**z-Transform**

- The DTFT provides a frequency-domain representation of discrete-time signals and LTI discrete-time systems.
- Because of the convergence condition, in many cases, the DTFT of a sequence may not exist.
- As a result, it is not possible to make use of such frequency-domain characterization in these cases.

**z-Transform**

- A generalization of the DTFT defined by
  \[ X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \]
  leads to the z-transform.
- z-transform may exist for many sequences for which the DTFT does not exist.
- Moreover, use of z-transform techniques permits simple algebraic manipulations.

**z-Transform**

- If we let \( z = re^{j\omega} \), then the z-transform reduces to
  \[ G(re^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n] r^{-n} e^{-j\omega n} \]
  - The above can be interpreted as the DTFT of the modified sequence \( \{g[n] r^{-n}\} \).
  - For \( r = 1 \) (i.e., \( |z| = 1 \)), z-transform reduces to its DTFT, provided the latter exists.

**z-Transform**

- Consequently, z-transform has become an important tool in the analysis and design of digital filters.
- For a given sequence \( g[n] \), its z-transform \( G(z) \) is defined as
  \[ G(z) = \sum_{n=-\infty}^{\infty} g[n] z^{-n} \]
  where \( z = \text{Re}(z) + j\text{Im}(z) \) is a complex variable.

**z-Transform**

- The contour \( |z| = 1 \) is a circle in the z-plane of unity radius and is called the unit circle.
- Like the DTFT, there are conditions on the convergence of the infinite series
  \[ \sum_{n=-\infty}^{\infty} g[n] z^{-n} \]
  - For a given sequence, the set \( R \) of values of \( z \) for which its z-transform converges is called the region of convergence (ROC).

**z-Transform**

- From our earlier discussion on the uniform convergence of the DTFT, it follows that the series
  \[ G(re^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n] r^{-n} e^{-j\omega n} \]
  converges if \( \{g[n] r^{-n}\} \) is absolutely summable, i.e., if
  \[ \sum_{n=-\infty}^{\infty} |g[n] r^{-n}| < \infty \]
**z-Transform**

- In general, the ROC $R$ of a $z$-transform of a sequence $g[n]$ is an annular region of the $z$-plane:
  \[ R_{g^-} < |z| < R_{g^+} \]
  where $0 \leq R_{g^-} < R_{g^+} \leq \infty$

- Note: The $z$-transform is a form of a Laurent series and is an analytic function at every point in the ROC

**Example** - Determine the $z$-transform $X(z)$ of the causal sequence $x[n] = \alpha^n \mu[n]$ and its ROC

- Now $X(z) = \sum_{n=-\infty}^{\infty} \alpha^n \mu[n] z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n}$

- The above power series converges to $X(z) = \frac{1}{1 - \alpha z^{-1}}$, for $|\alpha z^{-1}| < 1$

- ROC is the annular region $|z| > |\alpha|$
**z-Transform**

- The DTFT $G(e^{j\omega})$ of a sequence $g[n]$ converges uniformly if and only if the ROC of the z-transform $G(z)$ of $g[n]$ includes the unit circle.
- The existence of the DTFT does not always imply the existence of the z-transform.

**Example** - The finite energy sequence $h_{LP}[n] = \frac{\sin \omega_c n}{\pi n}$, $-\infty < n < \infty$ has a DTFT given by $H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \leq |\omega| \leq \omega_c \\ 0, & \omega_c < |\omega| \leq \pi \end{cases}$ which converges in the mean-square sense.

