Etica & Politica / Ethics & Politics, XI, 2009, 1, pp. 369-384

Meaning, Use, and Diagrams

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ABSTRACT
My starting point is two themes from Peirce: his familiar pragmatist conception of meaning focused on what follows from an application of a term rather than on what is the case if it is correctly applied, and his less familiar and rather startling claim that even purely deductive, logical reasoning is not merely formal but instead constructive or diagrammatic — and hence experimental, and fallible. My aim is to show, using Frege's two-dimensional logical language as a paradigm of a "constructive" logic in Peirce's sense, that taking this second theme into account in one's interpretation of the first yield a very different, and arguably more fruitful, conception of meaning than is usually ascribed to Peirce, not only a different conception of the role of inference in meaning than is found in, say, Brandom following Sellars, but also a very different understanding of the role of pragmatics in semantics than is standard in social practice theories.

Meaning, the pragmatist tells us, lies in use. This slogan has, however, come to have many uses in philosophy, and as a result has come to mean different things in different contexts. My interest here is in its use, and meaning, for Peirce. The aim is to clarify Peirce's idea that the content of a claim lies not in its truth conditions but instead in what follows if it is true, in its consequences, and to do so by developing a suggestion Peirce makes in the sixth of his Harvard lectures (delivered on 7 May 1903): that we should accept Kant's dictum that necessary reasoning only explicates the meaning of one's premises but "[reverse] the use to be made of it". Instead of starting with a conception of meaning on the basis of which to understand necessary reasoning, as Kant does, we are to start with actual instances of necessary reasoning in mathematics and "use the dictum that necessary reasoning only explicates the meanings of the terms of the premises to fix our ideas as to what we shall understand by the *meaning* of a term".[1] It is actual mathematical practice, in particular the necessary reasoning it involves, that is to teach us how to think about meaning. Because, on Peirce's view, necessary reasoning is essentially diagrammatic, an adequate understanding of his pragmatist conception of

[1] Charles Sanders Peirce, *The Essential Peirce*, vol. 2, ed. The Peirce Edition Project (Bloomington and Indianapolis: Indiana University Press, 1998), pp. 218-219.

meaning requires an understanding of diagrammatic reasoning. It is our use of diagrams in mathematics that is to teach us what we should understand by *meaning*—at least in mathematics.[2]

Peirce aims to achieve an understanding of meaning through reflection on the actual practice of mathematics. What he finds, we know, is that meaning should be understood in terms of consequences. This puts him in the company of Frege, as we will see, and at odds both with (say) Brandom, who espouses a form of inferentialism that embraces both circumstances and consequences, and McDowell, who adopts instead a Davidsonian theory of meaning in terms of truth—though it should be noted that neither Brandom nor McDowell is concerned with meaning in mathematics in particular. Their focus, at least for the most part, is our everyday, conversational uses of language.

But there are other differences as well between Peirce (and Frege), and Brandom, and McDowell, differences that reflect their very different conceptions of what our slogan that meaning lies in use itself means. Whereas for Brandom, we will see, the slogan provides the basis for the substantive theory construction Brandom undertakes in *Making It Explicit*, according to McDowell, it serves, more radically, to remind us that the sort of theory Brandom is after is profoundly misguided.[3] As McDowell understands it, the insight that is captured in the thought that meaning lies in use entails that the only task for the philosopher is a quietist diagnosis and cure. Peirce is no quietist. He wants to advance our philosophical understanding, and he thinks he can do that by reflecting on our mathematical practice. Unlike Brandom, however, he does so from within the practice itself. As the point might be put, Peirce's conception of the priority of use is modest, in McDowell's sense (to be explicated), without being quietistic.

Brandom describes *Making It Explicit* as "an investigation into the nature of *language*": "the aim is to offer sufficient conditions for a system of social practices to count as specifically *linguistic* practices".[4] The investigation begins with Kant's distinction between behavior in accordance with a rule (in the realm of nature, of causes) and action according to a conception of a rule (in

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[2] I have argued elsewhere that Peirce's conception of meaning in terms of inferential consequences, although apt for the case of mathematical concepts, should not be extended to the case of the everyday concepts of natural language. See my "Pragmatism and Objective Truth" in *New Pragmatists*, ed. Cheryl Misak (Oxford: Clarendon Press, 2007).

