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Are Mathematicians Better Described as Formalists or Pluralists?

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ABSTRACT. In this paper we try to convert the mathematician who calls himself, or herself, "a formalist" to a position we call "methodological pluralism". We show how the actual practice of mathematics fits methodological pluralism better than formalism while preserving the attractive aspects of formalism of *freedom* and *creativity*. Methodological pluralism is part of a larger, more general, pluralism, which is currently being developed as a position in the philosophy of mathematics in its own right.[1] Having said that, henceforth, in this paper, we abbreviate "methodological pluralism" with "pluralism".

1. Characterization of Formalism

We shall begin with Delefsen's characterisation of formalism, modify it to better fit the modern mathematician's conception, use Hilbert's two principles to set parameters on the notion of rigor, and end with a general description of

[1] For example see (Friend (2012))

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formalism. Formalism is a philosophy of mathematics which was developed in the late nineteenth century and at the beginning of the twentieth. To be more precise, it is not simply one philosophy. Following Detlefsen's careful characterization of formalism,[2] we can think of formalism as a family of positions, each member of which has some of the five following characteristics. For our characterisation of formalism we shall ignore (i). We shall then add Hilbert's two principles.

The following are the characteristics:

i. geometry no longer sets the standard for rigor. Instead, the standard is set by arithmetic[3].

ii. Formalists have a particular conception of what rigor is. For them rigor follows from an act of abstraction away from intuition instead of from an embedding into or onto a previously accepted theory (which is what we often find in pre-Hilbert and pre-Tarski presentations of geometry). This is a methodological constraint on the practice and presentation of mathematics.

iii. Formalists reject the idea that mathematical proof should be based on a "genetic" model of proof, because they do not believe that we have knowledge of mathematical truths[4] by having knowledge of their origins and causes (although this might, of course, be useful for some learning purposes). They replace the genetic model with axiomatic theories. Stipulating a set of axioms, or rules of inference, is what we need to have a mathematical theory.

iv. Formalism advocates "a nonrepresentational role for language in mathematical reasoning".[5]

v. Mathematicians have the freedom to create and work with different reasoning tools in order to get genuine knowledge. The idea that mathematical knowledge can be reached only by the use of contentual (in the sense of interpreted (semantically or ontologically)) reasoning is essentially alien to formalism.

Hilbert's principles 1 and 2[6] are:

[2] See Frege (1980, pp. 236-237).

[3] Detlefsen characterizes formalism as a position with all five charactersistics. We are less stringent in our definition

[4]The word "truth" is not to be taken too seriously here. It could be replaced by "content".

[5] Frege (1980, p. 237).

[6]Who best represents these characteristics is a separate historical question.

1. Mathematicians should aim to construct concepts and inferential methods, which are fruitful in mathematical practice.

2. It is necessary to make inferences as reliable as possible and to search on which basis this can be done.

We interpret (iv) and 2 to imply that we need a unique, specifed and tight proof theory for all our proofs. We ignore characteristic (i) (the one about arithmetic setting the standard for rigor above geometry) because we think that this has largely been eclipsed in present day mathematics - namely by the arithmetization of geometry by Hilbert and Tarski. However, since we also use Hilbert's two principles as characterising formalism, they will serve as parameters on what counts as rigor for the formalist.

Characteristics (iv) and (v) are important for us because the mathematicians who call themselves formalist tell us that they feel that as formalists they enjoy *freedom* and *creativity*. They are free to interpret the symbols as they choose – give them any interpretation which fits the formal constraints of rigor (characteristic (iv)). For Hilbert, and for our formalist, mathematics is only to be thought of as the collection of all mathematical theorems ever given in history, where the theorems are generated in axiomatic theories, and that is all.

2. How Formalist Mathematics Should Look

In the next section, we shall look at a test case, but before we should give some idea as to how we are to judge it. What should the practice of mathematics look like, according to the formalist? The inferential process (proof procedure) involved should be very strict. Each step should rigorously follow from the previous steps to make the proof gapless. A proof, for takes place within a theory. The theory contains axioms and a proof theory which allows inference moves. A acceptable proof, then consists in stating some axioms, using only allowed rules of inference of the formal system of proof of that theory, and deducing a conclusion. Natural deduction proofs in logic are perfect examples.

We deviate from making an acceptable proof when we:

(1) fail to specify which theory we are working in,

(2) import foreign axioms

(3) use rules of inference not in the proof theory

(4) fail to completely formalize our proofs (or show that we could do this in principle), or

(5) leave unexplained gaps in our reasoning.

The importance of sticking to the strict methodology is that if we have proved the theory to be consistent then by following the proof theory – the given methodology - we ensure continued consistency.[7]

3. The Test Case

The test case concerns "big projects". In these, mathematicians divide the main goal into different sub-goals each of which is itself divided into other sub-goals, or "cells". These cell-structure allows mathematicians (and computers) to work in parallel. Each cell works on specific problems (that are not always directly connected with the main goal, but that are necessary for its success). The success of the project depends on the success of the work of the cells. "Success" of the whole project consists in finding the solution to a problem, such as classifying mathematical theories. "Success" in a cell consists in proving theorems. Because "achieving the goal" is important, the mathematicians and computers working in a cell avail themselves of whatever it takes to prove the theorem assigned to that cell.

An actual example is the case of the classification of finite simple groups. This mathematical endeavor started more than a century ago and ended in 1983. It has been a collective work made of thousands of pages in books, articles and manuscripts written by many different mathematicians. This whole body of mathematical work represents the "solution" which gives the classification of finite simple groups. A theory in this sense is the collection of a very large number of different proofs made with different techniques on different topics. Because the whole collection of proofs is so large, it is "unsurveyable by a single human being".[8] Moreover, in the case of the classification of finite simple groups there is some controversy concerning the classification of the so-called "quasi-thin" groups.

