ESERCIZIO 1

16/11/12

(1) $E(s) = \frac{2\gamma}{s}$ $F(s) = -\frac{2\gamma e}{s}$

$$U(s) - U(s_0) = -\int_{s_0}^{s} F(s')\,ds' = 2\gamma e \int_{s_0}^{s} \frac{ds'}{s'}$$

$$U = 2\gamma e \ln \frac{s}{s_0} \qquad R_0 \le s, s_0 \le R$$

Poniamo per convenienza lo zero di energia a $s_0 = R_0$

$$\boxed{U = 2\gamma e \ln(s/R_0)} \qquad \beta\gamma e = \frac{1}{2}$$

$$Q_N(V,T) = \frac{1}{N!} q^N$$

Usiamo coordinate cilindriche z, s, φ

$$q = \frac{1}{\lambda^3}\int d\bar{r}\, e^{-\beta 2\gamma e \ln s/R_0}$$

$$= \frac{1}{\lambda^3}\int_0^L dz \int_{R_0}^{R} ds\, s \int d\varphi\, e^{-\ln s/R_0}$$

$$= \frac{1}{\lambda^3} L\, 2\pi \int_{R_0}^{R} ds\, s\, \frac{R_0}{s} = \frac{2\pi L}{\lambda^3} R_0(R - R_0)$$

$$Q_N(V,T) \simeq \left(\frac{e}{N}\, 2\pi \frac{L R_0 (R - R_0)}{\lambda^3}\right)^N$$

$$\boxed{A_N(V,T) = -k_B T N \ln\left[2\pi \frac{e}{N}\, \frac{L R_0 (R - R_0)}{\lambda^3}\right]}$$

(2)

② $dV = 2\pi R L\, dR$

$$P = -\frac{\partial A}{\partial V} = -\frac{1}{2\pi R L}\frac{\partial}{\partial R}\left[-N k_B T \ln\left(\frac{2\pi e}{N}\frac{L R_0 (R-R_0)}{\lambda^3}\right)\right]$$

$$= \frac{N k_B T}{2\pi R L}\frac{\partial}{\partial R}\ln(R-R_0) = \frac{N k_B T}{2\pi R (R-R_0) L}$$

$$\boxed{P = \frac{N k_B T}{2\pi R (R-R_0) L}}$$

③

$$\mu = \frac{\partial A}{\partial N} = \frac{\partial}{\partial N}\left\{N k_B T \ln \frac{N\lambda^3}{2\pi e L R_0 (R-R_0)}\right\}$$

$$= k_B T \ln \frac{N\lambda^3}{2\pi e L R_0 (R-R_0)} + k_B T$$

$$\boxed{\mu = k_B T \ln \frac{N\lambda^3}{2\pi L R_0 (R-R_0)}}$$

④ $$n(s) = N \frac{\int d\bar{r}_1\, e^{-2\beta e_f \ln\frac{s}{R_0}}\, \delta(r-\bar{r}_1)}{\int d\bar{r}_1\, e^{-2\beta e_f \ln\frac{s}{R_0}}}$$

$2\beta e_f = 1$

$$n(s) = \frac{N \quad R_0/s}{2\pi L R_0 (R - R_0)} = \frac{N}{2\pi L s (R - R_0)}$$

$$\boxed{n(s) = \frac{N}{2\pi L s (R - R_0)}}$$

$$\boxed{P = \frac{N k_B T}{2\pi L R (R - R_0)} = k_B T \rho(R)}$$

$$\boxed{\mu = k_B T \ln \frac{N \lambda^3}{2\pi L R_0 (R - R_0)} = k_B T \ln\left[\rho(R_0) \lambda^3\right]}$$