[3] See, for instance, John McDowell, "How not to read *Philosophical Investigations*: Brandom's Wittgenstein", in R. Haller and K. Puhl, eds., *Wittgenstein and the Future of Philosophy: A Reassessment after 50 Years* (Vienna: Holder, Pichler, Tempsky, 2002).

[4] Robert B. Brandom, *Making It Explicit: Reasoning, Representing, and Discursive Commitment* (Cambridge, Mass.: Harvard University Press, 1994), pp. xi and 7.

the realm of freedom, reasons). But because action according to a conception of a rule presupposes rather than explains both one's understanding of the rule and one's awareness of one's circumstances as circumstances in which the rule is correctly applied, Brandom also distinguishes, within the realm of freedom, between action that is explicitly undertaken according to a conception of a rule and action that is only implicitly so undertaken in practice. Thus for him, "there are three levels at which performances can be discussed: a level of norms explicit in rules and reasons, a level of norms implicit in practice, and a level of matter-of-factual regularities, individual and communal."[5] The fundamental, guiding thought of *Making It Explicit* is thatif anything is to be made of the Kantian insight that there is a fundamental normative dimension to the application of concepts (and hence to the significance of discursive or propositionally contentful intentional states and performances), an account is needed of what it is for norms to be implicit in practices. Such practices must be construed both as not having to involve explicit rules and as distinct from mere regularities.[6]

Brandom argues that explicit rules of linguistic usage are intelligible only on the basis of rules implicit in practice that govern the use of explicit rules. The task of the theorist is to make explicit the form such a practice might take, a set of sufficient conditions for practices to count as linguistic, and so for the words that are uttered to count as meaning this or that. The task is to *say* in a way that does not presuppose any grasp of the nature of meaningful utterances—that is, as if from the outside, or side-ways on—what something would have to be able to *do* in order to count as saying. Brandom thinks, in other words, that a theory of meaning must be robust or full-blooded in McDowell's sense:
"if a theory of meaning is full-blooded with respect to a given concept, that means that it describes a practical capacity such that to acquire it would be to acquire the concept—and it effects this description not necessarily without employing the concept, but . . . 'as from outside' its role as a determinant of content . . . one can acquire the concept by being taught the practical capacity that the theory describes; and the theory describes that practical capacity without helping itself to the notion of contents in which the concept figures."[7]

But, McDowell argues, such a full-blooded or robust theory is not to be had; there is and can be no perspective outside of our practices from which to

[5] *Making It Explicit*, p. 46.
[6] *Making It Explicit*, p. 29
[7] John McDowell, "In Defense of Modesty", reprinted in *Meaning, Knowledge, and Reality* (Cambridge, Mass.: Harvard University Press, 1998), 91-92.

explicate them, no "side-ways on" view from which to make explicit that which (according to Brandom) is implicit in our linguistic practice. To say that meaning lies in use just is to say, according to McDowell, that meaning lies open to view in use. Nothing is hidden. And because nothing is hidden there is nothing in need of explication (save for the fact that we think that something is hidden, and so in need of explication): "the outward aspect of linguistic behavior is essentially content involving, so that the mind's role in speech is, as it were, on the surface—part of what one presents to others, not something that is at best a hypothesis for them".[8] If McDowell is right, the only account of meaning that is possible is a modest theory, where "a modest theory of meaning, by design, starts in the midst of content".[9] Davidson's conception of a theory of meaning in terms of truth conditions is just such a theory insofar as it sets out the truth conditions of sentences in terms that are antecedently understood.

