COMPITO DELLO 01/06/18

ESERCIZIO 1

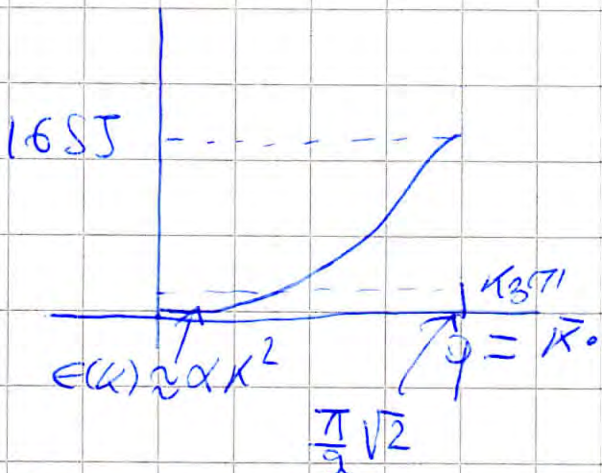
1. Let's start from the spin wave energy (eqs. 3.25, 3.28, 3.29)

$$\frac{E(T)}{V} = \frac{1}{V} \sum_{\bar{k} \in FBZ} \epsilon(\bar{k})\, n(\bar{k}) = \frac{1}{V} \sum_{\bar{k} \in F.BZ.} \frac{\epsilon(\bar{k})}{e^{\beta \epsilon(\bar{k})} - 1}$$

with

$$\epsilon(\bar{k}) = 2S \sum_{\bar{R},\, p.v.} J(\bar{R}) \sin^2\left(\frac{\bar{k}\cdot\bar{R}}{2}\right)$$

A qualitative plot from $\Gamma(\bar{k}=0)$ to a face (centre of) of the F.C.C dodecahedron, chosen for instance as $\bar{k} = \frac{\pi}{a}(1,1,0)$ is



$q = \bar{k}\cdot\hat{n}$, $n = \frac{1}{\sqrt{2}}(1,1,0)$

Clearly for $k_B T \ll SJ$ appreciable contribution to the energy come from $\beta\epsilon(\bar{k}) \lesssim 1$, i.e. for k where $\epsilon(k) \simeq \alpha k^2$ (i); so, taking $\epsilon(k) = \alpha k^2$ the k-sum can be extended to the entire k-space (ii) with negligible error (i).

(2) $\kappa a \ll 1$

$$\epsilon(\bar{\kappa}) \simeq 2S \sum_{\bar{R},\rho\sigma} J \frac{(\bar{\kappa}\cdot\bar{R})^2}{4} \quad , \bar{R} = \frac{a}{2}(\sigma,\sigma',\sigma'') \;, \sigma,\sigma',\sigma'' = \pm 1$$

$$\sum_{\bar{R},\rho\sigma} (\bar{\kappa}\cdot\bar{R})^2 = \frac{a^2}{4} \sum_{\sigma,\sigma',\sigma''} (\kappa_x\sigma + \kappa_y\sigma' + \kappa_z\sigma'')^2$$

$$= \frac{a^2}{4} \sum_{\sigma,\sigma',\sigma''} \left[\kappa_x^2 + \kappa_y^2 + \kappa_z^2 + 2(\kappa_x\kappa_y\sigma\sigma' + \kappa_x\kappa_z\sigma\sigma'' + \kappa_y\kappa_z\sigma'\sigma'')\right]$$

$$= \frac{a^2}{4} 8(\kappa_x^2 + \kappa_y^2 + \kappa_z^2) = 2a^2\kappa^2$$

$$\boxed{\epsilon(\bar{\kappa}) = SJ\kappa^2 a^2 \equiv \alpha\kappa^2}$$

(3) Let's start from the energy $E(T)$, small T,

$$u(T) = \frac{E(T)}{V} = \frac{1}{V} \sum_{\bar{\kappa}} \frac{\alpha\kappa^2}{e^{\beta\alpha\kappa^2} - 1}$$

