Exercise 1 09-06-2011

① $M\ddot{u}(n,t) = -G[2u(n,t) - u(n+1,t) - u(n-1,t)]$

$u(n,t) \sim e^{i(qna - \omega_q t)} \Longrightarrow$

$M\omega_q^2 = G[2 - (e^{iqa} + e^{-iqa})] = 4G\sin^2(\frac{qa}{2})$

$\omega_q = 2\sqrt{G/M}\,|\sin(\frac{qa}{2})| \sim c|q|,\ q \to 0\ ,\ c = a\sqrt{G/M}$

$$\boxed{u_{\pm}(qn,t) = e^{i(qna \mp \omega_q t)} \quad \Big|\quad -\frac{\pi}{a} < q \leq \pi/a}$$

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②

$u(n,t) = \sum_q a(q)\,[-e^{i(qna - \omega_q t)} + e^{i(qna + \omega_q t)}]$

$= -i\,\frac{\pi}{L}\,\frac{la}{10}\,\frac{L}{2\pi}\int_{-\infty}^{+\infty} dq\,[-e^{i(qna - ct|q|)} + e^{i(qna + ct|q|)}]$

$\times\ e^{-|q|l}$

$u(n,t) = -i\,\frac{la}{20}\int_{-\infty}^{+\infty} dq\,[-e^{-(l+ict)|q|}\,e^{iqna}$
$+ e^{-(l-ict)|q|}\,e^{iqna}]$

$= -i\frac{la}{20}\,[-I_1(ct,na) + I_1(-ct,na)] \equiv -i\frac{la}{20}\,I_2(ct,na)$

$I_1(y,x) = \int_{-\infty}^{+\infty} dq\, e^{-(l+iy)|q|}\, e^{iqx}$

As $e^{-(\ell+iy)|q|}$ is even in q,

$$I_1(y,x) = \int_{-\infty}^{+\infty} dq\, e^{-(\ell+iy)|q|} \cos(qx)$$

$$= 2\int_0^{\infty} dq\, e^{-(\ell+iy)q} \cos(qx) =$$

$$\int_0^{\infty} dq \left[e^{-(\ell+iy)q+iqx} + e^{-(\ell+iy)q-iqx} \right]$$

$$I_0 = \int_0^{\infty} dq\, e^{-q(\ell+iy-ix)} = - \frac{e^{-q(\ell+iy-ix)}}{\ell+i(y-x)} \Bigg|_0^{\infty}$$

$$= \frac{1}{\ell+i(y-x)} \equiv I_0(y,x)$$

$$I_1(y,x) = I_0(y,x) + I_0(y,-x)$$

$$I_2(y,x) = -I_1(y,x) + I_1(-y,x)$$

$$= -I_0(y,x) - I_0(y,-x) + I_0(-y,x) + I_0(-y,-x)$$

$$= -\frac{1}{\ell+i(y-x)} - \frac{1}{\ell+i(y+x)} + \frac{1}{\ell-i(y+x)} + \frac{1}{\ell-i(y-x)}$$

$$= \frac{i\,2(y-x)}{\ell^2+(y-x)^2} + \frac{i\,2(y+x)}{\ell^2+(y+x)^2}$$

$$u(n,t) = \frac{\ell a}{10} \left[\frac{ct-na}{\ell^2+(ct-na)^2} + \frac{ct+na}{\ell^2+(ct+na)^2} \right]$$

$$\boxed{u(x,t) = \frac{a}{10}\left[\frac{\frac{ct-xa}{\ell}}{1+\left(\frac{ct-xa}{\ell}\right)^2} + \frac{\frac{ct+xa}{\ell}}{1+\left(\frac{ct+xa}{\ell}\right)^2}\right]}$$

③ $$\boxed{u(x,0) = \frac{a}{10}\left[\frac{-xa/\ell}{1+(xa/\ell)^2} + \frac{xa/\ell}{1+(xa/\ell)^2}\right] = 0}$$

