COMPITO DEL 20-12-11

Esercizio 1: Gas ideale di Fermioni in 1D a $T=0$

①

L'Hamiltoniana di singola particella è

$$h^{(1)} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$$

e la parte orbitale della funzione d'onda di singola particella soddisfa

$$-\frac{\hbar^2}{2m}\varphi''(x) = E\varphi(x) \Rightarrow \varphi''(x) + k^2\varphi(x) = 0$$

$$k^2 = \frac{2m}{\hbar^2}E$$

$$\varphi(x) = A\sin kx + B\cos kx \qquad \varphi(0)=0 \Rightarrow B=0$$

$$\varphi(L)=0 \Rightarrow k = \frac{\pi}{L}n$$

$$n \in N$$

- orbitali $\boxed{\phi_n(x,s_z) = \sqrt{\frac{2}{L}}\sin\left(\frac{\pi}{L}nx\right)\chi_{s_z}}$

  $n \in N \quad (n = 1,2,3,\ldots)$

- energie

$$\boxed{E_n = \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2 n^2}$$

② I K_n sono spaziati di π/L e le energie $\propto K_n^2 \propto n^2$; occupando ciascun K_n con 2 elettroni ($S_z = \pm\hbar/2$) a partire dal più piccolo ($n=1$) otteniamo

$$N = \frac{2K_F}{\pi/L} \Rightarrow \boxed{K_F = \frac{N}{2}\frac{\pi}{L} = M\frac{\pi}{L}}$$

$$\boxed{\epsilon_F = \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2 M^2 = \frac{\hbar^2}{2m}\pi^2\left(\frac{\tilde{n}}{2}\right)^2} \qquad \tilde{n} = \frac{N}{L}$$

③ $$E = 2\frac{\hbar^2}{2m}\frac{\pi^2}{L^2}\sum_{n=1}^{M} n^2 = \frac{\hbar^2}{2m}\frac{\pi^2}{L^2}\frac{1}{3}M(M+1)(2M+1)$$

$$E = \frac{\hbar^2}{24m}\frac{\pi^2}{L^2}N(N+2)(N+1)$$

$$\boxed{\frac{E}{N} = \frac{\hbar^2}{24m}\frac{\pi^2}{L^2}(N+1)(N+2)}$$

④ $$\mu = \frac{\partial E}{\partial N} = \frac{\hbar^2}{24m}\frac{\pi^2}{L^2}\frac{\partial}{\partial N}\left[N^3+3N^2+2N\right]$$

$$\mu = \frac{\hbar^2}{24m}\frac{\pi^2}{L^2}\left[3N^2+6N+2\right] = \frac{\hbar^2}{2m}\frac{\pi^2}{L^2}\left(\frac{N}{2}\right)^2\left[1+\frac{2}{N}+\frac{2}{3N^2}\right]$$

$$\boxed{\mu = \epsilon_F\left[1+\frac{2}{N}+\frac{2}{3N^2}\right]}$$

$$\boxed{\lim_{N\to\infty}\mu(N) = \epsilon_F} \qquad \boxed{\frac{N}{L} = \tilde{n} = \text{cost.}}$$

Esercizio 2 : Fononi alla Debye in un cristallo 1D.

(1)
$$Q = \sum_{\{n_k\}} \exp\left[-\beta \sum_k \hbar\omega_k n_k\right].$$
$$= \prod_k \sum_{n_k=0}^{\infty} \exp\left[-\beta\hbar\omega_k n_k\right]$$
$$= \prod_k \sum_{n_k=0}^{\infty} \left(\exp\left[-\beta\hbar\omega_k\right]\right)^{n_k}$$
$$\boxed{Q = \prod_k \frac{1}{1-e^{-\beta\hbar\omega_k}}}$$

(2)
$$E = \frac{\partial(\beta A)}{\partial\beta} = \frac{\partial}{\partial\beta}(-\ln Q) = \frac{\partial}{\partial\beta}\sum_k \ln\left(1-e^{-\beta\hbar\omega_k}\right)$$
$$E = \sum_k \frac{\hbar\omega_k e^{-\beta\hbar\omega_k}}{1-e^{-\beta\hbar\omega_k}} = \sum_k \frac{\hbar\omega_k}{e^{\beta\hbar\omega_k}-1}$$

For $L\to\infty$ (P.B.C.)
$$\frac{E}{L} = \frac{1}{L}\int_{-k_D}^{k_D} \frac{dk}{\frac{2\pi}{L}} \frac{\hbar c|k|}{e^{\beta\hbar c|k|}-1}$$
$$= \frac{2}{2\pi}\int_0^{k_D} dk \frac{\hbar ck}{e^{\beta\hbar ck}-1} \qquad \boxed{t = \beta\hbar ck}$$
$$\frac{E}{L} = \frac{1}{\pi}\frac{(k_B T)^2}{\hbar c}\int_0^{T_D/T} dt \frac{t}{e^t-1} \qquad \frac{\hbar c k_D}{k_B T} = \frac{T_D}{T}$$
$$\lim_{T\to 0}\frac{E}{L} \simeq \frac{1}{\pi}\frac{(k_B T)^2}{\hbar c}\int_0^{\infty} dt \frac{t}{e^t-1} = \frac{\pi^{\cancel{2}}}{6}\frac{1}{\cancel{\pi}}\frac{(k_B T)^2}{\hbar c}$$

• $T\to 0$
$$\boxed{\frac{E}{L} = \frac{\pi}{6}\frac{(k_B T)^2}{\hbar c}}$$

③ $$C_V = \frac{\partial(E/L)}{\partial(T)} = \frac{\partial}{\partial(T)}\left(\frac{\pi}{6}\frac{(k_B T)^2}{\hbar c}\right) = \frac{\pi}{3}\frac{k_B^2 T}{\hbar c}$$

$$\boxed{C_V = \frac{\pi}{3}\frac{k_B T}{\hbar c} k_B}$$

④ Ricordando $-k_D \leq k \leq k_D$ e $\Delta k = 2\pi/L$

$$N = \frac{2 k_D}{2\pi/L} \Rightarrow k_D = \frac{\pi}{L} N = \pi n$$

$$\boxed{k_D = \pi n}$$

$$k_D = \frac{\pi}{100 \text{Å}} = \frac{\pi}{10^{-6}\,\text{cm}} = 3.14 \times 10^6\,\text{cm}^{-1}$$

$$\boxed{k_D = 3.14 \times 10^6\,\text{cm}^{-1}}$$