Adherents of modest theories of meaning tend to be quietists. They tend to think that once one sees that a theory of meaning must be modest, that there is no sideways-on perspective from which to theorize in the manner of Brandom, one will see that there is no task for philosophy but the therapeutic one of uncovering the confusions that led us to think that there was a philosophical issue in need of resolution or explication in the first place. Peirce, I have suggested, is neither immodest nor quietist. On his view, we are to look to use, not sideways on, as if from outside meaning and content, but precisely as contentful. The task is to gain insight into the nature of meaning (at least for the case of mathematics), to say something substantive about what meaning is, by reflecting on use, actual mathematical practice. It is to this task that we now turn.

According to the received view, due to Russell, the modern symbolic logic of relations enables us to *dispense* with diagrams in our reasoning.[10] The logic of relations is not, or at least is not supposed to be, just more of the same. Peirce came to think that this is a mistake. [11] According to him, "all necessary

[8] "In Defense of Modesty", p. 100.

[9] "In Defense of Modesty", p. 104.

[10] See Bertrand Russell, *Principles of Mathematics* (London: George Allen and Unwin, 1903), especially §§4 and 434.

[11] Even Peirce himself at first thought that reasoning in logic is essentially symbolic rather than diagrammatic. Only after careful study did he come to hold that this is wrong, and so to think that his own "analyses of reasoning surpass in thoroughness all that has ever been done in print, whether in words or in symbols,—all that De Morgan, Dedekind, Schröder, Peano, Russell, and others have ever done,—to such a degree as to remind one of the difference between a pencil sketch of a scene and a photograph of it" (EP ii 206).

reasoning is diagrammatic; and the assurance furnished by all other reasoning must be based upon necessary reasoning. In this sense, all reasoning depends directly or indirectly upon diagrams".[12] Indeed, Peirce came to hold, the logic of relations itself reveals the diagrammatic character of all reasoning: "all reasoning of the degree of intricacy of elementary geometry . . . involves the logic of relations, so that without that logic the gist of reasoning cannot be stated . . . That feature, obtrusive enough in reasoning about relations, is that in all reasoning there must be something amounting to a diagram before the mind's eye, and that the act of inference consists in *observing* a relation between parts of that diagram that had not entered into the design of its construction."[13]

Mathematical reasoning is, for Peirce, paradigmatic of necessary reasoning, but reasoning with diagrams is constitutive of mathematical reasoning; so according to Peirce, "Demonstration or Deductive Argumentation is best learned from Book I of Euclid's *Elements*".[14] To understand the nature of deductive reasoning, we must understand Euclid's reasoning in the *Elements* — say, in the very first proposition.

In the first proposition of Book I of the *Elements*, we are taught to construct an equilateral triangle on a given finite straight line. The demonstration begins with a setting out: let AB be the given straight line. A statement of what is to be done follows: to construct an equilateral triangle on AB. Then the construction is given:

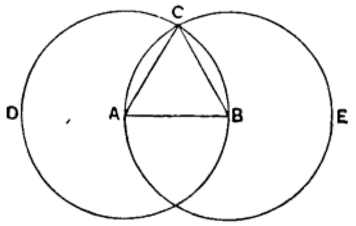
[C1.] With center A and distance AB let the circle BCD be described [licensed by the third postulate].

[C2.] With center B and distance BA let the circle ACE be described [again, by the third postulate].

[C3.] From point C, in which the circles cut one another, to the points A, B let the straight lines CA, CB be joined [licensed by the first postulate on the assumption that there is such a point C].

[12] Charles Sanders Peirce, *The New Elements of Mathematics*, Vol. 4, Mathematical Philosophy, ed. Carolyn Eisele (Atlantic Highlands N.J.: Humanities Press, 1976), p. 314.
[13] *New Elements*, vol. 4, p. 353. See also Charles Sanders Peirce, *Collected Papers of Charles Sanders Peirce*, vol. 3, ed. C. Hartshorne, P. Weiss, and A. Burke (Cambridge, Mass.: Harvard University Press, 1931), p. 350.
[14] *Essential Peirce*, vol. 2, p. 441.

The diagram that results is this.[15]



The *apodeixis*, which Peirce describes as "[tracing] out the reasons why a certain relation must always subsist between the parts of the diagram,"[16] then follows:

[A1.] Given that A is the center of circle CDB, AC is equal to AB licensed by the definition of a circle].

