Exercise 1   8-05-13 $\hbar = m = e = 1$

①

$$\epsilon_\sigma = \frac{\sum_{|\vec{k}_\sigma| < k_{F\sigma}} k^2/2}{\sum_{|\vec{k}_\sigma| < k_{F\sigma}}} = \frac{\frac{1}{2}\int_0^{k_{F\sigma}} dk\, k\, k^2}{\int_0^{k_{F\sigma}} dk\, k} = \frac{\frac{1}{2}\frac{k_{F\sigma}^4}{4}}{\frac{k_{F\sigma}^2}{2}} = \frac{1}{4} k_{F\sigma}^2$$

$$\boxed{\epsilon_\sigma = \frac{1}{2}\epsilon_{F\sigma} = \frac{1}{4} k_{F\sigma}^2}$$

$$\frac{\pi k_{F\sigma}^2}{\frac{(2\pi)^2}{A}} = N_\sigma = \frac{k_{F\sigma}^2}{4\pi} A \Rightarrow \boxed{k_{F\sigma}^2 = 4\pi \frac{N_\sigma}{A} \equiv 4\pi n_\sigma}$$

$$t = \frac{\sum_\sigma N_\sigma \epsilon_\sigma}{N} = \sum_\sigma \frac{n_\sigma \epsilon_\sigma}{n} = \sum_\sigma \frac{n_\sigma}{n}\frac{1}{4} 4\pi n_\sigma$$

$$t = \pi\left[\frac{n_\uparrow^2}{n} + \frac{n_\downarrow^2}{n}\right]$$

③

$$e(n,\zeta,B) = \frac{1}{N}\left\{Nt + \frac{U N_\uparrow N_\downarrow}{N} + \mu_B B (N_\uparrow - N_\downarrow)\right\}$$

$$= t + U n_\uparrow n_\downarrow + \mu_B B (n_\uparrow - n_\downarrow)\frac{1}{n}$$

$$= \pi\left(\frac{n_\uparrow^2}{n} + \frac{n_\downarrow^2}{n}\right) + \frac{U}{n^2} n_\uparrow n_\downarrow + \mu_B B\left(\frac{n_\uparrow}{n} - \frac{n_\downarrow}{n}\right)$$

$$e(n,\zeta,B) = \frac{\pi}{4} n\left((1+\zeta)^2 + (1-\zeta)^2\right) + \frac{1}{4} U (1+\zeta)(1-\zeta) + \frac{\mu_B B}{2}(1+\zeta-1+\zeta)$$

$$n = n_\uparrow + n_\downarrow, \quad n\zeta = n_\uparrow - n_\downarrow \Rightarrow n_\sigma = \frac{n}{2}(1+\sigma\zeta) \qquad \sigma = \pm 1$$

$$\boxed{e(n,\zeta,B) = \frac{\pi}{2} n (1+\zeta^2) + \frac{U}{4}(1-\zeta^2) + \mu_B B \zeta}$$

4. Let's minimize $e(n,\zeta,B)$ with respect to ζ, $-1 \leq \zeta \leq 1$

$$e(n,\zeta,B) = \frac{\pi}{2} n + \frac{U}{4} + \left(\frac{\pi}{2} n - \frac{U}{4}\right)\zeta^2 + \mu_B B \zeta$$

$$\frac{\partial e}{\partial \zeta} = 0 = \zeta\left(\pi n - \frac{U}{2}\right) + \mu_B B$$

$$\Rightarrow \zeta(n,B) = -\frac{\mu_B B}{\pi n - \frac{U}{2}} \qquad B > 0$$

$$\frac{\partial^2 e}{\partial \zeta^2} = \pi n - \frac{U}{2} \gtrless 0 \quad \text{for } n \gtrless \frac{U}{2\pi}$$

- there is a local minimum when $n > \frac{U}{2\pi}$ at

$$\zeta^* = -\frac{\mu_B B}{\pi n - \frac{U}{2}} \geq -1 \Rightarrow B \leq \frac{1}{\mu_B}\left(\pi n - \frac{U}{2}\right)$$

$$e(n,\zeta^*,B) = \frac{\pi}{2} n + \frac{U}{4} + \left(\frac{\pi}{2} n - \frac{U}{4}\right)\frac{(\mu_B B)^2}{\left(\pi n - \frac{U}{2}\right)^2} - \frac{(\mu_B B)^2}{\pi n - \frac{U}{2}}$$

$$\boxed{e(n,B) = \frac{\pi}{2} n + \frac{U}{4} - \frac{1}{2}\frac{(\mu_B B)^2}{\pi n - \frac{U}{2}}} \quad \boxed{B \leq \frac{\mu_B}{\pi n - \frac{U}{2}} = B_1}$$

