ESERCIZIO 1 - 29.04.16 $\hbar = m = e = 1$; 1D Fermions

①

$$t_\sigma \quad \frac{\sum_{|k_\sigma|<k_{F\sigma}} k^2/2}{\sum_{|k_\sigma|<k_{F\sigma}}} = \frac{\frac{1}{2}\int_{-k_{F\sigma}}^{k_{F\sigma}} dk\, k^2}{\int_{-k_{F\sigma}}^{k_{F\sigma}} dk} = \frac{\frac{1}{2}\cdot 2\cdot \frac{k_F^3}{3}}{2\, k_F} = \frac{k_{F\sigma}^2}{6}$$

$$\frac{2k_{F\sigma}}{\frac{2\pi}{L}} = N_\sigma = \frac{L\, k_{F\sigma}}{\pi} \Rightarrow k_{F\sigma} = \pi n_\sigma\ , \quad n_\sigma = \frac{N_\sigma}{L}\ ; \quad \boxed{t_\sigma = \frac{1}{6}\pi^2 n_\sigma^2}$$

③

$$n = n_+ + n_-\ , \quad n\zeta = n_+ - n_- \Rightarrow n_\sigma = \frac{n}{2}(1+\sigma\zeta)\ , \quad \sigma = \pm$$

$$e(n,\zeta,B) = \frac{1}{N}\left[N_+ t_+ + N_- t_- + \frac{U}{N} N_+ N_- + \mu_B B (N_+ - N_-)\right]$$

$$= \frac{\pi^2}{6}\left[\frac{n_+^3}{n} + \frac{n_-^3}{n}\right] + \frac{U}{4}\frac{n_+ n_-}{n^2} + \mu_B B\left[\frac{n_+}{n} - \frac{n_-}{n}\right]$$

$$= \frac{\pi^2}{48} n^2\left[(1+\zeta)^3 + (1-\zeta)^3\right] + \frac{U}{4}(1-\zeta^2) + \mu_B B\zeta$$

$$= \frac{\pi^2}{48} n^2\left[2 + 6\zeta^2\right] + \frac{U}{4}(1-\zeta^2) + \mu_B B\zeta$$

$$\boxed{e(n,\zeta,B) = \frac{\pi^2}{24} n^2 + \frac{U}{4} + \left[\frac{\pi^2}{8} n^2 - \frac{U}{4}\right]\zeta^2 + \mu_B B\zeta}$$

$$\equiv \alpha + \beta\zeta^2 + \gamma\zeta = \beta\left(\zeta + \frac{\gamma}{2\beta}\right)^2 - \left(\frac{\gamma}{2\beta}\right)^2 + \alpha$$

$e(n,\zeta,B)$ is a parabola with a minimum (maximum) at $\zeta = -\frac{\gamma}{2\beta}$

there is a mimum for $\beta = \frac{\pi^2}{8}n^2 - \frac{U}{4} > 0$ and a maximum for

$\beta = \frac{\pi^2 n^2}{8} - \frac{U}{4} < 0$

(4) $\beta > 0$ $[U < \pi^2 n^2/2]$

$$J = -\frac{\gamma}{2\beta} = -\frac{\mu_B B}{2\left[\frac{\pi^2}{8}n^2 - \frac{U}{4}\right]} \geq -1 \Rightarrow B \leq \frac{\pi^2 n^2/4 - U/2}{\mu_B}$$

When $B = B_c = \frac{\pi^2 n^2/4 - U/2}{\mu_B}$ the minimum is for $J = -1$. For $B > B_c$

the minimum is for $J < -1$ and $e(n, J, B)$ has positive first

derivative for $-1 \leq J \leq 1$, therefore in this range the minimum is at $J = -1$. To summarize:

$$B \leq B_c \Rightarrow J = -\frac{\mu_B B}{2\left(\frac{\pi^2}{8}n^2 - \frac{U}{4}\right)}$$

$$B \geq B_c \Rightarrow J = -1$$

$\beta < 0$ $U > 2\pi^2 n^2$

$e(n, J, B) = -|\beta|\left(J - \frac{\gamma}{2|\beta|}\right)^2 - \left(\frac{\gamma}{2\beta}\right)^2 + \alpha$, and it is evident that the

the minimum is at $J = -1$, for any value of $B > 0$.

⑤ Large density means $\frac{\pi^2}{8}n^2 > \frac{U}{4}$ $[\beta > 0]$ $\Rightarrow$

$$\zeta = -\frac{\mu_B}{\pi^2 n^2/4 - U/2} B, \quad B < B_c = \frac{\pi^2 n^2/4 - U/2}{\mu_B}$$

$$= -1, \quad B \geq B_c$$

⑥ Small density means $\frac{\pi^2}{8}n^2 < \frac{U}{4}$ $[\beta < 0]$ $\Rightarrow$

$$\zeta = -1$$

② See next page and just replace $\bar{r}_1, \bar{r}_2$ with x_1, x_2 and A with L!

