[2]\sin(3\omega) + 2h[3]\sin(2\omega) \} \]

• The amplitude response \( \tilde{H}(\omega) \) for each of the four types of linear-phase FIR filters can become negative over certain frequency ranges, typically in the stopband

Linear-Phase FIR Transfer Functions

• The corresponding frequency response is given by
  \[ H(e^{j\theta}) = e^{-j4\theta}e^{-j\pi/2} \{ 2h[0]\sin(\omega) + 2h[1]\sin(\omega/2) + 2h[2]\sin(3\omega) + 2h[3]\sin(2\omega) \} \]
Linear-Phase FIR Transfer Functions

- The magnitude and phase responses of the linear-phase FIR are given by
  \[ |H(e^{j\omega})| = |\tilde{H}(\omega)| \]
  \[ \Theta(\omega) = \begin{cases} \frac{-N\omega}{2} + \beta, & \text{for } \tilde{H}(\omega) \geq 0 \\ \frac{-N\omega}{2} + \beta - \pi, & \text{for } \tilde{H}(\omega) < 0 \end{cases} \]
- The group delay in each case is
  \[ \tau(\omega) = \frac{N}{2} \]

Zero Locations of Linear-Phase FIR Transfer Functions

- Consider first an FIR filter with a symmetric impulse response: \( h[n] = h[N-n] \)
- Its transfer function can be written as
  \[ H(z) = \sum_{n=0}^{N} h[n]z^{-n} = \sum_{n=0}^{N} h[N-n]z^{-n} \]
  By making a change of variable \( m = N-n \), we can write
  \[ \sum_{n=0}^{N} h[n-n]z^{-n} = \sum_{m=0}^{N} h[m]z^{-N+m} = z^{-N} \sum_{m=0}^{N} h[m]z^{m} \]

Zero Locations of Linear-Phase FIR Transfer Functions

- Now consider first an FIR filter with an antisymmetric impulse response: \( h[n] = -h[N-n] \)
- Its transfer function can be written as
  \[ H(z) = \sum_{n=0}^{N} h[n]z^{-n} = -\sum_{n=0}^{N} h[N-n]z^{-n} \]
  By making a change of variable \( m = N-n \), we get
  \[ \sum_{n=0}^{N} h[N-n]z^{-n} = -\sum_{m=0}^{N} h[m]z^{-N+m} = -z^{-N}H(z^{-1}) \]

Zero Locations of Linear-Phase FIR Transfer Functions

- Note that, even though the group delay is constant, since in general \( |H(e^{j\beta})| \) is not a constant, the output waveform is not a replica of the input waveform
- An FIR filter with a frequency response that is a real function of \( \omega \) is often called a zero-phase filter
- Such a filter must have a noncausal impulse response
Zero Locations of Linear-Phase FIR Transfer Functions

- It follows from the relation \( H(z) = \pm z^{-N} H(z^{-1}) \) that if \( z = e^{j\phi} \) is a zero of \( H(z) \), so is \( z = 1/z^{*} \).
- Moreover, for an FIR filter with a real impulse response, the zeros of \( H(z) \) occur in complex conjugate pairs.
- Hence, a zero at \( z = e^{j\phi} \) is associated with a zero at \( z = e^{-j\phi} \).

- Thus, a complex zero that is not on the unit circle is associated with a set of 4 zeros given by \( z = re^{\pm j\phi} \) and \( z = 1/\bar{z}^{\pm} \).
- A zero on the unit circle appears as a pair \( z = e^{j\phi} \) as its reciprocal is also its complex conjugate.

- Since a zero at \( z = \pm 1 \) is its own reciprocal, it can appear only singly.
- Now a Type 2 FIR filter satisfies \( H(z) = z^{-N} H(z^{-1}) \) with degree \( N \) odd.
- Hence \( H(-1) = (-1)^{-N} H(-1) = -H(-1) \), implying \( H(-1) = 0 \), i.e., \( H(z) \) must have a zero at \( z = -1 \).

- Likewise, a Type 3 or 4 FIR filter satisfies \( H(z) = -z^{-N} H(z^{-1}) \).
- Thus \( H(1) = -(1)^{-N} H(1) = -H(1) \) implying that \( H(z) \) must have a zero at \( z = 1 \).
- On the other hand, only the Type 3 FIR filter is restricted to have a zero at \( z = -1 \) since here the degree \( N \) is even and hence, \( H(-1) = -(1)^{-N} H(-1) = -H(-1) \).

- Summarizing
  (1) Type 1 FIR filter: Either an even number or no zeros at \( z = 1 \) and \( z = -1 \).
  (2) Type 2 FIR filter: Either an even number or no zeros at \( z = 1 \), and an odd number of zeros at \( z = -1 \).
  (3) Type 3 FIR filter: An odd number of zeros at \( z = 1 \) and \( z = -1 \).
Zero Locations of Linear-Phase FIR Transfer Functions

(4) Type 4 FIR filter: An odd number of zeros at \( z = 1 \), and either an even number or no zeros at \( z = -1 \)
- The presence of zeros at \( z = \pm 1 \) leads to the following limitations on the use of these linear-phase transfer functions for designing frequency-selective filters

Zero Locations of Linear-Phase FIR Transfer Functions

- A Type 2 FIR filter cannot be used to design a highpass filter since it always has a zero at \( z = 1 \)
- A Type 3 FIR filter has zeros at both \( z = 1 \) and \( z = -1 \), and hence cannot be used to design either a lowpass or a highpass or a bandstop filter

Zero Locations of Linear-Phase FIR Transfer Functions

- A Type 4 FIR filter is not appropriate to design a lowpass filter due to the presence of a zero at \( z = 1 \)
- Type 1 FIR filter has no such restrictions and can be used to design almost any type of filter

Bounded Real Transfer Functions

- A causal stable real-coefficient transfer function \( H(e^{j\omega}) \) is defined as a bounded real (BR) transfer function if
  \[ |H(e^{j\omega})| \leq 1 \quad \text{for all values of } \omega \]
- Let \( x[n] \) and \( y[n] \) denote, respectively, the input and output of a digital filter characterized by a BR transfer function \( H(e^{j\omega}) \) with \( X(e^{j\omega}) \) and \( Y(e^{j\omega}) \) denoting their DTFTs

Bounded Real Transfer Functions

- Then the condition \( |H(e^{j\omega})| \leq 1 \) implies that
  \[ |Y(e^{j\omega})|^2 \leq |X(e^{j\omega})|^2 \]
- Integrating the above from \(-\pi\) to \(\pi\), and applying Parseval’s relation we get
  \[ \sum_{n=-\infty}^{\infty} |x[n]|^2 \leq \sum_{n=-\infty}^{\infty} |y[n]|^2 \]

Bounded Real Transfer Functions

- Thus, for all finite-energy inputs, the output energy is less than or equal to the input energy implying that a digital filter characterized by a BR transfer function can be viewed as a passive structure
- If \( |H(e^{j\omega})| = 1 \), then the output energy is equal to the input energy, and such a digital filter is therefore a lossless system
Bounded Real Transfer Functions

- A causal stable real-coefficient transfer function $H(z)$ with $|H(e^{j\omega})|=1$ is thus called a lossless bounded real (LBR) transfer function.
- The BR and LBR transfer functions are the keys to the realization of digital filters with low coefficient sensitivity.