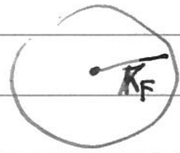
Exercise 1 - MAY 6, 2011

(1) $$2 \cdot \frac{\pi k_F^2}{(2\pi)^2} = n \qquad \boxed{k_F = \sqrt{2\pi n}}$$



Note: the energy level $\epsilon_{\bar{p}} = \alpha p$ depends on the modulus of the momentum!

(2) $$\boxed{E_F = \alpha p_F = \alpha \hbar \sqrt{2\pi n}}$$

(3) $$E = \sum_{\substack{\sigma = \pm 1 \\ p < p_F}} \alpha p = 2 \int \frac{d\bar{p}}{\hbar^2 (2\pi)^2 / A} \alpha p = \frac{A \cdot 4\pi}{(2\pi)^2} \int_0^{p_F} dp\, p\, \alpha p \frac{1}{\hbar^2}$$

$$= \frac{A}{\pi} \frac{\alpha}{\hbar^2} \frac{p_F^3}{3} = \frac{A\alpha}{3\pi} p_F k_F^2 = \frac{A\alpha}{3\cancel{\pi}} p_F 2\cancel{\pi} n$$

$$= \frac{2}{3} N \alpha p_F$$

$$\boxed{E = N \frac{2}{3} \epsilon_F = N \frac{2}{3} \alpha p_F}$$

$$\boxed{\epsilon(n) = \frac{2}{3} \alpha \hbar \sqrt{2\pi n}}$$

(4) $$E[n] = \int d\bar{r}\, \epsilon(n(\bar{r}))\, n(\bar{r}) + \int d\bar{r}\, n(\bar{r})\, v(\bar{r})$$

5 $$\frac{\delta E}{\delta n(\vec r)} = \frac{2}{3}\alpha\hbar\,\frac{3}{2}\sqrt{2\pi n(\vec r)} + v(\vec r) = \mu$$

$$\sqrt{n(\vec r)} = \frac{1}{\alpha\hbar\sqrt{2\pi}}\left[\mu - v(\vec r)\right] \qquad \mu - v(\vec r) \geq 0$$

$$\boxed{n(\vec r) = \frac{1}{2\pi\alpha^2\hbar^2}\left[\mu - v(\vec r)\right]^2}$$

6 $$n(\vec r) = \frac{1}{2\pi\alpha^2\hbar^2}\,\frac{v_0^{\,2}}{qr}\sin qr$$

$$= \frac{1}{2\pi\cancel{\alpha^2\hbar^2}}\,\frac{\cancel{\alpha^2\hbar^2}q^{\cancel{2}}}{2\cancel{q}r}\,\frac{M}{2}\sin qr$$

$$n(r) = \frac{M}{16\pi}\,q\,\frac{\sin qr}{r} \qquad 0 \leq r \leq \frac{\pi}{q}$$

$$= 0 \qquad r > \frac{\pi}{q}$$

$$N = \int_0^{\pi/q} dr\, r\, 2\pi\,\frac{M}{16\pi}\,q\,\frac{\sin qr}{r}$$

$$= \frac{M}{8}\int_0^{\pi} dx\,\sin x \qquad x = qr$$

$$= \frac{M}{8}\left[-\cos x\right]_0^{\pi} = \frac{M}{4} \qquad \boxed{N = \frac{M}{4}}$$

If instead we set $v_0 = \sqrt{\frac{M}{2}}\,\alpha\hbar q$ $\quad \boxed{N = M}$

Exercise 2 - MAY 6, 2011

① $g_v(\epsilon) = 2 \sum_{\vec{k}} \delta(\epsilon - \epsilon_v(\vec{k})) \cdot \frac{1}{A}$

$= 2 \frac{A}{(2\pi)^2} \frac{1}{A} \int d\vec{k}\, \delta(\epsilon - \epsilon_v + \frac{\hbar^2}{2m_v}(\vec{k}-\vec{k}^*)^2)$