② $\frac{U}{n}v(\bar{r}_i-\bar{r}_j) = \frac{U}{n}\delta(\bar{r}_i-\bar{r}_j)$

$$\tilde{V} = \frac{1}{2}\sum_{i\neq j}' v(\bar{r}_i-\bar{r}_j) \qquad V = \frac{U}{n}\tilde{V}$$

$$\langle SD|\tilde{V}|SD\rangle = \frac{1}{2}\sum_{\alpha\beta}\{\langle\alpha\beta|v|\alpha\beta\rangle - \langle\beta\alpha|v|\alpha\beta\rangle\} \equiv \langle\tilde{V}\rangle$$

$\alpha \equiv (\sigma, \bar{k}_\sigma) \qquad \beta \equiv (\sigma', \bar{q}_{\sigma'})$

$$\psi_{\sigma\bar{k}_\sigma}(s,\bar{r}) = \frac{1}{\sqrt{A}}\delta_{\sigma s}\, e^{i\bar{k}_\sigma\cdot\bar{r}}$$

$$\langle\tilde{V}\rangle = \frac{1}{2}\sum_{\substack{\sigma\sigma' \\ \bar{k}_\sigma,\bar{q}_{\sigma'}}}'\sum_{ss'}\int d\bar{r}_1\int d\bar{r}_2 \times \frac{1}{A^2}\delta(\bar{r}_1-\bar{r}_2)$$

$$\Big[\delta_{\sigma s}^2\delta_{\sigma' s'}^2 e^{i(\bar{k}_\sigma\bar{r}_1+\bar{q}_{\sigma'}r_2-\bar{k}_\sigma\bar{r}_1-\bar{q}_{\sigma'}\bar{r}_2)} - \delta_{\sigma s}\delta_{\sigma' s'}\delta_{\sigma' s}\delta_{\sigma s'}\, e^{i(\bar{k}_\sigma\cdot\bar{r}_1+\bar{q}_{\sigma'}r_2-\bar{q}_{\sigma'}\cdot r_1-\bar{k}_\sigma\cdot\bar{r}_2)}\Big]$$

$$= \frac{1}{2A^2}\int d\bar{r}_1 \sum_{\substack{\sigma\sigma' \\ \bar{k}_\sigma,\bar{q}_{\sigma'}}}'[1-\delta_{\sigma\sigma'}]$$

$$= \frac{1}{2A}\Big\{\sum_{\sigma\bar{k}_\sigma}'\sum_{\sigma'\bar{q}_{\sigma'}}' - \sum_{\sigma}'\sum_{k_\sigma}'\sum_{\bar{q}_\sigma}'\Big\}$$

$$= \frac{1}{2A}\Big\{N^2 - \sum_\sigma' N_\sigma^2\Big\} = \frac{1}{2A}\{(N_\uparrow+N_\downarrow)^2 - N_\uparrow^2 - N_\downarrow^2\}$$

$$= \frac{1}{2A}\cdot 2N_\uparrow N_\downarrow = \frac{N_\uparrow N_\downarrow}{A}$$

$$\langle V\rangle = \frac{U}{n}\langle\tilde{V}\rangle = \frac{U}{n}\frac{N_\uparrow N_\downarrow}{A} = U\frac{N_\uparrow N_\downarrow}{n}$$

ESERCIZIO 2 - 29.04.16

We shall refer to the attached older exercise exold

① Same as ①-exold

②

$$g_c(E) = 2A^{-1}\sum_{\bar{q}} \delta\left[E - \epsilon_c - (\hbar^2/2)(q_x^2/m_x + q_y^2/m_y)\right]$$

$$= \frac{2}{A}\int \frac{d\bar{q}}{(2\pi)^2/A}\,\delta\left[E - \epsilon_c - (\hbar^2/2)(q_x^2/m_x + q_y^2/m_y)\right]$$

$$= \frac{1}{2\pi^2}\,\frac{2}{\hbar^2}\sqrt{m_x m_y}\int d\bar{p}\,\delta(E - \epsilon_c - p^2)$$

$$= \frac{m_c}{\pi^2\hbar^2}\,2\pi\int dp\,p\,\delta(E - \epsilon_c - p^2)$$

$$= \frac{m_c}{\pi\hbar^2}\int dx\,\delta(E - \epsilon_c - x)$$

$$\boxed{g_c(E) = \frac{m_c}{\pi\hbar^2}\,\theta(E - \epsilon_c)}$$

$$\boxed{\frac{\hbar^2 q_x^2}{2m_x} \equiv p_x^2 \qquad \frac{\hbar^2 q_y^2}{2m_y} \equiv p_y^2}$$

$$\boxed{m_c = \sqrt{m_x m_y}}$$

$$\boxed{x = p^2}$$

③ and ④ as in exold

(5)

$$N_C = \frac{\sqrt{0{,}2} \times 9{,}11 \times 10^{-28}}{3{,}14 \times (1.05)^2 \times 10^{-54}} \times 1{,}38 \times 10^{-16} \times 300 = 4{,}87 \times 10^{12}\ cm^{-2}$$

$$P_v = \frac{m_v}{m_c} N_c = \frac{0{,}4}{\sqrt{2}} \times 4.87 \times 10^{12} = 4{,}36 \times 10^{12}\ cm^{-2}$$