[A2.] Given that B is the center of circle CAE, BC is equal to BA [again by the definition of a circle].

[A3.] Given that AC equals AB and BC equals BA, we can infer that AC equals BC because what are equal to the same are equal to each other [Common Notion 1].

[A4.] Given that AB, BC, and AC are equal to one another, the triangle ABC is equilateral [by the definition of equilateral triangle, on the assumption that there is such a triangle ABC].

This triangle was constructed on the given finite straight line AB as required, and so we are done.

The first thing to note about this little demonstration, at least as Peirce understands it, is that the reasoning is general throughout.[17] A drawing that one makes (for instance, of a line or circle) is not an *instance* of the relevant geometrical figure but instead functions as a Peircean icon exhibiting the content of the concept of some geometrical figure. Because geometrical figures such as circles, lines, and squares are defined by the relations of their parts, and these relations can be iconically represented in a diagram, one can exhibit in a drawing those contents themselves. The drawing of (say) a circle, in the context of a Euclidian demonstration, looks circular not because it is an instance of a circle but because it iconically represents *what it is to be* a circle, namely a plane figure all points on the circumference of which are equidistant

[15] This image is taken from Thomas L. Heath's edition of *The Thirteen Books of the Elements*, 3 vols. (Toronto, Ont.: Dover, first ed. 1908, second ed. 1956), vol. 1, p. 241.

[16] *Essential Peirce*, vol. 2, p. 303.

[17] This point is argued at much greater length in my "Diagrammatic Reasoning in Euclid's *Elements*".

from a center. The drawing, in other words, denotes a circle though no circle in particular: it is "what is called a General sign; that is, it denotes a General object".[18] As we would say, it denotes a concept, but it does so not symbolically, by signifying the concept using a merely conventional symbol, but instead iconically, through a homomorphism between the spatial relations among the parts of the drawing, on the one hand, and the constitutive relations among the parts of circles, on the other.

Now, according to Peirce, "diagrams . . . show, — as literally *show* as a Percept shows the Perceptual Judgment to be true, — that a consequence does follow, and more marvelously yet, that it *would* follow under all varieties of circumstance accompanying the premises".[19] A diagram, on his account, shows something that is not only true but also necessary. Indeed, Peirce rejects the sort of symbolic account that logicians give of, say, syllogistic reasoning precisely because conceived symbolically, rather than iconically, a syllogism "would fail to furnish Evidence" where this evidence "consists in the fact that the truth of the conclusion is *perceived*, in all its generality, and in the generality the how and why of the truth is perceived".[20] Having drawn a particular diagram according to certain specifications, it is then possible to "[trace] out the reasons why a certain relation must always subsist between the parts of the diagram".[21] But how exactly is this to work? In particular, how does one move, say, in proposition I.1, from a claim about radii of circles to a claim about a triangle?

It is a familiar, distinctive, and even notorious feature of reasoning with a Euclidean diagram that one can read off of a diagram more than was put into it, for example, the existence of a point at the cut of two lines. Peirce takes this reading off as itself an inference: "the act of inference consists in *observing* a relation between parts of that diagram that had not entered into the design of its construction".[22] Now in a sense we do this already in using, say, Euler diagrams. If I draw a circle, then another inside it, and a third inside that second circle, I can then observe a relation between the first circle and the third circle that had not entered into the design of my construction. In Euclid the idea is much more radical: in the course of a Euclidean demonstration not only new relations between given parts but wholly new geometrical entities emerge. As we will see in more detail below, the parts of a Euclidean diagram

[18] *New Elements*, vol. 4, p. 315, n. 1.

[19] *New Elements*, vol. 4, p. 318.

[20] *New Elements*, vol. 4, p. 317.

[21] *Essential Peirce*, vol. 2, p. 303.

[22] *New Elements*, vol. 4, p. 353

are not merely parts of the diagram taken as a whole; they are parts of the icons of geometrical entities discernable *within* the diagram, parts that can be combined and recombined to form new wholes, that is, icons of new geometrical entities. This possibility, which has no analogue in Euler diagrams, is critical to an adequate understanding of the process of reasoning in Euclidean demonstrations with diagrams.