④

$$\dot{u}(x,t) = \frac{c}{\ell}\frac{a}{10}\left[\frac{1}{1+\left(\frac{ct-xa}{\ell}\right)^2} + \frac{1}{1+\left(\frac{ct+xa}{\ell}\right)^2}\right.$$

$$\left. - \frac{+2\left(\frac{ct-xa}{\ell}\right)^2}{\left[1+\left(\frac{ct-xa}{\ell}\right)^2\right]^2} - \frac{+2\left(\frac{ct+xa}{\ell}\right)^2}{1+\left(\frac{ct+xa}{\ell}\right)^2}\right]$$

$$\dot{u}(x,0) = \frac{c}{\ell}\frac{a}{10}\left[\frac{2}{1+\left(\frac{xa}{\ell}\right)^2} - \frac{4\left(\frac{xa}{\ell}\right)^2}{\left[1+\left(\frac{xa}{\ell}\right)^2\right]^2}\right]$$

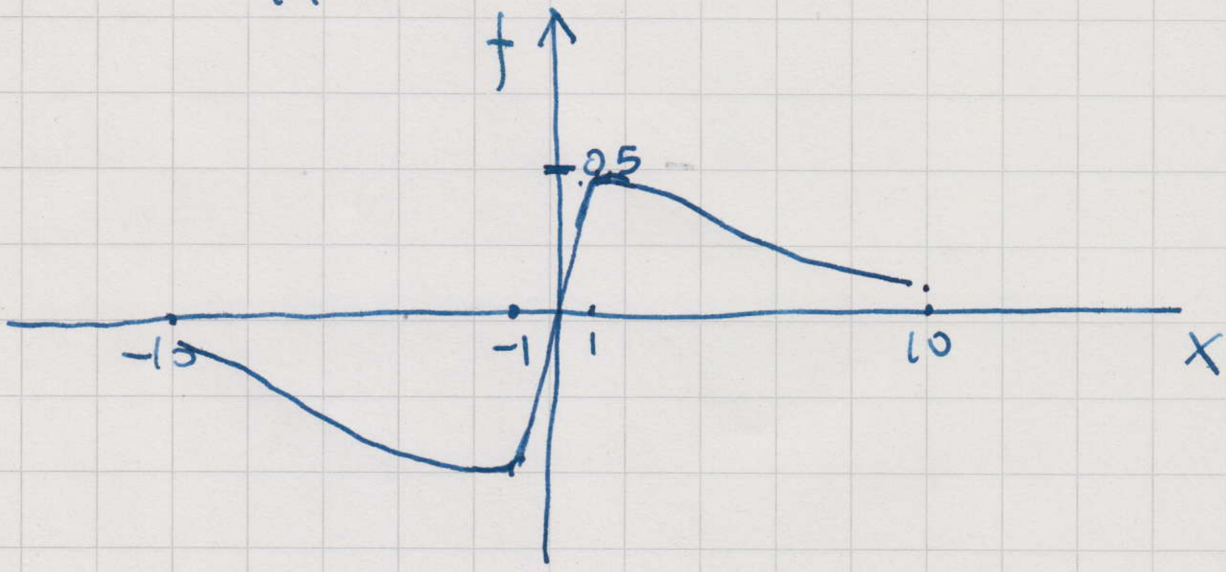
$$= \frac{c}{\ell}\frac{a}{10}\,2\,\frac{1-\left(\frac{xa}{\ell}\right)^2}{\left[1+\left(\frac{xa}{\ell}\right)^2\right]^2}$$

$$\boxed{\dot{u}(x,0) = \frac{c}{\ell}\frac{a}{5}\frac{1-(xa/\ell)^2}{\left[1+(xa/\ell)^2\right]^2}}$$

⑤ $$u\left(n, \frac{ma}{c}\right) = \frac{a}{10}\left[\frac{(m-n)a/\ell}{1+[(m-n)a/\ell]^2} + \frac{(m+n)a/\ell}{1+[(m+n)a/\ell]^2}\right]$$

$$\tilde{u}(n,m) = \frac{u(n, \frac{ma}{c})}{a/10} = f\left((m-n)\frac{a}{\ell}\right) + f\left((m+n)\frac{a}{\ell}\right)$$

$$f(x) = \frac{x}{1+x^2} = -f(-x) \qquad f'(x) = 0 \Rightarrow x = \pm 1$$



- Are displaced the atoms in two regions, centred at $n = m$ and $n = -m$, of width