$= \frac{1}{2\pi^2} \int d\vec{q}\, \delta(\epsilon - \epsilon_v + \frac{\hbar^2}{2m_v} q^2)$, $\vec{q} = \vec{k} - \vec{k}^*$

$g_v = \frac{1}{2\pi^2} \left(\sqrt{\frac{2m_v}{\hbar^2}}\right)^2 \int d\vec{Q}\, \delta(\epsilon - \epsilon_v + Q^2)$ $\quad Q = \sqrt{\frac{\hbar^2}{2m_v}}\, q$

$= \frac{2m_v}{2\pi^2 \hbar^2} 2\pi \int_0 dQ\, Q\, \delta(\epsilon - \epsilon_v + Q^2)$

$= \frac{m_v}{\pi \hbar^2} \int_0 d(Q^2)\, \delta(\epsilon - \epsilon_v + Q^2)$

$$\boxed{g_v(\epsilon) = \frac{m_v}{\pi \hbar^2} \theta(\epsilon_v - \epsilon)}$$

② Using the manipulation of the previous calculation (replacing m_v with $-m_c$) we get

$g_c(\epsilon) = \frac{m_c}{\pi \hbar^2} \int_0 d(Q^2)\, \delta(\epsilon - \epsilon_c - Q^2)$

$$\boxed{g_c = \frac{m_c}{\pi \hbar^2} \theta(\epsilon - \epsilon_c)}$$

3

$$N_c = \int_{\epsilon_c}^{\infty} d\epsilon \, g_c(\epsilon) \, e^{-\beta(\epsilon-\epsilon_c)}$$

$$= \frac{m_c}{\pi\hbar^2} \int_{\epsilon_c}^{\infty} d\epsilon \, \theta(\epsilon-\epsilon_c) \, e^{-\beta(\epsilon-\epsilon_c)}$$

$$= \frac{m_c}{\pi\hbar^2} \int_{0}^{\infty} dy \, e^{-\beta y} = \frac{m_c}{\pi\hbar^2} \frac{1}{\beta}$$

$$\boxed{N_c = \frac{m_c}{\pi\hbar^2} k_B T}$$

4

$$P_v = \int_{-\infty}^{\epsilon_v} d\epsilon \, g_v(\epsilon) \, e^{-\beta(\epsilon_v-\epsilon)}$$

$$= \frac{m_v}{\pi\hbar^2} \int_{-\infty}^{\epsilon_v} d\epsilon \, \theta(\epsilon_v-\epsilon) \, e^{-\beta(\epsilon_v-\epsilon)}$$

$$= \frac{m_v}{\pi\hbar^2} \int_{-\infty}^{0} dy \, e^{\beta y} = \frac{m_v}{\pi\hbar^2} \frac{1}{\beta}$$

$$\boxed{P_v = \frac{m_v}{\pi\hbar^2} k_B T}$$

5

$$N_c = \frac{0.067 \times 9.11 \times 10^{-28}}{3.14 \times (1.05)^2 \times 10^{-54}} \, 1.38 \times 10^{-16} \times 300$$

$$\boxed{N_c = 0.730 \times 10^{12} \text{ cm}^{-2}} \qquad P_v = \frac{m_v}{m_c} N_c = \frac{0.38}{0.067} N_c$$

$$\boxed{P_v = 4.14 \times 10^{12} \text{ cm}^{-2}}$$

6 $n_i = \sqrt{N_c P_v}\ e^{-E_g/2k_B q'}$

$= \sqrt{0.730 \times 4.14 \times 10^{12}}\ e^{-\frac{1.6\times10^{-12}}{2\times1.38\times10^{-16}\times300}}$

$= 1.74 \times 10^{+12}\ e^{-19.3}$

$= 1.74 \times 10^{12} \times 4.15 \times 10^{-9}$

$= 7.22 \times 10^{3}$

$\boxed{n_i = 7.22 \times 10^{3}\ cm^{-2}}$