In the diagram of proposition I.1 one sees certain lines now as icons of radii, as required to determine that they are equal in length, and now as icons of sides of a triangle, as required in order to draw the conclusion that one has drawn an equilateral triangle on a given straight line. The cogency of the reasoning clearly requires both perspectives. But if it does then the Euclidean diagram, that is, the array of drawn lines, functions in a way that is very different from the way (say) an Euler diagram (more exactly, the array of lines making up such a diagram) functions. By contrast with an Euler diagram, the various collections of lines and points in a Euclidean diagram are icons of, say, circles, or other particular sorts of geometrical figures, *only when viewed a certain way*, only when, as Kant would think of it, the manifold display (or a portion of it) is synthesized under some particular concept, say, that of a circle, or of a triangle.[23] The Euclidean drawing, as certain marks on the page, has (intrinsically) the *potential* to be regarded in radically different ways (each of which is fully determinate albeit general). It is just this potential that is actualized in the course of reasoning, as one sees lines now as radii and now as sides of a triangle.

I have suggested that the drawn points and lines discernable in a Euclidean diagram function as icons of geometrical figures only relative to a way of regarding those drawn points and lines. What is from one perspective, say, a radius of a circle is from another a side of a triangle. It follows directly that Euclidean diagrams involve three distinct levels of articulation. At the lowest level are the signs for the primitive parts, the points, lines, angles, and areas out of which geometrical figures are constructed. At the second level are constructions, out of these signs, of icons for the (concepts of) geometrical objects themselves, the objects that form the subject matter of geometry, all of which are wholes of those primitive parts. At this level we find points as endpoints of lines, as points of intersection of lines, and as centers of circles; we

[23] Significantly, the thought that a system of signs may function not by way of combinations of primitive signs that have designation independent of any context of use, but instead by appeal to primitive signs that designate only given a context of use, is developed also in Sun-Joo Shin's reading of Peirce's notation for his alpha logic in *The Iconic Logic of Peirce's Graphs* (Cambridge, Mass, and London: Bradford Books, MIT Press, 2002).

find angles of various sorts that are limited by lines that are also parts of those angles; and we find figures of various sorts. At the third level, finally, is the whole diagram, which is not itself an icon of any geometrical figure but within which can be discerned various (icons for) second-level objects depending on how one configures various collections of drawn lines within the diagram. It is these three levels of articulation, which enable a variety of different analyses or carvings of the various parts of the diagram, that account for the ways in which one can radically reconfigure parts of intermediate wholes into new (intermediate) wholes in the course of a Euclidean demonstration, and thereby demonstrate significant and often surprising geometrical truths.

In Euclid's first proposition, a line that is at first taken iconically to represent a radius of a circle is later taken iconically to represent a side of a triangle. And as already noted, it is *only* because the drawing can be regarded in these different ways that one can determine, first, that certain lines are equal (because they can be regarded as radii of a single circle) and then *conclude* that a certain triangle is equilateral (because its sides are equal in length). The demonstration is fruitful, a real extension of our knowledge, for just this reason: because we were able to see a part of one whole as combined with a part of another whole to form an utterly new and hitherto unavailable whole, we were able to discover something that was simply not there, even implicitly, in the materials with which we began. Regarded one way two lines are seen to be equal (because they are radii of a single circle), regarded another they are seen to be the sides of a triangle. It is thus a *perceptual* skill that is required if one is to understand a course of diagrammatic reasoning in Euclid.

