COMPITO DEL 18 12 15 ESERCIZIO 1

① $g(E) = \frac{2}{L} \sum_p \delta(E - \alpha|p|)$, $-\infty < p < \infty$

$$g(E) = \frac{2}{L} \int_{-\infty}^{\infty} \frac{dp}{\frac{h}{L}} \delta(E - \alpha|p|) = \frac{2}{h} \, 2 \int_0^{\infty} d\varepsilon \, \delta(E - \alpha p) \qquad \boxed{t = \alpha p}$$

$$= \frac{4}{h\alpha} \int_0^{\infty} dt \, \delta(E - t) = \frac{4}{h\alpha} \theta(E)$$

② $$\rho = \int d\varepsilon \, g(E) \frac{1}{\frac{e^{\beta E}}{z} + 1} = \frac{4}{h\alpha} \int_0^{\infty} d\varepsilon \frac{1}{\frac{e^{\beta E}}{z} + 1} = \frac{4 k_B T}{h\alpha} \int_0^{\infty} dy \frac{z e^{-y}}{1 + z e^{-y}}$$

$$= \frac{1}{\ell} \int_0^{\infty} dy \, \frac{\partial}{\partial y} \left[- \ln(1 + z e^{-y}) \right] = \frac{1}{\ell} \ln(1 + z), \qquad \ell = \frac{h\alpha}{4 k_B T}$$

$$\boxed{\rho = \frac{1}{\ell} \ln(1 + z)}$$

③ $$z = e^{\rho \ell} - 1 = e^{\beta\mu}$$

$$\beta\mu = \ln(e^{\rho\ell} - 1)$$

$$\mu = k_B T \ln\left[e^{\rho\ell}(1 - e^{-\rho\ell})\right] = k_B T \rho \ell + k_B T \ln(1 - e^{-\rho\ell})$$

$$= \rho \frac{h\alpha}{4} + k_B T \ln\left(1 - e^{-\rho \frac{h\alpha}{4 k_B T}}\right)$$

$$\lim_{T \to 0} \mu(T, \rho) = \rho \frac{h\alpha}{4} \equiv \mu_0(\rho)$$

(4) $$N = 2\sum_{|p|<p_F} = 2\int_{-p_F}^{p_F}\frac{dp}{\frac{h}{L}} = \frac{2L}{h}\,2\int_0^{p_F} dp = \frac{4L}{h}\,p_F$$

$$p_F = \frac{N}{L}\frac{h}{4} \qquad \epsilon_F = \rho\frac{h}{4}\alpha$$

$$\boxed{\mu_0(\rho) = \epsilon_F(\rho)}$$

ESERCIZIO 2

① $$g(E) = \frac{1}{V}\sum_{\bar{p}} \delta\left(E - \frac{p^2}{2m}\right) = \frac{1}{V}\int \frac{dp}{\left(\frac{h}{L}\right)^5}\,\delta\left(E - \frac{p^2}{2m}\right)$$

$$= \frac{1}{h^5}\int d\Omega \int_0^\infty dp\, p^4\, \delta\left(E - \frac{p^2}{2m}\right) = \frac{\Omega_5}{h^5}(2m)^{5/2}\int_0^\infty d\sqrt{s}\, s^2\, \delta(E-s)$$

$$= \frac{8\pi^2}{3h^5}(2m)^{5/2}\int_0^\infty ds\, \frac{s^{3/2}}{2}\,\delta(E-s) = \frac{4\pi^2}{3h^5}(2m)^{5/2}E^{3/2}\theta(E) = CE^{3/2}\theta(E)$$

② $$\rho = \frac{\langle N\rangle}{V} = \frac{1}{V}\frac{z}{1-z} + \int_0^\infty dE\, C E^{3/2}\frac{1}{\frac{e^{\beta E}}{z} - 1} \equiv \rho_0 + \rho_1$$

$$\rho_1 = C(k_B T)^{\frac{5}{2}}\int_0^\infty dy\, \frac{y^{3/2}}{\frac{e^y}{z}-1} = C(k_B T)^{\frac{5}{2}}\int_0^\infty dy\, y^{3/2}\frac{z e^{-y}}{1-ze^{-y}} =$$

$$C(k_B T)^{\frac{5}{2}}\sum_{n=0}^\infty (-1)^n z^{n+1}\int_0^\infty dy\, y^{3/2} e^{-(n+1)y} = C(k_B T)^{\frac{5}{2}}\sum_{n=1}^\infty (-)^{n-1} z^n \int_0^\infty dy\, y^{3/2} e^{-ny}$$

$$= C(k_B T)^{5/2}\sum_{n=1}^\infty (-)^{n-1}\frac{z^n}{n^{5/2}}\int_0^\infty ds\, s^{3/2} e^{-s} = (k_B T)^{\frac{5}{2}} C\, \Gamma\left(\frac{5}{2}\right)\sum_{n=1}^\infty (-1)^{n-1}\frac{z^n}{n^{5/2}}$$

$$= \frac{4\pi^2}{3h^5}(2m)^{5/2}(k_B T)^{\frac{5}{2}}\frac{3}{2}\frac{1}{2}\sqrt{\pi}\, g_{5/2}(z) = \frac{1}{\lambda^5} g_{5/2}(z)$$

$$\rho = \frac{1}{V}\frac{z}{1-z} + \frac{1}{\lambda^5} g_{5/2}(z)$$

· Nota $g_{5/2}(1) < \infty$, quindi c'è condensazione

③ $z\frac{\partial}{\partial z}\langle N\rangle = \frac{z}{1-z} + \frac{z^2}{(1-z)^2} + \frac{V}{\lambda^3} g_{3/2}(z) = \langle N^2\rangle - \langle N\rangle^2$

$$\rho > \rho_c(T) = \frac{g_{5/2}(1)}{\lambda^3(T)}$$

④ $V \to \infty \quad z \simeq 1 - \frac{1}{N_0}$

$$\frac{z}{1-z} \simeq \frac{1-\frac{1}{N_0}}{1-1+\frac{1}{N_0}} = N_0 - 1 \simeq N_0 , \quad \langle N\rangle \simeq N_0 + \frac{V}{\lambda^3} g_{3/2}(1) = N_0 + V\rho_c = N_0 + N_c$$

$$\langle N^2\rangle - \langle N\rangle^2 \simeq -N_0 + N_0^2 + \frac{V}{\lambda^3} g_{3/2}(1)$$

$$\frac{\langle N^2\rangle - \langle N\rangle^2}{\langle N\rangle^2} = \frac{N_0^2 - N_0 + \frac{V}{\lambda^3} g_{3/2}(1)}{\left(N_0 + \frac{V}{\lambda^3} g_{3/2}(1)\right)^2} \simeq \frac{N_0^2}{\langle N\rangle^2} = \frac{\rho_0^2}{\rho^2}$$

$$\sqrt{\frac{\langle N^2\rangle - \langle N\rangle^2}{\langle N\rangle^2}} \simeq \frac{\rho_0}{\rho} \sim 1$$

Le fluttuazioni non sono soppresse, sono macroscopiche!