COMPITO 17/11/16

ESERCIZIO 1.

$$\Sigma_1(N,V,E) = \frac{1}{N!\, h^{3N}} \int_V d\bar{r}_1 \ldots \int_V d\bar{r}_N \times$$

$$\times \int d\bar{p}_1 \ldots \int d\bar{p}_N \, \theta\left(E - c\sum_i |\bar{p}_i|\right)$$

$$= \frac{V^N}{N!\, h^{3N}} (8\pi)^N \left(\frac{E}{c}\right)^{3N} \frac{1}{3N!}$$

$$\simeq \left[\frac{8\pi V}{h^3}\left(\frac{E}{c}\right)^3\right]^N \left(\frac{e}{N}\right)^N \left(\frac{e}{3N}\right)^{3N}$$

$$= \left[8\pi \frac{V}{N}\left(\frac{E}{3hcN}\right)^3\right]^N e^{4N}$$

$$S = k_B \ln \Sigma_1 = N k_B \ln \left[\frac{8\pi V e^4}{N}\left(\frac{E}{3hcN}\right)^3\right]$$

(1) $$\boxed{S(N,V,E) = N k_B \ln \left[8\pi e^4 \frac{V}{N}\left(\frac{E}{3hcN}\right)^3\right]}$$

(2)

$$\frac{1}{T} = \frac{\partial S}{\partial E}\Big|_{N,V} = \frac{\partial}{\partial E}\left\{ N k_B \ln\left[E^3\right]\right\}$$

$$= \frac{3Nk_B}{E} \qquad \boxed{T = \frac{E}{3Nk_B}}$$

$$\boxed{E = 3Nk_BT} \qquad \boxed{S(N,V,T) = Nk_B \ln\left[\frac{8\pi e^4 V}{N}\left(\frac{k_BT}{hc}\right)^3\right]}$$

(3) $dE = TdS - PdV + \mu\, dN$

$$\mu = -T\frac{\partial S}{\partial N}\Big|_{E,V} = -T\frac{\partial}{\partial N}\left\{Nk_B \ln\left[\frac{8\pi e^4 V}{N}\left(\frac{E}{3hcN}\right)^3\right]\right\}$$

$$= -T\left\{k_B \ln\left[8\pi e^4 \frac{V}{N}\left(\frac{E}{3hcN}\right)^3\right] + Nk_B\frac{\partial}{\partial N}\ln\frac{1}{N^4}\right\}$$

$$= -T\left\{k_B \ln\left[8\pi e^4\frac{V}{N}\left(\frac{E}{3hcN}\right)^3\right] - 4Nk_B\frac{1}{N}\right\}$$

$$= -T\left\{k_B \ln\left[8\pi \cancel{e^4}\frac{V}{N}\left(\frac{E}{3hcN}\right)^3\right] \cancel{- k_B \ln e^4}\right\}$$

$$\mu = k_BT \ln\left[\frac{N/V}{8\pi}\left(\frac{3Nk_BT}{3hcN}\right)^3\right]$$

$$\mu = k_B T \ln\left[\frac{\rho}{8\pi}\left(\frac{hc}{k_B T}\right)^3\right] = k_B T \ln\left[\frac{1}{8\pi}l^3\rho\right]$$

$$\boxed{\mu = k_B T \ln\left[\frac{1}{8\pi}l^3\rho\right] = k_B T \ln\left[\tilde{l}^3\rho\right]}$$

$$l = \frac{hc}{k_B T} \qquad \tilde{l} = \frac{hc}{2\pi^{\frac{1}{3}} k_B T}$$

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(4)

$$P = T\frac{\partial S}{\partial V}\Big|_{E,N} = T\frac{\partial}{\partial V}\left[N k_B \ln V\right]$$

$$= \frac{N k_B T}{V}$$

$$\boxed{P = \frac{N k_B T}{V}}$$

ESERCIZIO 2.

$$H_1 = \frac{p^2}{2m} + v\,\ln\left(\frac{s}{a}\right) \qquad \bar{r} = (\bar{s}, z)$$

$$\bar{s} = s(\cos\varphi, \sin\varphi)$$

$$H = \sum_{i=1}^{N} H_1(\bar{p}_i, \bar{r}_i)$$

$$Q_N(V, T) = \frac{q^N}{N!}$$

$$q = \frac{1}{h^3}\left[\int d\bar{p}\, e^{-\beta\frac{p^2}{2m}} \cdot \int_0^L dz\, 2\pi \int_a^R ds\, s\, e^{-\beta v_0 \ln\frac{s}{a}}\right]$$

$$= \frac{1}{\lambda^3} \cdot 2\pi L \int_a^R ds\, s\, e^{-\beta v_0 \ln\frac{s}{a}}$$