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$$\mathcal{H} = \sum_{i=1}^{N} h_i \qquad h = \frac{p^2}{2m} + k_B T \frac{z}{\ell}\left[1 + \lambda \frac{z}{L_z}\right]$$

$$Q_N = \frac{1}{N!} q^N \qquad q = \frac{1}{h^3}\int d\bar{p} \int_V d\bar{r}\, e^{-\beta h}$$

$$q = \frac{1}{h^3}\int d\bar{p}\, e^{-\beta p^2/2m} \int_V d\bar{r}\, e^{-\frac{z}{\ell} - \lambda \frac{z^2}{\ell L_z}} = q_1 \cdot q_2$$

$$q_1 = \frac{1}{\lambda_T^3} \qquad \frac{1}{\lambda_T} = \sqrt{\frac{2\pi m k_B T}{h^2}}$$

$$q_2 = A_b \int_0^{L_z} dz\, e^{-z/\ell}\, e^{-\frac{z^2}{\ell L_z}\lambda}$$

$$\simeq A_b \int_0^{\infty} dz\, e^{-z/\ell}\left[1 - \lambda \frac{z^2}{\ell L_z}\right]$$

$$= A_b \ell \int_0^{\infty} dy\left[e^{-y} - \lambda \frac{\ell}{L_z} y^2 e^{-y}\right] \qquad y = \frac{z}{\ell}$$

$$\int_0^{\infty} dy\, y^2 e^{-y} = \left[\frac{d^2}{d\alpha^2}\int_0^{\infty} dy\, e^{-\alpha y}\right]_{\alpha=1} = \left[\frac{d^2}{d\alpha^2}\frac{1}{\alpha}\right]_{\alpha=1} = \left.\frac{2}{\alpha^3}\right|_{\alpha=1} = 2$$

$$q_2 = A_b \ell\left[1 - 2\lambda \frac{\ell}{L_z}\right]$$

$$Q_N = \frac{1}{N!}(q_1 \cdot q_2)^N = \frac{1}{N!}\left(\frac{A_b \ell}{\lambda_T^3}\right)^N \left(1 - 2\lambda \frac{\ell}{L_z}\right)^N$$

① $Q_N \simeq \frac{1}{N!}\left(\frac{A_b \ell}{\lambda_T^3}\right)^N \left(1 - 2\lambda \frac{\ell}{L_z} N\right)$

$$\boxed{Q_N \simeq \left(\frac{e A_b \ell}{N \lambda_T^3}\right)^N \left(1 - 2\lambda \frac{\ell}{L_z} N\right)}$$

② $A = -k_B T\left[N \ln \frac{e A_b \ell}{N \lambda_T^3} + \ln\left(1 - 2\lambda \frac{\ell}{L_z} N\right)\right]$

$$\boxed{A \simeq -N k_B T\left[\ln \frac{e A_b \ell}{N \lambda_T^3} - 2\lambda \frac{\ell}{L_z}\right]}$$

③ $\ell \sim \beta^{-1}$ $\quad 1/\lambda_T^3 \sim \beta^{-3/2}$ $\quad \ell/\lambda_T^3 = C' \beta^{-5/2}$

$$E = \frac{\partial \beta A}{\partial \beta} = -\frac{\partial}{\partial \beta}\left[\ln \beta^{-5/2} - 2\lambda \frac{C'}{L_z}\beta^{-1}\right] N$$

$$= \frac{5}{2} N \frac{1}{\beta} - 2N\lambda \frac{\ell}{L_z}\frac{1}{\beta} \qquad \ell = C'\beta^{-1}$$

$$\boxed{E = N\left[\frac{5}{2} k_B T - \lambda \cdot 2 \frac{k_B^2 T^2}{m g L_z}\right]} \qquad \ell = \frac{k_B T}{m g}$$

④ $C_V = \frac{\partial E}{\partial T} = N k_B\left[\frac{5}{2} - \lambda 4 \frac{k_B T}{m g L_z}\right]$

$$\boxed{C_V = N k_B\left[\frac{5}{2} - 4\lambda \frac{\ell}{L_z}\right]}$$