**Table 3.8: Commonly Used z-Transform Pairs**

<table>
<thead>
<tr>
<th>Sequence</th>
<th>z-Transform</th>
<th>ROC</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta[n]$</td>
<td>1</td>
<td>All values of $z$</td>
</tr>
<tr>
<td>$\mu[n]$</td>
<td>$\frac{1}{1 - z^{-1}}$</td>
<td>$</td>
</tr>
<tr>
<td>$a^n[n]$</td>
<td>$\frac{1}{1 - a z^{-1}}$</td>
<td>$</td>
</tr>
<tr>
<td>$(r^m \cos \omega_0 n)\mu[n]$</td>
<td>$\frac{1 - (r \cos \omega_0) z^{-1}}{1 - (2r \cos \omega_0) z^{-1} + r^2 z^{-2}}$</td>
<td>$</td>
</tr>
<tr>
<td>$(r^m \sin \omega_0 n)\mu[n]$</td>
<td>$\frac{1 - (r \sin \omega_0) z^{-1}}{1 - (2r \cos \omega_0) z^{-1} + r^2 z^{-2}}$</td>
<td>$</td>
</tr>
</tbody>
</table>

**Rational z-Transforms**

- In the case of LTI discrete-time systems we are concerned with in this course, all pertinent z-transforms are rational functions of $z^{-1}$.
- That is, they are ratios of two polynomials in $z^{-1}$:

$$G(z) = \frac{P(z)}{D(z)} = \frac{P_0 + P_1 z^{-1} + \cdots + P_M z^{-(M-1)} + P_M z^{-M}}{D_0 + D_1 z^{-1} + \cdots + D_N z^{-(N-1)} + D_N z^{-N}}$$

**Rational z-Transforms**

- The degree of the numerator polynomial $P(z)$ is $M$ and the degree of the denominator polynomial $D(z)$ is $N$.
- An alternate representation of a rational z-transform is as a ratio of two polynomials in $z^*$:

$$G(z) = z^{(N-M)} \frac{P_0 z^M + P_1 z^{M-1} + \cdots + P_M z + P_M}{D_0 z^N + D_1 z^{N-1} + \cdots + D_N z + D_N}$$
Rational $z$-Transforms

• A rational $z$-transform can be alternately written in factored form as

$$G(z) = \frac{p_0 \prod_{n=1}^{M} (1 - \xi_n z^{-1})}{d_0 \prod_{l=1}^{N} (1 - \lambda_l z^{-1})}$$

$$= z^{(N-M)} \frac{p_0 \prod_{n=1}^{M} (z - \xi_n)}{d_0 \prod_{l=1}^{N} (z - \lambda_l)}$$

Rational $z$-Transforms

• At a root $z = \xi$ of the numerator polynomial $G(\xi) = 0$, and as a result, these values of $z$ are known as the zeros of $G(z)$

• At a root $z = \lambda$ of the denominator polynomial $G(\lambda) \to \infty$, and as a result, these values of $z$ are known as the poles of $G(z)$

Rational $z$-Transforms

• Consider

$$G(z) = z^{(N-M)} \frac{p_0 \prod_{n=1}^{M} (z - \xi_n)}{d_0 \prod_{l=1}^{N} (z - \lambda_l)}$$

• Note $G(z)$ has $M$ finite zeros and $N$ finite poles

• If $N > M$ there are additional $N - M$ zeros at $z = 0$ (the origin in the $z$-plane)

• If $N < M$ there are additional $M - N$ poles at $z = 0$

Rational $z$-Transforms

• Example - The $z$-transform

$$\mu(z) = \frac{1}{1 - z^{-1}}, \text{ for } |z| > 1$$

has a zero at $z = 0$ and a pole at $z = 1$

Rational $z$-Transforms

• A physical interpretation of the concepts of poles and zeros can be given by plotting the log-magnitude $20 \log_{10}|G(z)|$ as shown on next slide for

$$G(z) = \frac{1 - 2.4 z^{-1} + 2.88 z^{-2}}{1 - 0.8 z^{-1} + 0.64 z^{-2}}$$
Rational z-Transforms

- Observe that the magnitude plot exhibits very large peaks around the points $z = 0.4 \pm j0.6928$ which are the poles of $G(z)$
- It also exhibits very narrow and deep wells around the location of the zeros at $z = 1.2 \pm j1.2$