(6)

$$n_i = \sqrt{P_v N_c}\, e^{-E_g/2k_BT} = \sqrt{4.36 \times 4{,}87} \times 10^{12}\, e^{-\frac{2.15}{2 \times 8{,}62 \times 10^{-5} \times 300}}$$

$$= 4{,}61 \times 10^{12}\, e^{-41{,}6} = 4{,}61 \times 10^{12} \times 8.58 \times 10^{-19} = 3{,}96 \times 10^{-6}$$

Exercise 2 - MAY 6, 2011

1 $$g_v(\epsilon) = 2 \sum_{\vec{k}} \delta(\epsilon - \epsilon_v(\vec{k})) \cdot \frac{1}{A}$$

$$= 2 \frac{A}{(2\pi)^2} \frac{1}{A} \int d\vec{k}\, \delta\left(\epsilon - \epsilon_v + \frac{\hbar^2}{2m_v}(\vec{k}-\vec{k}^*)^2\right)$$

$$= \frac{1}{2\pi^2} \int d\bar{q}\, \delta\left(\epsilon - \epsilon_v + \frac{\hbar^2}{2m_v} q^2\right), \quad \bar{q} = \vec{k} - \vec{k}^*$$

$$g_v = \frac{1}{2\pi^2}\left(\sqrt{\frac{2m_v}{\hbar^2}}\right)^2 \int d\bar{Q}\, \delta(\epsilon - \epsilon_v + Q^2) \qquad Q = \sqrt{\frac{\hbar^2}{2m_v}}\, q$$

$$= \frac{2m_v}{2\pi^2\hbar^2}\, 2\pi \int_0 dQ\, Q\, \delta(\epsilon - \epsilon_v + Q^2)$$

$$= \frac{m_v}{\pi\hbar^2} \int_0 d(Q^2)\, \delta(\epsilon - \epsilon_v + Q^2)$$

$$\boxed{g_v(\epsilon) = \frac{m_v}{\pi\hbar^2}\, \vartheta(\epsilon_v - \epsilon)}$$

2 Using the manipulation of the previous calculation (replacing m_v with $-m_c$) we get

$$g_c(\epsilon) = \frac{m_c}{\pi\hbar^2} \int_0 d(Q^2)\, \delta(\epsilon - \epsilon_c - Q^2)$$

$$\boxed{g_c = \frac{m_c}{\pi\hbar^2}\, \vartheta(\epsilon - \epsilon_c)}$$

(3)
$$N_c = \int_{\epsilon_c}^{\infty} d\epsilon \, g_c(\epsilon) \, e^{-\beta(\epsilon-\epsilon_c)}$$

$$= \frac{m_c}{\pi\hbar^2} \int_{\epsilon_c}^{\infty} d\epsilon \, \theta(\epsilon-\epsilon_c) \, e^{-\beta(\epsilon-\epsilon_c)}$$

$$= \frac{m_c}{\pi\hbar^2} \int_{0}^{\infty} dy \, e^{-\beta y} = \frac{m_c}{\pi\hbar^2} \frac{1}{\beta}$$

$$\boxed{N_c = \frac{m_c}{\pi\hbar^2} k_B T}$$

(4)
$$P_v = \int_{-\infty}^{\epsilon_v} d\epsilon \, g_v(\epsilon) \, e^{-\beta(\epsilon_v-\epsilon)}$$

$$= \frac{m_v}{\pi\hbar^2} \int_{-\infty}^{\epsilon_v} d\epsilon \, \theta(\epsilon_v-\epsilon) \, e^{-\beta(\epsilon_v-\epsilon)}$$

$$= \frac{m_v}{\pi\hbar^2} \int_{-\infty}^{0} dy \, e^{\beta y} = \frac{m_v}{\pi\hbar^2} \frac{1}{\beta}$$

$$\boxed{P_v = \frac{m_v}{\pi\hbar^2} k_B T}$$

(5)
$$N_c = \frac{0.067 \times 9.11 \times 10^{-28}}{3.14 \times (1.05)^2 \times 10^{-54}} \, 1.38 \times 10^{-16} \times 300$$

$$\boxed{N_c = 0.730 \times 10^{12} \ \text{cm}^{-2}} \qquad P_v = \frac{m_v}{m_c} N_c = \frac{0.38}{0.067} N_c$$

$$\boxed{P_v = 4.14 \times 10^{12} \ \text{cm}^{-2}}$$

6 $n_i = \sqrt{N_c N_v}\, e^{-E_g/2k_B T}$

$= \sqrt{0.730 \times 4.14} \times 10^{12}\, e^{-\frac{1.6\times10^{-12}}{2\times1.38\times10^{-16}\times300}}$

$= 1.74 \times 10^{+12}\, e^{-19.3}$

$= 1.74 \times 10^{12} \times 4.15 \times 10^{-9}$

$= 7.22 \times 10^{3}$

$\boxed{n_i = 7.22 \times 10^{3}\ \mathrm{cm}^{-2}}$