Necessary reasoning, as Peirce understands it, essentially involves perception, the capacity literally to see that the consequence follows, indeed must follow. At least for the case of a Euclidean demonstration, this seems deeply right. In order to follow, that is, to understand, a Euclidean demonstration, one must be able to see various drawn lines and points now as parts of one iconic figure and now as parts of another. The drawn lines are assigned very different significances at different stages in one's reasoning; they are *interpreted* differently depending on the context of lines they are taken, at a given stage in one's reasoning, to figure in. It is just this, I have suggested, that explains the fruitfulness of a Euclidean demonstration, the fact that its conclusion is, as Kant would say, synthetic a priori. In Euclid, the desired conclusion is contained in the diagram, drawn according to one's starting point and the postulates, not merely implicitly, needing only to be made explicit (as in a deductive proof on the standard construal or in an Euler diagram), but instead only potentially. The potential of the diagram to demonstrate the conclusion is made actual only through a course of reasoning in the diagram, that is, by a series of successive refigurings of what it is that is being iconically represented by various parts of the diagram. Parts of wholes must be taken apart and combined with parts of other wholes to make quite new wholes. And this is possible (again) because the various parts of the diagram signify geometrical objects only relative to ways of regarding those parts. A given line must actually be construed now as an iconic representation of (say) a radius of a circle and now as an iconic representation of a side of a triangle, if the demonstration is to succeed. The diagram, more exactly, its proper parts, must be actualized, now as this iconic representation and now as that, through one's recognition of them as such representations, if the result is to emerge from what is given. Only a course of thinking through the diagram can actualize the truth that it potentially contains.

The question now of course is, given this account of the use of diagrams in Euclid's mathematical practice, what can we infer about meaning? Notice, first, that meaning, in one obvious sense, is given by definitions in Euclid; we are *told* what it is to be a circle, or a triangle, or a square. But we are also provided with postulates setting out the primitive constructions that are allowed in the system. Together these definitions and postulates (as well as any previously demonstrated constructions, which function in later demonstrations as derived postulates) determine what we might think of as the circumstances and consequences of applying (in a broad sense) various terms for geometrical entities to collections of lines and points in a diagram. A finite line length provides the circumstance for the introduction of something correctly described as a circle, and a consequence of this introduced figure is that any two of its radii are equal in length. More generally, it is the postulates, whether primitive or derived, that constrain what can be put into a diagram, the definitions that determine what can be taken out. But if that is so, then although the definitions provide both necessary and sufficient conditions for something's being, say, a circle, what is exhibited *in* the diagram should be conceived in terms of consequences alone.

This is most obvious in the case of *reductio* demonstrations in Euclid. The diagram—for instance, that of two circles that, impossibly, cut each other at four points, as in Proposition III.10—encodes, iconically, not something that is, or could be, the case, but instead (as Frege would say) everything necessary for a correct inference on the supposition needed for the demonstration. Although one does need to know the sufficient conditions for something to be, say, an equilateral triangle in order to see one in the diagram, and so to infer, as in Proposition I.1, something about equilateral triangles, what must be put *into* the diagram as the basis for such an inference, what must be displayed in it for the purposes of diagrammatic reasoning, is not what is sufficient but what is necessary, that is, inferential consequences, not what is true but what follows. What the necessary reasoning of a Euclidean demonstration requires is the laying out in a diagram of the contents, that is, the inferential consequences, of the various concepts of Euclidean geometry.

According to our account, reasoning in Euclid functions by exhibiting the contents of concepts of geometrical figures, that is, their inferential consequences, in such a way that various parts of those contents can be combined and recombined to realize something new. One comes in this way to see a necessary relation among concepts that was in no way present, even implicitly, in one's starting point. But of course the contents, or inferential consequences, of most mathematical concepts cannot be exhibited in this way. Nevertheless, I want to suggest, they can be exhibited, and because they can, we can think of deductive reasoning in other areas of mathematics as similarly diagrammatic, as involving an essentially perceptual skill. Insofar as even strictly deductive reasoning does involve such a skill, it follows that in mathematics generally meaning as it is exhibited in the course of a proof concerns consequences, not, at least in the first instance, truth conditions.

According to the standard understanding of symbolic languages, the primitive signs of the language are meaningful independent of any context of use. They are given by an interpretation or model assigning to each primitive sign some semantic value, the contribution it makes to the truth conditions of sentences in which it occurs. Frege asks us to envisage a radically different sort of symbolic language, one that functions, as we might say, *diagrammatically*, that is, like a Euclidean diagram insofar as the primitive signs only express a sense independent of their occurrence in a sentence (and relative to an analysis).[24] Much as a particular drawn line is an icon of a radius of a circle (or of a side of a triangle) only in the context of a diagram and relative to a way of regarding it, so only in the context of a sentence and relative to an analysis do the signs of Frege's language, whether simple or complex, designate.