ROC of a Rational z-Transform

- ROC of a z-transform is an important concept
- Without the knowledge of the ROC, there is no unique relationship between a sequence and its z-transform
- Hence, the z-transform must always be specified with its ROC

ROC of a Rational z-Transform

- Moreover, if the ROC of a z-transform includes the unit circle, the DTFT of the sequence is obtained by simply evaluating the z-transform on the unit circle
- There is a relationship between the ROC of the z-transform of the impulse response of a causal LTI discrete-time system and its BIBO stability

ROC of a Rational z-Transform

- In this plot, the ROC, shown as the shaded area, is the region of the z-plane just outside the circle centered at the origin and going through the pole at $z = 1$

ROC of a Rational z-Transform

- Example - The z-transform $H(z)$ of the sequence $h[n] = (-0.6)^n \mu[n]$ is given by $H(z) = \frac{1}{1 + 0.6z^{-1}}$, $|z| > 0.6$
- Here the ROC is just outside the circle going through the point $z = -0.6$
ROC of a Rational z-Transform

- A sequence can be one of the following types: finite-length, right-sided, left-sided and two-sided
- In general, the ROC depends on the type of the sequence of interest

Example - Consider a finite-length sequence \( g[n] \) defined for \(-M \leq n \leq N\), where \( M \) and \( N \) are non-negative integers and \( \|g[n]\| < \infty \)
- Its z-transform is given by
\[
G(z) = \sum_{n=-M}^{N} g[n] z^{-n} = \sum_{0}^{N-M} g[n-M] z^{N+M-n}
\]

ROC of a Rational z-Transform

- Example - A right-sided sequence with nonzero sample values for \( n \geq 0 \) is sometimes called a causal sequence
- Consider a causal sequence \( u_1[n] \)
- Its z-transform is given by
\[
U_1(z) = \sum_{n=0}^{\infty} u_1[n] z^{-n}
\]

ROC of a Rational z-Transform

- Example - A left-sided sequence with nonzero sample values for \( n \leq 0 \) is sometimes called a anticausal sequence
- Consider an anticausal sequence \( v_1[n] \)
- Its z-transform is given by
\[
V_1(z) = \sum_{n=-\infty}^{0} v_1[n] z^{-n}
\]
ROC of a Rational $z$-Transform

- It can be shown that $V_1(z)$ converges interior to a circle $|z| = R_3$, including the point $z = 0$
- On the other hand, a left-sided sequence with nonzero sample values only for $n \leq N$ with $N$ nonnegative has a $z$-transform $V_2(z)$ with $N$ poles at $z = 0$
- The ROC of $V_2(z)$ is interior to a circle $|z| = R_4$, excluding the point $z = 0$

- Example - The $z$-transform of a two-sided sequence $w[n]$ can be expressed as
  \[ W(z) = \sum_{n=-\infty}^{\infty} w[n] z^{-n} = \sum_{n=0}^{\infty} w[n] z^{-n} + \sum_{n=-\infty}^{0} w[n] z^{-n} \]
  - The first term on the RHS, $\sum_{n=0}^{\infty} w[n] z^{-n}$, can be interpreted as the $z$-transform of a right-sided sequence and it thus converges exterior to the circle $|z| = R_5$
- If $R_S < R_6$, there is an overlapping ROC given by $R_S < |z| < R_6$
- If $R_S > R_6$, there is no overlap and the $z$-transform does not exist

- The ROC of a rational $z$-transform cannot contain any poles and is bounded by the poles
- To show that the $z$-transform is bounded by the poles, assume that the $z$-transform $X(z)$ has simple poles at $z = \alpha$ and $z = \beta$
- Assume that the corresponding sequence $x[n]$ is a right-sided sequence
ROC of a Rational z-Transform

- Then \( x[n] \) has the form
  \[
  x[n] = (r_1 \alpha^n + r_2 \beta^n) \mu(n - N_o),
  \]
  where \( N_o \) is a positive or negative integer.