$\boxed{B > B_1}$ $$e(n,B) = \frac{\pi}{2} n + \frac{U}{4} + \left(\frac{\pi}{2} n - \frac{U}{4}\right)(-1)^2 - \mu_B B$$

$$\boxed{e(n,B) = \pi n - \mu_B B} \quad \boxed{B > \mu_B / (\pi n - U/2)}$$

- When $n < U/2\pi$ there is no local minimum at J^* and it is evident that e is minimum a $J = -1$, the polarization that makes minima both

$$\left(\frac{\pi}{2} n - \frac{U}{4}\right) J^2 \quad \text{and} \quad \mu_B B J \ !$$

$$\boxed{e(n, B) = \pi n - \mu_B B \quad \Big| \quad n < U/2\pi}$$

(5) $$J = -\frac{\mu_B}{\pi n - \frac{U}{2}} B \qquad B < \frac{\mu_B}{\pi n - \frac{U}{2}}$$

$$= -1 \qquad B > \frac{\mu_B}{\pi n - \frac{U}{2}}$$

$$\Bigg\} \ n > \frac{U}{2\pi}$$

(6) $$J = -1 \qquad n < \frac{U}{2\pi}$$

(2) $\frac{U}{n} v(\bar{r}_i - \bar{r}_j) = \frac{U}{n} \delta(\bar{r}_i - \bar{r}_j)$

$$\tilde{V} = \frac{1}{2} \sum_{i \neq j} v(\bar{r}_i - \bar{r}_j) \qquad V = \frac{U}{n} \tilde{V}$$

$$\langle SD | \tilde{V} | SD \rangle = \frac{1}{2} \sum_{\alpha\beta} \{ \langle \alpha\beta | v | \alpha\beta \rangle - \langle \beta\alpha | v | \alpha\beta \rangle \} \equiv \langle \tilde{V} \rangle$$

$$\alpha \equiv (\sigma, \bar{k}_\sigma) \qquad \beta \equiv (\sigma', \bar{q}_{\sigma'})$$

$$\psi_{\sigma \bar{k}_\sigma}(s, \bar{r}) = \frac{1}{\sqrt{A}} \delta_{\sigma s} e^{i \bar{k}_\sigma \cdot \bar{r}}$$

$$\langle \tilde{V} \rangle = \frac{1}{2} \sum_{\substack{\sigma\sigma' \\ \bar{k}_\sigma, \bar{q}_{\sigma'}}} \sum_{ss'} \int d\bar{r}_1 \int d\bar{r}_2 \times \frac{1}{A^2} \delta(\bar{r}_1 - \bar{r}_2)$$

$$\Big[\delta_{\sigma s}^2 \delta_{\sigma' s'}^2 e^{i(\bar{k}_\sigma \bar{r}_1 + \bar{q}_{\sigma'} r_2 - \bar{k}_\sigma \bar{r}_1 - \bar{q}_{\sigma'} \bar{r}_2)}$$

$$- \delta_{\sigma s} \delta_{\sigma' s'} \delta_{\sigma' s} \delta_{\sigma s'} e^{i(\bar{k}_\sigma \cdot \bar{r}_1 + \bar{q}_{\sigma'} r_2 - \bar{q}_{\sigma'} \cdot r_1 - \bar{k}_\sigma \cdot \bar{r}_2)} \Big]$$

$$= \frac{1}{2A^2} \int d\bar{r}_1 \sum_{\substack{\sigma\sigma' \\ \bar{k}_\sigma, \bar{q}_{\sigma'}}} [1 - \delta_{\sigma\sigma'}]$$

$$= \frac{1}{2A} \Big\{ \sum_{\sigma \bar{k}_\sigma} \sum_{\sigma' \bar{q}_{\sigma'}} - \sum_{\sigma} \sum_{k_\sigma} \sum_{\bar{q}_\sigma} \Big\}$$

$$= \frac{1}{2A} \Big\{ N^2 - \sum_\sigma N_\sigma^2 \Big\} = \frac{1}{2A} \{ (N_\uparrow + N_\downarrow)^2 - N_\uparrow^2 - N_\downarrow^2 \}$$

$$= \frac{1}{2A} \cdot 2 N_\uparrow N_\downarrow = \frac{N_\uparrow N_\downarrow}{A}$$

$$\langle V \rangle = \frac{U}{n} \langle \tilde{V} \rangle = \frac{U}{n} \frac{N_\uparrow N_\downarrow}{A} = U \frac{N_\uparrow N_\downarrow}{n}$$

