I COMPITO - 18-11-13

ESERCIZIO 1

1

$$q = \frac{1}{h^3} \int d\bar{p}\, e^{-\beta A p^s} \int d\bar{r}\, e^{-\beta B r^t}$$

$$\int d\bar{y}\, e^{-\alpha y^\gamma} = 4\pi \int_0^\infty dy\, y^2 e^{-\alpha y^\gamma} =$$

tutto lo spazio

$$= 4\pi \frac{1}{\alpha^{3/\gamma}} \int_0^\infty dx\, x^2 e^{-x^\gamma} = \left(\frac{1}{\alpha}\right)^{3/\gamma} C_{2\gamma}$$

Segue

$$q = \frac{1}{h^3} C_{2s} C_{2t} \left(\frac{K_B T}{A}\right)^{3/s} \left(\frac{K_B T}{B}\right)^{3/t}$$

$$Q_N(V,T) = \frac{1}{N!} \left[\frac{C_{2s} C_{2t}}{h^3} \frac{(K_B T)^{(3/s + 3/t)}}{A^{3/s} B^{3/t}} \right]^N$$

$$\equiv Q_N(T) \; !$$

$$A \simeq - N K_B T \ln \left[\frac{e}{N} \frac{C_{2s} C_{2t}}{h^3} \frac{(K_B T)^{(3/s + 3/t)}}{A^{3/s} B^{3/t}} \right]$$

2 $$E = \frac{\partial \beta A}{\partial \beta} = -N \frac{\partial}{\partial \beta} \ln \left[\left(\frac{1}{\beta}\right)^{3/s + 3/t} \right]$$

$$= N \frac{\partial}{\partial \beta} \left[\frac{3}{s} + \frac{3}{t} \right] \ln \beta = N k_B T \left[\frac{3}{s} + \frac{3}{t} \right]$$

$$E = N k_B T \left[\frac{3}{s} + \frac{3}{t} \right]$$

3 $$h = \sum_{i=1}^{3} \left[A p_i^2 + B q_i^2 \right]$$

$$\frac{\partial h}{\partial p_j} = 2 A p_j \qquad p_j \frac{\partial h}{\partial p_j} = 2 A p_j^2$$

$$\langle A p_j^2 \rangle = \frac{1}{2} \langle p_j \frac{\partial h}{\partial p_j} \rangle = \frac{k_B T}{2} = \langle B q_k^2 \rangle$$

$$\langle h \rangle = 6 \cdot \frac{k_B T}{2} = 3 k_B T \qquad \langle \mathcal{H} \rangle = 3 N k_B T$$

Il risultato del punto 2 per $s = t = 2$ dà lo stesso valore.

4 $$\langle r^2 \rangle = \frac{\int d\bar{r}\, r^2 e^{-\beta B r^t}}{\int d\bar{r}\, e^{-\beta B r^t}} = \frac{4\pi \int_0^\infty dr\, r^4 e^{-\alpha r^t}}{4\pi \int_0^\infty dr\, r^2 e^{-\alpha r^t}}$$

$$= \frac{\left(\frac{1}{\alpha}\right)^{5/t} C_{4t}}{\left(\frac{1}{\alpha}\right)^{3/t} C_{2t}} = \frac{C_{4t}}{C_{2t}} \left(\frac{1}{\alpha}\right)^{2/t}$$

$$\alpha = \beta B = \frac{B}{k_B T}$$

$$\langle r^2 \rangle = \frac{C_{4t}}{C_{2t}} \left(\frac{k_B T}{B}\right)^{2/t} \sim T^{2/t}$$

ESERCIZIO 2

(1) $\mathcal{H} = \sum_i h^{(1)}(\bar{p}_i, \bar{r}_i)$, $h^{(1)}(\bar{p},\bar{r}) = \alpha p$

$$Q_N(V,T) = \frac{1}{N!} q^N \quad , \quad q = \frac{1}{h^3} \int d\bar{r} \int d\bar{p}\, e^{-\beta h^{(1)}(\bar{p},\bar{r})}$$

$$q = \frac{V}{h^3} \int d\bar{p}\, e^{-\beta\alpha|\bar{p}|} = \frac{V}{h^3} 4\pi \int_0^\infty dp\, p^2 e^{-\beta\alpha p}$$

$$q = \frac{4\pi V}{h^3} \frac{2(k_B T)^3}{\alpha^3} = 8\pi V \left(\frac{k_B T}{\alpha h}\right)^3 = \frac{V}{\ell^3}$$

$$\ell^3 = \left(\frac{\alpha h}{k_B T}\right)^3 \frac{1}{8\pi}$$

$$\boxed{Q_N(V,T) = \frac{1}{N!} \left[8\pi V \left(\frac{k_B T}{\alpha h}\right)^3\right]^N}$$

(2) $$Z = \sum_{N=0}^{\infty} e^{\beta\mu N} Q_N(V,T) = \sum_{N=0}^{\infty} \frac{e^{\beta\mu N}}{N!} q^N$$

$$= \sum_{N=0}^{\infty} \frac{\left(e^{\beta\mu} q\right)^N}{N!} = \exp\left[e^{\beta\mu} q\right] = \exp\left[e^{\beta\mu} \frac{V}{\ell^3}\right]$$

$$\int_0^\infty dx\, x^2 e^{-\gamma x} = \frac{\partial^2}{\partial\gamma^2} \int_0^\infty dx\, e^{-\gamma x} = \frac{\partial^2}{\partial\gamma^2} \frac{1}{\gamma} = \frac{2}{\gamma^3}$$

$$\boxed{Z = \exp\left[e^{\beta\mu}\frac{V}{\ell^3}\right] = \exp\left[e^{\beta\mu}\, 8\pi V\left(\frac{k_B T}{\alpha h}\right)^3\right]}$$

(3)

$$\frac{PV}{k_B T} = \ln Z = e^{\beta\mu}\, 8\pi V\left(\frac{k_B T}{\alpha h}\right)^3$$

$$\Rightarrow \quad e^{\beta\mu} = \frac{P}{8\pi}\,\frac{(\alpha h)^3}{(k_B T)^4}$$

$$\boxed{\mu = k_B T \ln\left[\frac{(\alpha h)^3}{8\pi (k_B T)^4} P\right]}$$

(4)

$$A = -k_B T \ln Q_N = -k_B T \ln\left\{\frac{1}{N!} q^N\right\}$$

$$\simeq -N k_B T \ln \frac{eq}{N} = -N k_B T \ln\left\{\frac{e}{N}\frac{V}{\ell^3}\right\}$$

$$\mu = \frac{\partial A}{\partial N} = -k_B T \ln\left\{\frac{e}{N}\frac{V}{\ell^3}\right\} + k_B T$$

$$\rho = \frac{N}{V} \qquad \boxed{\mu = k_B T \ln \frac{N\ell^3}{V}} = \boxed{k_B T \ln \rho \ell^3}$$

$$\rho \ell^3 = \left(\frac{\alpha h}{k_B T}\right)^3 \frac{P}{8\pi k_B T} = \ell^3 \frac{P}{k_B T}$$

$$\boxed{P = k_B T \rho}$$