- Now, the z-transform of the right-sided sequence \( \gamma^n \mu[n - N_o] \) exists if
  \[
  \sum_{n=N_o}^{\infty} \left| \gamma^n z^{-n} \right| < \infty
  \]
  for some \( z \).

ROC of a Rational z-Transform

- The condition
  \[
  \sum_{n=N_o}^{\infty} \left| \gamma^n z^{-n} \right| < \infty
  \]
  holds for \( |z| > |\gamma| \) but not for \( |z| \leq |\gamma| \).

- Therefore, the z-transform of
  \[
  x[n] = (r_1 \alpha^n + r_2 \beta^n) \mu(n - N_o),
  \]
  has an ROC defined by \( |\beta| < |z| \leq |\alpha| \).

ROC of a Rational z-Transform

- Likewise, the z-transform of a left-sided sequence
  \[
  x[n] = (r_1 \alpha^n + r_2 \beta^n) \mu[-n - N_o],
  \]
  has an ROC defined by \( 0 \leq |z| < |\alpha| \).

- Finally, for a two-sided sequence, some of the poles contribute to terms in the parent sequence for \( n < 0 \) and the other poles contribute to terms \( n \geq 0 \).

ROC of a Rational z-Transform

- In general, if the rational z-transform has \( N \) poles with \( R \) distinct magnitudes, then it has \( R + 1 \) ROCs.

- Thus, there are \( R + 1 \) distinct sequences with the same z-transform.

- Hence, a rational z-transform with a specified ROC has a unique sequence as its inverse z-transform.
ROC of a Rational z-Transform

- The ROC of a rational z-transform can be easily determined using MATLAB
  \[ [z, p, k] = tf2zp(num, den) \]
  determines the zeros, poles, and the gain constant of a rational z-transform with the numerator coefficients specified by the vector `num` and the denominator coefficients specified by the vector `den`.

- \([num, den] = zp2tf(z, p, k)\) implements the reverse process.

- The factored form of the z-transform can be obtained using
  \[ sos = zp2sos(z, p, k) \]
  where each second-order factor given as an \(\begin{bmatrix} a & b \\ a & b \end{bmatrix}\) matrix.

- The pole-zero plot is determined using the function `zplane`

**Example** - The pole-zero plot of
\[ G(z) = \frac{2z^4 + 16z^3 + 44z^2 + 56z + 32}{3z^4 + 3z^3 - 15z^2 + 18z - 12} \]
obtained using MATLAB is shown below.

**Inverse z-Transform**

- **General Expression**: Recall that, for \( z = re^{j\omega} \), the z-transform \( G(z) \) given by
  \[ G(z) = \sum_{n=-\infty}^{\infty} g[n]z^{-n} = \sum_{n=-\infty}^{\infty} g[n]r^{-n}e^{-jn\omega} \]
is merely the DTFT of the modified sequence \( g[n]r^{-n} \).

- Accordingly, the inverse DTFT is thus given by
  \[ g[n]r^{-n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} G(re^{j\omega})r^{j\omega}d\omega \]
Inverse z-Transform

• By making a change of variable \( z = re^{\theta} \), the previous equation can be converted into a contour integral given by

\[
g[n] = \frac{1}{2\pi j} \oint_C G(z) z^{n-1} dz
\]

where \( C \) is a counterclockwise contour of integration defined by \( |z| = r \).

Inverse z-Transform

• But the integral remains unchanged when \( z \) is replaced with any contour \( C \) encircling the point \( z = 0 \) in the ROC of \( G(z) \).

• The contour integral can be evaluated using the Cauchy’s residue theorem resulting in

\[
g[n] = \sum \text{residues of } G(z) z^{n-1} \text{ at the poles inside } C.
\]

• The above equation needs to be evaluated at all values of \( n \) and is not pursued here.