Consider, for example, this sentence of arithmetic: $2^4 = 16$. As we first learned to read this sentence, the various numerals designate numbers prior to and independent of their involvement in the sentence, and the sentence as a whole is read as exhibiting an arithmetical relation among those designated numbers: that two raised to the fourth power is equal to sixteen. Frege teaches us to read this sentence in a new way, as exhibiting only a sense, *Sinn*, that is, a Fregean thought, one that can be analyzed in various ways into function and argument, none of which are privileged. Taking '2' as marking the argument

[24] The reading of Frege that is assumed here is developed and defended in my *Frege's Logic* (Cambridge, Mass.: Harvard University Press, 2005).

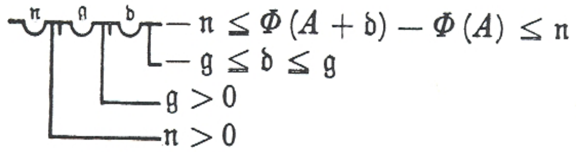
place, for example, the sentence expresses the judgment that two is a fourth root of sixteen; that is, on this analysis, the expression '$\xi^4 = 16$', in which 'ξ' marks the argument place, is a concept word designating the concept *fourth root of sixteen*. But other analyses are possible as well. We can, for instance, take '4' to mark the argument place, leaving '$2^\xi = 16$' as a complex sign designating the concept *logarithm of sixteen to the base two*. We can also take both '2' and 16' as marking the argument places of the relation '$\xi^4 = \zeta$', that is, the relation of a number to its fourth power. Or we can take '2^4' and '16' as the arguments for the relation of equality. And other analyses are possible as well. Sentences written in Frege's two-dimensional *Begriffsschrift* notation are essentially similar. The primitive signs of that language express senses independent of any context of use, senses that determine the senses of sentences formed by combining those primitive signs. Such sentences can then be variously analyzed into function and argument depending on how they are regarded, and each such analysis will yield truth conditions for the sentence taken as a whole. Independent of any analysis one has only the thought expressed and the truth-value designated.

Although we first learn to read arithmetical equations as built up out of antecedently meaningful parts, we can (on that basis) learn to read those same equations differently, as exhibiting Fregean senses, thoughts that are variously analyzable into function and argument. And the same is true, I have indicated, of a sentence in Frege's logical language. Notice, now, that on this new reading a sentence displays three levels of articulation. At the lowest level are the primitive signs out of which everything is composed. At the highest level is the whole sentence expressing a thought. Between these two levels, finally, are the various concept words and object names that are given relative to an analysis of the sentence. Much as one can see different figures in a Euclidean diagram depending on how one regards various collections of primitives of that system, so one can see different concept words in a sentence expressed in Frege's *Begriffsschrift* notation depending on how one regards its collection of primitives, depending, that is, on how one carves it into function and argument. Much as a drawn figure (as seen from one particular perspective) functions iconically in a Euclidian demonstration to exhibit relations of parts in a geometrical figure, so an array of primitive signs (conceived according to some particular analysis into function and argument) functions in Frege's notation to exhibit the content, the sense, of a concept word.

As conceived in standard quantificational logic, the content of, say, the mathematical notion of the continuity of a function at a point is given by the truth conditions of the claim that a function *f* is continuous at a point *a*:

$(\forall\varepsilon>0)(\exists\delta>0)(\forall x)(|x| < \delta \supset |f(x + a) - f(a)| < \varepsilon)$.

