I COMPITO – 09/11/18

ESERCIZIO 1

① $\mathcal{H}_1 = \dfrac{\bar{p}^2}{2m} - \bar{d}\cdot\bar{E}$, hamiltoniana ad un corpo

$$q = \frac{1}{h^2}\int \bar{p}\, e^{-\beta p^2/2m} \int dS \int_0^{2\pi} d\phi\, e^{\beta d E \cos\phi} = \frac{1}{\lambda^2} S \cdot 2\pi I_0(\beta d E)$$

Si è usato: $\int d\bar{p}\, e^{-\beta p^2/2m} = \int_{-\infty}^{\infty} dp_x\, e^{-\beta p_x^2/2m} \int_{-\infty}^{\infty} dp_y\, e^{-\beta p_y^2/2m} = \left(\int_{-\infty}^{\infty} dp\, e^{-\beta p^2/2m}\right)^2 = \left(\sqrt{2\pi m k_B T}\right)^2 = h^2\lambda$

$$Q_N(N,S,T,E) = \frac{q^N}{N!} \Rightarrow A(N,S,T,E) = -kT \ln\left(\frac{q^N}{N!}\right) \simeq -k_B T \ln\left(\frac{e q}{N}\right)^N = -N k_B T \ln\left[\frac{e S}{N\lambda^2} 2\pi I_0(\beta d E)\right]$$

$$\boxed{A(N,S,T,E) = -N k_B T \ln\left[\frac{e S}{N\lambda^2} 2\pi I_0(\beta d E)\right]}$$

② $E = \dfrac{\partial \beta A}{\partial \beta} = -N \dfrac{\partial}{\partial\beta} \ln\left[\dfrac{I_0(\beta d E)}{\beta}\right]$; nota: $\dfrac{1}{\lambda^2} = \dfrac{2\pi m k_B T}{h^2} = \dfrac{2\pi m}{h^2}\dfrac{1}{\beta}$

$$E = -N\left[dE \frac{I_0'(\beta d E)}{I_0(\beta d E)} - \frac{1}{\beta}\right] = -N d E\left[\frac{I_1(\beta d E)}{I_0(\beta d E)} - \frac{1}{\beta d E}\right]$$

$$\boxed{E = -N d E\left[\frac{I_1(\beta d E)}{I_0(\beta d E)} - \frac{1}{\beta d E}\right]}$$

③ $P = -\dfrac{1}{S}\dfrac{\partial A}{\partial E} = \dfrac{N k_B T}{S}\dfrac{\partial}{\partial E}\ln\left[I_0(\beta d E)\right] = \dfrac{N d}{S}\dfrac{I_1(\beta d E)}{I_0(\beta d E)}$

$$\boxed{P = \frac{N d}{S}\frac{I_1(\beta d E)}{I_0(\beta d E)}}$$

④

$$I_0(z) \simeq \frac{1}{2\pi}\int_0^{2\pi} d\phi \left[1 + z\cos\phi + \frac{1}{2}z^2\cos^2\phi\right] = \frac{1}{2\pi}\left[2\pi + z\cdot 0 + \frac{z^2}{2}\int_0^{2\pi} d\phi \, \frac{1+\cos 2\phi}{2}\right] = \frac{1}{2\pi}\left[2\pi + \frac{z^2}{4}\cdot 2\pi\right]$$

$I_0(z) \simeq 1 + \frac{z^2}{4}$, $z \ll 1$; $I_1(z) \simeq \frac{z}{2}$

$$\chi = \left.\frac{\partial P}{\partial E}\right|_{E=0} \simeq \frac{\partial}{\partial E}\left[\frac{Nd}{S} \frac{\beta d E/2}{1 + \frac{(\beta d E)^2}{4}}\right] \simeq \frac{Nd}{S}\frac{\partial}{\partial E}\frac{\beta d E}{2} = \frac{N\beta d^2}{2S}$$

$$\boxed{\chi = \frac{N d^2}{S\, 2 k_B T}}$$

ESERCIZIO 2

① $q = \frac{1}{h}\int_{-\infty}^{\infty} dp\, e^{-\beta p^2/2m} \cdot \int_a^L dx\, e^{-\beta U_0 \ln\frac{x}{a}} = \frac{a}{\lambda}\int_1^{L/a} d\frac{x}{a}\, e^{\ln\left[\left(\frac{x}{a}\right)^{-\beta U_0}\right]} = \frac{a}{\lambda}\int_1^{L/a} dy\, y^{-\beta U_0}\ ;\ y = \frac{x}{a}$

$q = \left[\frac{a}{\lambda}\frac{y^{1-\beta U_0}}{1-\beta U_0}\right]_1^{L/a} = \frac{a}{\lambda}\frac{1}{1-\gamma}\left[\left(\frac{L}{a}\right)^{1-\gamma} - 1\right];\quad \gamma = \beta U_0$