Inverse Transform by Partial-Fraction Expansion

• A rational \( z \)-transform \( G(z) \) with a causal inverse transform \( g[n] \) has an ROC that is exterior to a circle.

• Here it is more convenient to express \( G(z) \) in a partial-fraction expansion form and then determine \( g[n] \) by summing the inverse transform of the individual simpler terms in the expansion.

• A rational \( G(z) \) can be expressed as

\[
G(z) = \frac{P(z)}{D(z)} = \frac{\sum_{i=0}^{M} P_i z^{-i}}{\sum_{i=0}^{N} D_i z^{-i}}
\]

• If \( M \geq N \) then \( G(z) \) can be re-expressed as

\[
G(z) = \sum_{i=0}^{M-N} \eta_i z^{-\ell} + \frac{P(z)}{D(z)}
\]

where the degree of \( P(z) \) is less than \( N \).

Inverse Transform by Partial-Fraction Expansion

• Simple Poles: In most practical cases, the rational \( z \)-transform of interest \( G(z) \) is a proper fraction with simple poles.

• Let the poles of \( G(z) \) be at \( z = \lambda_k, 1 \leq k \leq N \).

• A partial-fraction expansion of \( G(z) \) is then of the form

\[
G(z) = \sum_{k=1}^{N} \frac{\rho_k}{1 - \lambda_k z^{-1}}
\]
Inverse Transform by Partial-Fraction Expansion

- The constants $\rho_l$ in the partial-fraction expansion are called the residues and are given by
  \[ \rho_l = (1 - \lambda_l z^{-1})G(z)\big|_{z=\lambda_l} \]
- Each term of the sum in partial-fraction expansion has an ROC given by $|z| > |\lambda_l|$ and, thus has an inverse transform of the form $\rho_l (\lambda_l)^n u[n]

Inverse Transform by Partial-Fraction Expansion

- Therefore, the inverse transform $g[n]$ of $G(z)$ is given by
  \[ g[n] = \sum_{l=1}^{N} \rho_l (\lambda_l)^n u[n] \]
- Note: The above approach with a slight modification can also be used to determine the inverse of a rational $z$-transform of a noncausal sequence.

Inverse Transform by Partial-Fraction Expansion

- **Example:** Let the $z$-transform $H(z)$ of a causal sequence $h[n]$ be given by
  \[ H(z) = \frac{z(z+2)}{(z-0.2)(z+0.6)} = \frac{1+2z^{-1}}{(1-0.2z^{-1})(1+0.6z^{-1})} \]
- A partial-fraction expansion of $H(z)$ is then of the form
  \[ H(z) = \frac{\rho_1}{1-0.2z^{-1}} + \frac{\rho_2}{1+0.6z^{-1}} \]
- Hence
  \[ H(z) = \frac{2.75}{1-0.2z^{-1}} - \frac{1.75}{1+0.6z^{-1}} \]
- The inverse transform of the above is therefore given by
  \[ h[n] = 2.75(0.2)^n u[n] - 1.75(-0.6)^n u[n] \]

Inverse Transform by Partial-Fraction Expansion

- **Multiple Poles:** If $G(z)$ has multiple poles, the partial-fraction expansion is of slightly different form
- Let the pole at $z = \nu$ be of multiplicity $L$ and the remaining $N - L$ poles be simple and at $z = \lambda_l$, $1 \leq l \leq N - L$
Inverse Transform by Partial-Fraction Expansion

- Then the partial-fraction expansion of $G(z)$ is of the form

$$G(z) = \sum_{i=0}^{M-N} \eta_i z^{-i} + \sum_{\ell=1}^{N-L} \frac{\rho_\ell}{1 - \lambda_\ell z^{-1}} + \sum_{i=1}^{L} \frac{\gamma_i}{1 - (1-v) z^{-1}}$$

where the constants $\gamma_i$ are computed using

$$\gamma_i = \frac{1}{(L-i)! (L-i)! d^{L-i}} \left[(1-v z^{-1})^i G(z)\right]_{z=v},$$

- The residues $\rho_\ell$ are calculated as before.