Exercise 2 8-5-13

1. An insulator, as the valence band is full and the conduction band is empty (2 electron / unit cell).

2. $E_v(\bar{q}) \simeq 2\gamma_v\left[-3+1-\frac{q_x^2a^2}{2}+2\left(1-\frac{q_x^2a^2}{8}\right)\times\left(1-\frac{3q_y^2a^2}{8}\right)\right]$

$$\simeq 2\gamma_v\left[-\frac{3}{4}q_x^2a^2-\frac{3}{4}q_y^2a^2\right]$$

$$\boxed{E_v(\bar{q}) \simeq -\frac{3}{2}\gamma_v a^2 q^2}$$

$$-2\Delta+E_c(\bar{q}) = -2\gamma_c\left[-\frac{3}{4}a^2\left(q_x^2+q_y^2\right)\right]$$

$$\boxed{E_c(\bar{q}) = 2\Delta+\frac{3}{2}\gamma_c a^2 q^2}$$

3.

$$g_v(E) = \frac{2}{A}\sum_{\bar{q}}{}' \delta\left(E-\left(-\frac{3}{2}\gamma_v a^2 q^2\right)\right)$$

$$= \frac{2}{\cancel{A}}\int\frac{d\bar{q}}{\frac{(2\pi)^2}{\cancel{A}}}\delta\left(E+\frac{3}{2}\gamma_v a^2 q^2\right)$$

$$g_v(E) = \frac{1}{\pi}\int dq\, q\,\delta\left(E+\frac{3}{2}\gamma_v a^2 q^2\right)$$

$$\Rightarrow \quad E = -\frac{3}{2}\gamma_v a^2 q^2$$

$$q^* = \sqrt{-\frac{2E}{3\gamma_v a^2}} \qquad E < 0$$

$$\boxed{g_v(E) = \frac{1}{\pi} \left.\frac{q\,\vartheta(-E)}{|3\gamma_v a^2 q|}\right|_{q^*} = \frac{\vartheta(-E)}{3\pi\gamma_v a^2}}$$

$$g_c(E) = \frac{2}{A}\sum_{\vec{q}} \delta\left(E - 2\Delta - \frac{3}{2}\gamma_c a^2 q^2\right)$$

$$= \frac{1}{\pi}\int dq\, q\, \delta\left(E - 2\Delta - \frac{3}{2}\gamma_c a^2 q^2\right)$$

$$E = 2\Delta + \frac{3}{2}\gamma_c a^2 q^2$$

$$q^* = \sqrt{\frac{E - 2\Delta}{\frac{3}{2}\gamma_c a^2}} \qquad E \geq 2\Delta$$

$$\boxed{g_c(E) = \frac{1}{\pi}\frac{q}{|3\gamma_c a^2 q|}\vartheta(E - 2\Delta) = \frac{1}{3\pi\gamma_c a^2}\vartheta(E - 2\Delta)}$$

(4) $n_c(\tau) = N_c(\tau)\, e^{-\beta(\epsilon_c - \mu)}$

$$N_c = \int g_c(E)\, e^{-\beta(E-\epsilon_c)}\, dE$$

$$= \int_{2\Delta}^{\infty} dE\, e^{-\beta(E-2\Delta)} \frac{1}{3\pi\gamma_c a^2}$$

$$= \int_0^{\infty} dE'\, e^{-\beta E'} \frac{1}{3\pi\gamma_c a^2} = \frac{k_B\tau}{3\pi\gamma_c a^2} \int_0^{\infty} dx\, e^{-x}$$

$$\boxed{N_c = \frac{k_B\tau}{3\pi\gamma_c a^2}}$$

$$p_v(\tau) = P_v(\tau)\, e^{-\beta(\mu-\epsilon_v)} \qquad \epsilon_v = 0$$

$$P_v = \int g_v(E)\, e^{+\beta(E-\epsilon_v)}\, dE$$

$$= \int_{-\infty}^{0} dE \frac{1}{3\pi\gamma_v a^2} e^{\beta E} = \frac{k_B\tau}{3\pi\gamma_v a^2}$$

$$\boxed{P_v = \frac{k_B\tau}{3\pi\gamma_v a^2}}$$

(5)

$$n_c = p_v$$

$$\Rightarrow \quad N_c \, e^{-\beta(2\Delta - \mu)} = P_v \, e^{-\beta\mu}$$

$$e^{2\beta\mu} = \frac{P_v}{N_c} e^{2\beta\Delta} = \frac{\gamma_c}{\gamma_v} e^{2\beta\Delta}$$

$$\beta\mu = \frac{1}{2} \ln \frac{\gamma_c}{\gamma_v} + \beta\Delta$$

$$\boxed{\mu = \frac{k_B T}{2} \ln \frac{\gamma_c}{\gamma_v} + \Delta}$$

(6) $\mu = \Delta + 2 k_B T$

$$\Rightarrow \quad \frac{1}{2} \ln \frac{\gamma_c}{\gamma_v} = 2$$

$$\ln \frac{\gamma_c}{\gamma_v} = 4$$

$$\boxed{\gamma_c = \gamma_v e^4}$$