$$= \frac{1}{V} \int \frac{d\bar{\kappa}}{\frac{(2\pi)^3}{V}} \frac{\alpha\kappa^2}{e^{\beta\alpha\kappa^2} - 1}$$

$$= \frac{1}{2\pi^2} \int_0^\infty d\kappa\, \kappa^2 \frac{\alpha\kappa^2}{e^{\beta\alpha\kappa^2} - 1} \qquad x = \sqrt{\frac{\alpha}{k_B T}}\,\kappa$$

$$= \frac{1}{2\pi^2} \alpha \left(\frac{k_B T}{\alpha}\right)^{5/2} \int_0^\infty dx \frac{x^4}{e^{x^2} - 1}$$

$$= \frac{\alpha}{2\pi^2} \left(\frac{k_B T}{\alpha}\right)^{5/2} \cdot I$$

We get

$$C_V^M = \frac{\partial U(T)}{\partial T} = \frac{5}{2} k_B \frac{\alpha}{2\pi^2}\left(\frac{k_B T}{\alpha}\right)^{3/2} \frac{1}{\alpha} I$$

Let's calculate I

$$I = \int_0^\infty dx\, x^4 e^{-x^2} \sum_{n=0}^\infty e^{-nx^2}$$

$$= \sum_{n=1}^\infty \frac{\partial^2}{\partial n^2} \int_0^\infty dx\, e^{-nx^2} = \sum_{n=1}^\infty \frac{\partial^2}{\partial n^2} \sqrt{\frac{\pi}{n}} \cdot \frac{1}{2}$$

$$= \frac{\sqrt{\pi}}{2} \sum_{n=1}^\infty \frac{3}{4} \frac{1}{n^{5/2}} = \frac{3\sqrt{\pi}}{8} \sum_{n=1}^\infty \frac{1}{n^{5/2}}$$

$$I = \frac{3\sqrt{\pi}}{8} \zeta\left(\frac{5}{2}\right)$$

$$\boxed{C_V^M = \sqrt{\pi}\, k_B \frac{15 \alpha}{32\pi^2} \zeta\left(\frac{5}{2}\right)\left(\frac{k_B T}{\alpha}\right)^{3/2} \frac{1}{\cancel{\alpha}}}$$

As $\alpha = SJa^2$,

$$C_V^M = k_B \sqrt{\pi} \frac{15\, \zeta(5/2)}{32\pi^2} \left(\frac{k_B T}{SJ}\right)^{3/2} \frac{1}{a^3}$$

and using $n = 2/a^3$ we finally get

$$C_V^M = \sqrt{\pi} \frac{15\, \zeta\left(\frac{5}{2}\right)}{64\pi^2} n k_B \left(\frac{k_B T}{SJ}\right)^{3/2}$$

4. $C_V^L = \frac{12\pi^4}{5} n k_B \left(\frac{T k_B}{k\Theta}\right)^3$

clearly at small temperatures C_V^M dominates as $C_V^M / C_V^L \sim T^{-3/2}$, $T \to 0$.

5. $C_V^L(T^*) = C_V^M(T^*)$ yields

$$1 = \frac{\frac{15\,\zeta(5/2)}{64\cdot\pi^{3/2}}\left(\frac{k_B T^*}{SJ}\right)^{3/2}}{\frac{12\pi^4}{5}\frac{(k_B T^*)^3}{(k_B\Theta)^3}} = \frac{25\cdot\zeta(\frac{5}{2})}{256\cdot\pi^{11/2}}\left(\frac{k_B\Theta}{SJ}\right)^{3/2}\left(\frac{\Theta}{T^*}\right)^{3/2}$$

$$T^* = \Theta\,\frac{k_B\Theta}{SJ}\left(\frac{25\,\zeta(\frac{5}{2})}{256\,\pi^{11/2}}\right)^{2/3}$$