Partial-Fraction Expansion Using MATLAB

- $[r, p, k] = \text{residuez}(\text{num}, \text{den})$ develops the partial-fraction expansion of a rational $z$-transform with numerator and denominator coefficients given by vectors `num` and `den`.
- Vector $r$ contains the residues.
- Vector $p$ contains the poles.
- Vector $k$ contains the constants $\eta_i$.

Inverse z-Transform via Long Division

- The $z$-transform $G(z)$ of a causal sequence $\{g[n]\}$ can be expanded in a power series in $z^{-1}$.
- In the series expansion, the coefficient multiplying the term $z^{-n}$ is then the $n$-th sample $g[n]$.
- For a rational $z$-transform expressed as a ratio of polynomials in $z$, the power series expansion can be obtained by long division.

Inverse z-Transform Using MATLAB

- The function `impz` can be used to find the inverse of a rational $z$-transform $G(z)$.
- The function computes the coefficients of the power series expansion of $G(z)$.
- The number of coefficients can either be user specified or determined automatically.
Table 3.9: z-Transform Properties

<table>
<thead>
<tr>
<th>Property</th>
<th>Sequence</th>
<th>z-Transform</th>
<th>ROC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conjugate</td>
<td>$x[n]e^{-j\omega n}$</td>
<td>$X(z^*)$</td>
<td>$</td>
</tr>
<tr>
<td>Translational</td>
<td>$x[n-n_0]$</td>
<td>$z^{-n_0}X(z)$</td>
<td>$</td>
</tr>
<tr>
<td>Linearity</td>
<td>$\alpha x[n] + \beta y[n]$</td>
<td>$\alpha X(z) + \beta Y(z)$</td>
<td>$R_R'$ overlaps $R_Y'$</td>
</tr>
<tr>
<td>Multiplication</td>
<td>$x[n]y[n]$</td>
<td>$X(z)Y(z)$</td>
<td>$R_R' \cap R_Y'$</td>
</tr>
<tr>
<td>Differentiation</td>
<td>$\frac{d}{dz}X(z)$</td>
<td>$\omega X(z)$</td>
<td>$</td>
</tr>
<tr>
<td>Convolution</td>
<td>$x[n] * h[n]$</td>
<td>$X(z)H(z)$</td>
<td>$R_X'$ overlaps $R_H'$</td>
</tr>
<tr>
<td>Modulation</td>
<td>$\mu[n]x[n]$</td>
<td>$\mu[n]X(z)$</td>
<td>$R_{\mu} \subseteq R_X'$</td>
</tr>
<tr>
<td>Periodic Relation</td>
<td>$\sum_{n=-\infty}^{\infty} x[n] = \sum_{k=-\infty}^{\infty} X(z^*)$</td>
<td>$\sum_{n=-\infty}^{\infty} x[n] = \sum_{k=-\infty}^{\infty} \mu[n]X(z^*)$</td>
<td>$</td>
</tr>
</tbody>
</table>

Note: If $R_X'$ encloses the region $R_Y'$, then $R_{\mu} = R_X'$.

z-Transform Properties

- **Example** - Consider the two-sided sequence $v[n] = \alpha^n u[n] - \beta^n u[-n-1]$.
- Let $x[n] = \alpha^n u[n]$ and $y[n] = -\beta^n u[-n-1]$ with $X(z)$ and $Y(z)$ denoting, respectively, their z-transforms.
- Now $X(z) = \frac{1}{1 - \alpha z^{-1}}$, $|z| > |\alpha|$ and $Y(z) = \frac{1}{1 - \beta z^{-1}}$, $|z| < |\beta|$.