$$\Delta n \frac{a}{\ell} \sim 20$$

Exercise 2   09-06-2011

① $\frac{4\pi}{3} k_{F\sigma}^3 = N_\sigma \cdot \frac{8\pi^3}{V} \Rightarrow k_{F\sigma} = (6\pi^2 n_\sigma)^{\frac{1}{3}}$

$$T_\sigma = \sum_{k<k_{F\sigma}} \hbar^2 k^2/2m = \frac{\hbar^2}{2m} \frac{V}{8\pi^3} \int_{k<k_{F\sigma}} d\bar{k}\, k^2$$

$$= \frac{\hbar^2}{2m} \frac{V}{2\pi^2} \int_0^{k_{F\sigma}} dk\, k^2 k^2$$

$$= \frac{\hbar^2}{2m} \frac{V}{2\pi^2} \frac{k_{F\sigma}^5}{5} = \frac{\hbar^2 k_{F\sigma}^2}{2m} \frac{1}{5} \frac{V}{2\pi^2} 6\pi^2 n_\sigma$$

$$\boxed{T_\sigma = \frac{3}{5} N_\sigma \frac{\hbar^2 k_{F\sigma}^2}{2m} = \frac{3}{5} N_\sigma \frac{\hbar^2}{2m} (6\pi^2)^{2/3} n_\sigma^{2/3}}$$

② $E_z = \mu_B (N_\uparrow - N_\downarrow) B \qquad \bar{B} = B\hat{z}$

③ $$T_\sigma = \frac{3}{5} \frac{\hbar^2}{2m} (6\pi^2)^{2/3} N_\sigma n_\sigma^{2/3}$$

$$= \frac{3}{5} \frac{\hbar^2}{2m} (6\pi^2)^{2/3} \frac{N}{2}(1+\sigma J) \left[\frac{n}{2}(1+\sigma J)\right]^{2/3}$$

$$= A N n (1+\sigma J)^{5/3} \qquad \sigma = \pm \text{ for } \uparrow,\downarrow$$

$$E = A N n^{2/3} \left[(1+J)^{5/3} + (1-J)^{5/3}\right] + \mu_B J B \cdot N$$

④ $$\frac{1}{N} \frac{\partial E}{\partial J} = \frac{5}{3} A n^{2/3} \left[(1+J)^{2/3} - (1-J)^{2/3}\right] + \mu_B B = 0$$

(5) $J \ll 1$

$$\frac{5}{3} A n^{2/3} \left[1 + \frac{2}{3} J - \frac{1}{9} J^2 - 1 + \frac{2}{3} J - \frac{1}{9} J^2 + O(J^3) \right] + \mu_B B = 0$$

$$\frac{5}{3} A n^{2/3} \, \frac{4}{3} J = - \mu_B B$$

$$\boxed{J = - \frac{9}{20} \frac{\mu_B}{A n^{2/3}} B}$$

(6) $$\chi = - \mu_B \left. \frac{dJ}{dB} \right|_{B=0} = \frac{9}{20} \frac{\mu_B^2}{A n^{2/3}}$$

$$A = \frac{\hbar^2}{2m} \cdot \frac{3}{5} (3\pi^2)^{2/3} \cdot \frac{1}{2}$$

$$A n^{2/3} = \frac{3}{5} \frac{\hbar^2}{2m} \frac{(3\pi^2 n)^{2/3}}{2} = \frac{3}{5} \frac{\hbar^2}{2m} \frac{k_F^2}{2} = \frac{3}{5} \frac{\epsilon_F}{2}$$

$$\chi = \frac{9}{20} \frac{\mu_B^2}{\frac{3}{10} \epsilon_F} = \frac{3}{2} \frac{\mu_B^2}{\epsilon_F} = \frac{1}{n} \frac{3n}{2\epsilon_F} \mu_B^2$$

$$\boxed{\chi = \frac{1}{n} g(\epsilon_F) \mu_B^2}$$

It coincides with A.M. result if one looks at $\tilde{\chi} = - \mu_B n \frac{dJ}{dB} = \frac{dM}{dB}$!