6. $$T^* = 3.88\times10^{-3}\;\Theta\,\frac{k_B\Theta}{SJ}$$

$$= 3.88\times10^{-3}\,(400)^2\,\frac{8.62\cdot10^{-5}\cdot2}{0.1}$$

$$= 1.07\ \mathrm{K}$$

Esercizio 2

1. $0 \le z \le a$

$$\nabla^2 \bar{B}(z) = \bar{B}''(z) = \frac{\bar{B}(z)}{\lambda^2}$$

$$\Rightarrow \bar{B}(z) = \bar{b}_1 e^{-z/\lambda} + \bar{b}_2 e^{+z/\lambda}$$

2. Define $\bar{B}_\perp = B_z \hat{z}$ $\bar{B}_\parallel = B_y \hat{y} + B_x \hat{x}$

$\nabla \cdot \bar{B} = 0$ yields the continuity of $\bar{B}_\perp$ at $z=0, a$.

$\nabla \times \bar{B} = \frac{4\pi}{c} \vec{J}$ yields continuity of $\bar{B}_\parallel$ at $z=0, a$, provided there are no surface currents at $z=0, a$, i.e. currents of zero thickness!

3. At $z=0$

$$\bar{b}_1 + \bar{b}_2 = \bar{B}_1 = B_1 \hat{y}$$

and at $z=a$

$$\bar{b}_1 e^{-a/\lambda} + \bar{b}_2 e^{+a/\lambda} = \bar{B}_2 = B_2 \hat{y}$$

Let's consider the x component of the above equations

$$\begin{cases} b_{1x} + b_{2x} = 0 \\ b_{1x} e^{-a/\lambda} + b_{2x} e^{a/\lambda} = 0 \end{cases}$$

$$\Rightarrow \left.\begin{array}{l} -b_{1x} = b_{2x} \\ b_{1x}(e^{-a/\lambda} - e^{a/\lambda}) = 0 \end{array}\right\} \Rightarrow b_{1x} = b_{2x} = 0$$

Similarly for the z components.

Let's consider now the y components

$$\begin{cases} b_{1y} + b_{2y} = B_1 \\ b_{1y} e^{-a/\lambda} + b_{2y} e^{a/\lambda} = B_2 \end{cases}$$

One gets

$$b_{1y} = \frac{-B_2 + B_1 e^{+a/\lambda}}{2 \sinh \frac{a}{\lambda}}$$

$$b_{2y} = \frac{-B_1 e^{-a/\lambda} + B_2}{2 \sinh \frac{a}{\lambda}}$$

$$B_y(z) = \frac{1}{2\sinh\frac{a}{\Lambda}}\Big\{ B_1 e^{-(z-a)/\Lambda} - B_2 e^{-z/\Lambda}$$
$$+ B_2 e^{z/\Lambda} - B_1 e^{(z-a)/\Lambda} \Big\}$$

$$B_y(z) = \frac{1}{\sinh\frac{a}{\Lambda}}\Big\{ B_2 \sinh\frac{z}{\Lambda} + B_1 \sinh\frac{a-z}{\Lambda} \Big\}$$

(4) We know from Maxwell eqs that

$$\bar{J} = \frac{c}{4\pi}\nabla\times\bar{B}$$

$$J_y = \frac{c}{4\pi}(\partial_z B_x - \partial_x B_z) = 0$$

$$J_z = \frac{c}{4\pi}(\partial_x B_y - \partial_y B_x) = 0$$

$$J_x = \frac{c}{4\pi}(\partial_y B_z - \partial_z B_y)$$
$$= -\frac{c}{4\pi}\frac{1}{\Lambda}\frac{1}{\sinh\frac{a}{\Lambda}}\Big\{ -B_1 \cosh\frac{(a-z)}{\Lambda} + B_2 \cosh\frac{z}{\Lambda} \Big\}$$

$$\bar{J} = \hat{y}\,\frac{c}{4\pi\Lambda\sinh\frac{a}{\Lambda}}\Big\{ B_1 \cosh\frac{a-z}{\Lambda} - B_2 \cosh\frac{z}{\Lambda} \Big\}$$