z-Transform Properties

- **Example** - Determine the z-transform and its ROC of the causal sequence $x[n] = r^n (\cos \omega_n) u[n]$.
- We can express $x[n] = v[n] + v^*[n]$ where $v[n] = \frac{1}{2} r^n e^{j\omega_n} u[n] = \frac{1}{2} \alpha^n u[n]$.
- The z-transform of $v[n]$ is given by $V(z) = \frac{1}{2} \frac{1}{1 - \alpha z^{-1}} = \frac{1}{2 - r e^{j\omega_n}}$, $|z| > |\alpha| = r$.

or,

$$X(z) = \frac{1- (r \cos \omega_n) z^{-1}}{1 - (2 r \cos \omega_n) z^{-1} + r^2 z^{-2}}, \quad |z| > r$$

- **Example** - Determine the z-transform $Y(z)$ and the ROC of the sequence $y[n] = (n+1)\alpha^n u[n]$.
- We can write $y[n] = n x[n] + x[n]$ where $x[n] = \alpha^n u[n]$. 

**z-Transform Properties**

- Using the linearity property we arrive at $V(z) = X(z) + Y(z) = \frac{1}{1 - \alpha z^{-1}} + \frac{1}{1 - \beta z^{-1}}$.
- The ROC of $V(z)$ is given by the overlap regions of $|z| > |\alpha|$ and $|z| < |\beta|$.
- If $|\alpha| < |\beta|$, then there is an overlap and the ROC is an annular region $|\alpha| < |z| < |\beta|$.
- If $|\alpha| > |\beta|$, then there is no overlap and $V(z)$ does not exist.

- Using the conjugation property we obtain the z-transform of $v^*[n]$ as $V^*(z^*) = \frac{1}{2} \frac{1}{1 - \alpha^* z^{-1}} = \frac{1}{2} \frac{1}{1 - r e^{-j\omega_n}} z^{-1}$.
- Finally, using the linearity property we get $X(z) = V(z) + V^*(z^*)$.

- Example - Consider the two-sided sequence $v[n] = \alpha^n u[n] - \beta^n u[-n-1]$.
- Let $x[n] = \alpha^n u[n]$ and $y[n] = -\beta^n u[-n-1]$ with $X(z)$ and $Y(z)$ denoting, respectively, their z-transforms.

z-Transform Properties

- Example - Determine the z-transform and its ROC of the causal sequence $x[n] = r^n (\cos \omega_n) u[n]$.

z-Transform Properties

- Example - Determine the z-transform $Y(z)$ and the ROC of the sequence $y[n] = (n+1)\alpha^n u[n]$. 

- We can write $y[n] = n x[n] + x[n]$ where $x[n] = \alpha^n u[n]$. 

- Example - Consider the two-sided sequence $v[n] = \alpha^n u[n] - \beta^n u[-n-1]$.
- Let $x[n] = \alpha^n u[n]$ and $y[n] = -\beta^n u[-n-1]$ with $X(z)$ and $Y(z)$ denoting, respectively, their z-transforms.
- Now $X(z) = \frac{1}{1 - \alpha z^{-1}}$, $|z| > |\alpha|$ and $Y(z) = \frac{1}{1 - \beta z^{-1}}$, $|z| < |\beta|$.
Now, the z-transform \( X(z) \) of \( x[n] = \alpha^n u[n] \) is given by
\[
X(z) = \frac{1}{1 - \alpha z^{-1}}, \quad |z| > |\alpha|
\]
Using the differentiation property, we arrive at the z-transform of \( n x[n] \) as
\[
-\zeta \frac{dX(z)}{dz} = \frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}, \quad |z| > |\alpha|
\]
Using the linearity property we finally obtain
\[
Y(z) = \frac{1}{1 - \alpha z^{-1}} + \frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}
\]
\[
= \frac{1}{(1 - \alpha z^{-1})^2}, \quad |z| > |\alpha|
\]