$Q_N = \frac{q^N}{N!} \simeq \left(\frac{e\,q}{N}\right)^N \Rightarrow A(N,L,T) = -N k_B T \ln\left[\frac{e}{N}\frac{a}{\lambda}\frac{1}{1-\gamma}\left(\left(\frac{L}{a}\right)^{1-\gamma} - 1\right)\right]$

$$\boxed{A = -N k_B T \ln\left[\frac{e\,a}{N\lambda}\frac{1}{1-\beta U_0}\left(\left(\frac{L}{a}\right)^{1-\beta U_0} - 1\right)\right]}$$

② $\langle x\rangle = \langle x_1\rangle = \frac{\int_a^L dx\, x\, e^{-\gamma\ln(x/a)}}{\int_a^L dx_1\, e^{-\gamma\ln(x/a)}} = \frac{a^2\int_1^{L/a} dy\, y^{1-\gamma}}{a\int_1^{L/a} dy\, y^{-\gamma}} = a\,\frac{\frac{1}{2-\gamma}\left[\left(\frac{L}{a}\right)^{2-\gamma} - 1\right]}{\frac{1}{1-\gamma}\left[\left(\frac{L}{a}\right)^{1-\gamma} - 1\right]} = a\,\frac{1-\gamma}{2-\gamma}\,\frac{\left(\frac{L}{a}\right)^{2-\gamma} - 1}{\left(\frac{L}{a}\right)^{1-\gamma} - 1}$

$$\boxed{\langle x\rangle = a\,\frac{1-\beta U_0}{2-\beta U_0}\,\frac{\left(\frac{L}{a}\right)^{2-\beta U_0} - 1}{\left(\frac{L}{a}\right)^{1-\beta U_0} - 1}}$$

③ (i) $\gamma = \beta U_0 \gg 1$

$\langle x\rangle = a\,\frac{\gamma-1}{\gamma-2}\,\frac{1-\left(\frac{a}{L}\right)^{\gamma-2}}{1-\left(\frac{a}{L}\right)^{\gamma-1}} \simeq a\,\frac{1-\left(\frac{a}{L}\right)^{\gamma}}{1-\left(\frac{a}{L}\right)^{\gamma}} \simeq a \qquad \boxed{\langle x\rangle \simeq a} \qquad L \gg a$

(ii) $\gamma = \beta U_0 \ll 1$

$\langle x\rangle = a\,\frac{1-\gamma}{2-\gamma}\,\frac{\left(\frac{L}{a}\right)^{2-\gamma} - 1}{\left(\frac{L}{a}\right)^{1-\gamma} - 1} \simeq a\,\frac{1}{2}\,\frac{\left(\frac{L}{a}\right)^{2} - 1}{\left(\frac{L}{a}\right)^{1} - 1} \simeq \frac{a}{2}\,\frac{\left(\frac{L}{a}\right)^{2}}{\left(\frac{L}{a}\right)} = \frac{L}{2} \qquad \boxed{\langle x\rangle \simeq \frac{L}{2}} \qquad L \gg a$

④

$$P = -\frac{\partial A}{\partial L} = N k_B T \frac{\partial}{\partial L} \ln\left[\left(\frac{L}{a}\right)^{1-\beta U_0} - 1\right] = N k_B T \frac{(1-\beta U_0)\left(\frac{L}{a}\right)^{-\beta U_0}}{\left(\frac{L}{a}\right)^{1-\beta U_0} - 1} \cdot \frac{1}{a} = \frac{N k_B T}{L}(1-\beta U_0)\frac{\left(\frac{L}{a}\right)^{1-\beta U_0}}{\left(\frac{L}{a}\right)^{1-\beta U_0} - 1}$$

$$\boxed{P = \frac{N k_B T}{L}(1-\beta U_0)\frac{(L/a)^{1-\beta U_0}}{(L/a)^{1-\beta U_0} - 1}}$$

(i) $\beta U_0 >> 1$

$$P \simeq \frac{N k_B T}{L}(-\beta U_0)\frac{\left(\frac{a}{L}\right)^{\beta U_0}}{\left(\frac{a}{L}\right)^{\beta U_0} - 1} = \frac{N k_B T}{L}\beta U_0 \frac{\left(\frac{a}{L}\right)^{\beta U_0}}{1 - \left(\frac{a}{L}\right)^{\beta U_0}} \simeq \frac{N k_B T}{L}\beta U_0 \left(\frac{a}{L}\right)^{\beta U_0} \xrightarrow[L \to \infty]{} 0$$

(ii) $\beta U_0 << 1$

$$P \simeq \frac{N k_B T}{L} \cdot 1 \cdot \frac{L/a}{\frac{L}{a} - 1} \simeq \frac{N k_B T}{L}$$