Condensed Matter Physics II. – A.A. 2016-2017, June 7, 2017

(*time 3 hours*)

Solve the following two exercises.

NOTE:

- Give all details which help in understanding the proposed solution. Answers which only contain the final result or not enough detail will be judged insufficient and discarded;
- If you are requested to give evaluation/estimates, do so using 3 significant figures.

Exercise 1: *Heisenberg ferromagnet on a square lattice with nearest neighbors coupling*

Consider S=1/2 spins on a square lattice (2D) with a ferromagnetic nearest neighbors coupling J. The lattice has lattice parameter a and a non vanishing magnetic field H is present.

1. Write down the expression of the excitation energy $\epsilon(H, \mathbf{k})$ of spin waves.
2. As we are interested with low temperature T, consider to leading order in k the expansion of $\epsilon(H, \mathbf{k})$ for small k. You should get to leading order in k an expression like $\epsilon(H, \mathbf{k}) \simeq g\mu_B H + c \times k^2$. Explicitely calculate the coefficient c.
3. Implement a *Debye* like approximation in which you replace $\epsilon(H, \mathbf{k})$ with the expression valid for small k at all $k < k_D$, where the disk of radious k_D contains exactly N values of k and N is the number of sites in the 2D crystal. Calculate the energy density of state for the spin waves in such an approximation. Define the *Debye* temperature from $\epsilon_D = K_B T_D = \epsilon(H = 0, k_D)$.
4. Specialize eq. (33.30) of the textbook to the present 2D case to obtain $M(H, T)$. You just need to replace the volume V with the surface A, and the excitation energy $\epsilon(\mathbf{k})$ with $\epsilon(H, \mathbf{k})$ found above. You can then replace the k-integral with an energy integral, employing the energy density of states found for the spin waves.
5. Explicitely calculate $M(H, T)$ by performing the energy integral found above. The necessary integral is an elementary one. Recall $\int dx(e^x - 1)^{-1} = ln(1 - e^{-x})$.
6. Choosing $4\pi J = 1eV$, estimate the *Debye* temperature T_D and show that at $0 < T \ll T_D$ when you decrease H there is a small but finite $H(T)$ at which $M(H, T)$ vanishes, implying that in 2D there is no spontaneous magnetization at any finite temperature.

Exercise 2: *Cooper pai in p-wave*

Consider the addition of 2 electrons to a normal metal (a full Fermi sea, i.e., a system of non interacting electrons filling a sphereof radious k_F in k-space .

The added electrons attract each other and one can treat the problem of the extra electrons as obeying the equation:

$$\left[-\frac{\hbar^2\nabla_1^2}{2m} - \frac{\hbar^2\nabla_2^2}{2m} + U(\mathbf{r_1} - \mathbf{r_2}) - E\right]\Psi(\mathbf{r_1}, \mathbf{r_2}) = 0. \tag{1}$$

1. Rewrite eq. (1) for the ground state. Assume that the ground state has the center of mass at rest and expand $\Psi(\mathbf{r_1}, \mathbf{r_2}) = \phi_0(\mathbf{r})$, with $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, in plane waves outside the Fermi sphere, i.e., $\phi_0(\mathbf{r}) = \sum_{\mathbf{k}} g(\mathbf{k}) exp(i\mathbf{k} \cdot \mathbf{r})$.
2. Assume that the matrix element between normalized plave waves $\langle \mathbf{q}|U|\mathbf{k}\rangle$ is non vanishing with value $\langle \mathbf{q}|U|\mathbf{k}\rangle = -(U_1/V)\hat{q} \cdot \hat{k}$, only for plane waves satisfying $\epsilon_F \le \epsilon^{(0)}(q), \epsilon^0(k) \le \epsilon_F + \hbar\omega_D$, and ω_D is a typical Debye frequency and $U_1 > 0$.
3. Say what is the symmetry of $g(\mathbf{k})$ for singlet and triplet.
4. What is the minimum energy for the singlet?
5. For the triplet use the addition formulae reported below expanding $g(\mathbf{k})$ in spherical harmonics, $g(\mathbf{k}) = \sum_l \sum_{m=-l}^{l} g_{lm}(k) Y_{lm}(\hat{k})$, and obtain which $g_{lm}(k)$ are non-zero and the equation they satisfy.
6. Solve the equation obtained above for the binding energy by rewriting it in terms of the density of state of the non-interacting Fermi sea, and exploiting the fact that $\hbar\omega_D \ll \epsilon_F$.

NOTA:

- If you are requested to give evaluation/estimates, do so using 3 significant figures.
- The addition formula provides the cosine of the angle between 2 given directions in terms of spherical harmonics as follows:

  $$\hat{q} \cdot \hat{k} = \frac{4\pi}{3} \sum_{m=-1}^{1} Y_{1,m}^*(\hat{q}) Y_{1,m}(\hat{k}) = \frac{4\pi}{3} \sum_{m=-1}^{1} Y_{1,m}^*(\hat{k}) Y_{1,m}(\hat{q}).$$

  Notice that the m sum is from -1 to 1!

  Also, recall that $Y_{lm}(\hat{k}) = Y_{lm}(\theta, \phi)$ form an orthonormal set of functions with respect to integration in the 3D solid angle, where θ and ϕ are respectively the polar and azimuthal angles. You may conveniently use $Y_{lm}(\hat{q}) = Y_{lm}(\theta', \phi')$.