poniamo $\epsilon = \beta v_0 = v_0 / k_B T$

$$q = \frac{2\pi L}{\lambda^3} \int_a^R ds\, s\, e^{+\ln\left(\frac{s}{a}\right)^{-\epsilon}}$$

$$= \frac{2\pi L}{\lambda^3} \int_a^R ds\, s\left(\frac{a}{s}\right)^{\epsilon} = \frac{2\pi a^2 L}{\lambda^3} \int^{R/a} dy\, y^{1-\epsilon}$$

$$= \frac{2\pi a^2 L}{\lambda^3} \left.\frac{y^{2-\epsilon}}{2-\epsilon}\right|_1^{R/a} = \frac{2\pi a^2 L}{\lambda^3} \frac{\left(R/a\right)^{2-\epsilon} - 1}{2-\epsilon}$$

$$y = \frac{s}{a}$$

$$Q_N(V,T) \simeq \left(\frac{e}{N}\right)^N \left[\frac{2\pi a^2 L}{\lambda^3} \frac{\left(R/a\right)^{2-\epsilon} - 1}{2-\epsilon}\right]^N$$

$$A = -k_B T \ln Q_N$$

① $$\boxed{A = -N k_B T \ln \left[\frac{e\, 2\pi a^2 L}{\lambda^3 N} \frac{\left(R/a\right)^{2-\epsilon} - 1}{2-\epsilon}\right]}$$

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②

$$\langle s_1 \rangle = \frac{\int_a^R ds_1\, s_1\, s_1\, e^{-\beta U_0 \ln \frac{s}{a}}}{\int_a^R ds_1\, s_1\, e^{-\beta U_0 \ln \frac{s}{a}}}$$

$$= \frac{\int_a^R ds_1\, s_1^2 \left(\frac{a}{s}\right)^\epsilon}{\int_a^R ds\, s_1 \left(\frac{a}{s}\right)^\epsilon} = \frac{a^3 \int_1^{R/a} dy\, y^{2-\epsilon}}{a^2 \int_1^{R/a} dy\, y^{1-\epsilon}}$$

$$= a \frac{\left[y^{3-\epsilon}/(3-\epsilon)\right]_1^{R/a}}{\left[y^{2-\epsilon}/(2-\epsilon)\right]_1^{R/a}}$$

$$\langle s_1 \rangle = a \, \frac{2-\epsilon}{3-\epsilon} \, \frac{\left(\frac{R}{a}\right)^{3-\epsilon} - 1}{\left(\frac{R}{a}\right)^{2-\epsilon} - 1}$$

(i) $\epsilon \gg 1 \qquad 3-\epsilon < 0, \quad 2-\epsilon < 0$

$$\left(\frac{R}{a}\right)^{3-\epsilon} = \left(\frac{a}{R}\right)^{\epsilon-3} \ll 1$$

$$\left(\frac{R}{a}\right)^{2-\epsilon} = \left(\frac{a}{R}\right)^{\epsilon-2} \ll 1$$

$$\boxed{\langle s_1 \rangle \simeq a \, \frac{\epsilon-2}{\epsilon-3} \simeq a}$$

(ii) $\epsilon \ll 1 \qquad 3-\epsilon \simeq 3 \qquad 2-\epsilon \simeq 2$

$$\left(\frac{R}{a}\right)^{3} \gg 1 \qquad \left(\frac{R}{a}\right)^{2} \gg 1$$

$$\langle s_1 \rangle \simeq a \, \frac{2}{3} \, \frac{(R/a)^{3-\epsilon}}{(R/a)^{2-\epsilon}} = \frac{2}{3} a \cdot \frac{R}{a} = \frac{2}{3} R$$

$$\boxed{\langle s_1 \rangle \simeq \frac{2}{3} R}$$

(4) $\delta V = 2\pi R L \, dR$

$$P = -\frac{1}{2\pi R L}\frac{\partial A}{\partial R} = +\frac{N k_B T}{2\pi R L}\frac{\partial}{\partial R}\left\{\ln\left[\left(\frac{R}{a}\right)^{2-\epsilon}-1\right]\right\}$$

$$= +\frac{N k_B T}{2\pi R L}\frac{\left(\frac{R}{a}\right)^{1-\epsilon}}{\left(\frac{R}{a}\right)^{2-\epsilon}-1}\cdot(2-\epsilon)\frac{1}{a}$$

$$\boxed{P = \frac{N k_B T}{2\pi R L a}\frac{(R/a)^{1-\epsilon}}{\left[\left(\frac{R}{a}\right)^{2-\epsilon}-1\right]/(2-\epsilon)}}$$