(5) We put $\bar{B}_1 = \bar{B}_2 = B_0$, so

$$\bar{B}(z) = \hat{y} B_0 \frac{\sinh\frac{z}{\Lambda} + \sinh\frac{a-z}{\Lambda}}{\sinh\frac{a}{\Lambda}}$$

Inside the superconductor $\bar{B}_0 + 4\pi\bar{M} = \bar{B}(z)$

$$\bar{M} = -\frac{1}{4\pi}\left[\bar{B}_0 - \bar{B}(z)\right]$$

$$\bar{M} = -\hat{y}\frac{B_0}{4\pi}\left[1 - \frac{\sinh\frac{z}{\Lambda} + \sinh\frac{a-z}{\Lambda}}{\sinh\frac{a}{\Lambda}}\right]$$

(6)

$$\langle\bar{M}\rangle = -\hat{y}\frac{B_0}{4\pi}\left[1 - \frac{\Lambda\left\{\cosh\frac{a}{\Lambda} - 1 - \cosh\frac{a-z}{\Lambda}\Big|_0^a\right\}}{a\sinh\frac{a}{\Lambda}}\right]$$

$$\langle\bar{M}\rangle = -\hat{y}\frac{B_0}{4\pi}\left[1 - \frac{\Lambda}{a}\frac{2\cosh\frac{a}{\Lambda} - 2}{\sinh\frac{a}{\Lambda}}\right]$$

$$\chi = \frac{\partial\langle M\rangle}{\partial B_0} = -\frac{1}{4\pi}\left[1-\frac{2\Lambda}{a}\,\frac{\cosh\frac{a}{\Lambda}-1}{\sinh\frac{a}{\Lambda}}\right]$$

(i) $\frac{a}{\Lambda} \gg 1$ $\quad \frac{\cosh\frac{a}{\Lambda}-1}{\sinh\frac{a}{\Lambda}} \simeq 1$

$$\chi \simeq -\frac{1}{4\pi}\left[1-\frac{2\Lambda}{a}\right] \simeq -\frac{1}{4\pi}$$

(ii) $\frac{a}{\Lambda} \ll 1$ $\quad \frac{\cosh(\frac{a}{\Lambda})-1}{\sinh\frac{a}{\Lambda}} \simeq \frac{\frac{1}{2}\left(\frac{a}{\Lambda}\right)^2}{\frac{a}{\Lambda}} = \frac{a}{2\Lambda}$

to next order $\quad \frac{\cosh\frac{a}{\Lambda}-1}{\sinh\frac{a}{\Lambda}} \simeq \frac{\frac{1}{2}\left(\frac{a}{\Lambda}\right)^2+\frac{1}{24}\left(\frac{a}{\Lambda}\right)^4}{\frac{a}{\Lambda}+\frac{1}{6}\left(\frac{a}{\Lambda}\right)^3}$

$$= \frac{1}{2}\frac{a}{\Lambda}\,\frac{1+\left(\frac{a}{\Lambda}\right)^2\frac{1}{12}}{1+\frac{1}{6}\left(\frac{a}{\Lambda}\right)^2} \simeq \frac{1}{2}\frac{a}{\Lambda}\left[1+\left(\frac{a}{\Lambda}\right)^2\left(\frac{1}{12}-\frac{1}{6}\right)\right]$$

$$= \frac{a}{2\Lambda}\left[1-\frac{1}{12}\left(\frac{a}{\Lambda}\right)^2\right]$$

$$\chi \simeq -\frac{1}{4\pi}\left[1-1+\frac{1}{12}\left(\frac{a}{\Lambda}\right)^2\right] = -\frac{1}{48\pi}\left(\frac{a}{\Lambda}\right)^2$$