The work of the mathematician, and Abel prize winner, Jean-Pierre Serre showed how this could be regarded as a gap in the larger proof of "the theo-

[7] In some cases, of course, we only have relative consistency.

[8] Otte (1990, p. 61).

ry".[9] It counts as a gap because of the length and the structural complexity of this mathematical body. The classification of quasi-thin groups is a key step for the main goal of the classification of finite simple groups. Quasi-thin groups were announced as "classified" in the early 80's by Geoff Mason, but in fact they were not. In Mason's proof critical gaps were due to the "proliferation" of unexpected groups. Only in 2004 did Michael Ashbacher and Stephen Smith give a complete proof in a two-volume book of more than 1200 pages. The criticism of Serre addresses the fact that the dishomogeneity of the general proof for finite simple groups does not allow us to prevent further gaps that are not yet discovered and fixed. These gaps, however, do not seem to have brought discredit to the results.

This is a typical example of a work of mathematics which hardly fits into a strict formalist framework. It is hardly a "demonstration" because it is unsurveyable. But this is the least of the complaints, since what we mean by "surveyable", and "demonstrable" can be flexibly interpreted to fit this case. What is more damning is that we see examples of mathematicians deviating from a strict axiomatic system.

In our test case (and in many other instances as well) we witness what we shall call "deviant" proofs. These are "proofs" where mathematicians use steps which deviate from the rigorous set of rules methodologies and axioms agreed to in advance. Of course, shortcuts can be useful to speed up a proof without any danger of inconsistency. But, strictly speaking, a straight shortcut can be proved as a lemma, and therefore, the steps could be filled in upon request. Here we are interested in something else. *Deviant* shortcuts or detours can help to circumvent an impasse which *could not be overridden* with the standard steps agreed upon in advance, and this is what Serre found in our test case. We suggest that the test case is not isolated. It is an illustration of a general trend we find in present day mathematics.

One might think that our argument has mis-fired. After all, the classification of all finite simple groups was never meant to be "carried out within a formal theory", with axioms and rules of inference given in advance. Rather, the work is carried out at the meta-level. Either said mathematicians are not formalists or they are formalists in bad faith. They insist on the attractive aspects of formalism while ignoring the constraints.

Here is our counter-argument. It is correct to say that no umbrella formal theory was agreed upon in advance for "proving all the theorems". However,

[9] See, for instance, Raussen e Skau (2003).

even if the work is being done at the meta-level, this should not entail that all standards are dropped. The classification does require careful definitions, it does require proofs – that a particular group or class of groups falls under a particular classification. The standards of classification, ensuring that we are not arbitrary in our classification, is ensured by the rigour of the meta-theory. There are well known mathematical techniques for this These proofs – even if they are carried out at the meta-level, are still proofs, which are checked for correctness and so on. There are several “big” mathematical projects being carried out today, and they show the features we are interested in – a lack of adherence to one “method” of proof, and therefore run the risk of inconsistency in methodologies.

4. The Proble: Diagnosis and Solutions

So what has gone wrong? The reasons the mathematicians are attracted to formalism were that it (1) allows for the creativity and freedom of mathematicians and (2) avoids heavy foundational philosophical disputes about ontology and correct universal methodology. We propose keeping these two attractive ideas in mind and present pluralism in methodology as a neat philosophical replacement of formalism.

The pluralist is not set on giving one axiomatic system in advance. It is quite alright to mix and match methodologies and theories in mathematics. However, when we do this, we should be vigilant. The pluralist makes two recommendations.

(1) Know what counts as a acceptable proof within a mathematical theory Axiomatization is fine as a starting point, but axiomatization is thought of as just a point about being explicit. This is important because the acceptable proof will set a standard which will be used to signal deviation. Since the pluralist allows deviance in proof, the trick is to know when we are being deviant. Vigilance can be made quite systematic, by means of a protocol.

(2) We should bear in mind that when revising, correcting, being critical of a result or being skeptical, we should look first to the deviant steps. Then, after that, we show that there is no internal inconsistency in adopting the deviant methods.

However, the analysis of the pluralist does not stop here. A deviant proof might also be a call to revise or re-evaluate the original meta-theory. This reflects the idea that the pluralist thinks of axiomatic theories as an exercise in being explicit, not as giving the truth. The pluralist attitude twards truth in mathematics guaranatees the freedom, so cherished by formalists.

For example, if our mathematical model of some part of physics (the application of mathematics to "reality") predicted a certain outcome, and we found that the outcome was not what was predicted by the mathematics, then we look first for an error in calculation, second, we look at any deviant elements in the making of the faulty prediction, but thirdly, we might look at revising the whole theory – making a new one, adding axioms, adding rules of inference, modifying or eliminating existing ones, etc. This is where the pluralist goes beyond the formalist. Since pluralism does not encounter the problems we saw with formalism, the adoption of pluralism is a net gain for the mathematician.

Methodological pluralism better describes mathematicians' practice than does formalism. We do not need methodological rigidity to guarantee consistency when we can use "reality", physical theory or another mathematical theory to sanction methodological deviance in proof.[10] Rather than hold mathematicians *to* a rigid standard of rigor, pluralists *use* the standard of rigor to make us aware of deviance. This is simply practical and reflects actual practice.

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[10] Pluralist does not insist on consistency, since, e. g., she endorses paraconsistent theories. However, she endorses and encourages cross-checking with other theories.

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