This sentence, read as it is normally read, is composed of antecedently meaningful parts as specified in a standard semantics for the language, parts that are combined into a whole according to the syntactic rules of the language. The concept of continuity, on this account, is nothing over and above its parts in a given logical array. That same concept, as Frege conceives it, is defined in *Begriffsschrift* thus:[25]



This sentence, read as Frege comes to read such sentences, *only* exhibits a sense, a Fregean thought, independent of any analysis—though in fact, in this case, an analysis is already indicated. We are to take the Greek letters 'Φ' and 'A' to mark the argument places for a higher-level concept word. What we have in this totality of signs, then, is an expression that designates (relative to the given analysis) the concept of continuity. But this sign is complex; it has parts each of which expresses a sense. The whole then expresses a sense making it abundantly clear just what that content is.[26]

Furthermore, and this really is the essential point, if this complex expression occurs in the context of a whole sentence (if, that is, the arguments places are appropriately filled), then that whole sentence can *also* be differently analyzed, analyzed so as to yield different concept words, not the concept word for continuity at all. A fruitful proof in *Begriffsschrift* inevitably requires just that, different analyses, at different points in the proof, of the various sentences involved in the proof; in the course of the proof, one must conceive a sentence now in one way (under one analysis) and now in another. More specifically, and just as we find in the case of a diagram in Euclid, a thought must be seen one way (in light of one analysis) if one is to see why it is true, but quite differently if one is to see what can be made to follow from it. Reasoning in

[25] Frege presents this analysis of the concept of continuity in "Boole's Logical Calculus and the Concept-script", in Gottlob Frege, *Posthumous Writings*, ed. Hans Hermes, Friedrich Kambartel, and Friedrich Kaulbach, and trans. Peter Long and Roger White (Chicago: University of Chicago Press, 1979), p. 24.

[26] See "Boole's logical Calculus and the Concept-Script" for many more examples.

Frege's *Begriffsschrift* thus involves precisely the same sort of perceptual skill that is required in the diagrammatic reasoning of Euclid.

In Frege's system of notation, as in a diagram in Euclid, three levels of articulation are discernable in a sentence. At the lowest level are the primitive signs expressing senses. At the highest level are sentences expressing thoughts, that is, senses that are a function of the senses of the primitives that make up the thought. And finally, between these two extremes, are the concept words expressing senses and designating concepts that are discernable in light of an analysis of the thought expressed. The whole sentence is like a Euclidean diagram insofar as its parts can be seen now this way and now that. Furthermore, as I have indicated, these various perspectives are often required in the course of a deduction in Frege. But if that is right, then reasoning in Frege's system of logic, just as in Euclid, constitutively involves a perceptual skill, the capacity to see a given array of signs now one way and now another. And just as in Euclid, the course of reasoning is necessary to actualize the potential of the list of sentences to constitute a proof. The proof is not in the sentences but in what they enable one to see, namely, a certain relationship between premises and conclusion, the relation of entailment. Proof is not, then, reducible to a set of sentences, or to a relationship among the sentences of some set. A collection of sentences can constitute a proof only for someone who already understands the practice of proof, the use of signs in proving something. For, as in Euclid, the conclusion is not contained in the premises merely implicitly, needing only to be made explicit; it is contained in the premises *potentially*, a potential that is actualized only through a course of reasoning on the part of a thinker. As

Frege himself puts the point, "here, we are not simply taking out of the box again what we have just put into it. The conclusions we draw from it extend our knowledge, and ought therefore, on Kant's view, to be regarded as synthetic; and yet they can be proved by purely logical means, and are thus analytic. The truth is that they are contained in the definitions, but as plants are contained in their seed, not as beams are contained in a house."[27]

Only in a notation read diagrammatically, that is, as exhibiting a content that can be variously analyzed into function and argument, rather than symbolically (its primitives as having meaning, designation, independent of any context of use), can this be true.

Much as Peirce does, Frege thinks of meaning, as it is to be exhibited in an array of lines and symbols for the purposes of reasoning, in terms of

[27] Gottlob Frege, *The Foundations of Arithmetic*, trans. J. L. Austin (Evanston, Ill.: Northwestern University Press, 1980) §88.

consequences. What is displayed, or "fully expressed", in *Begriffsschrift* is "everything necessary for a correct inference".[28] Although truth conditions can be formulated for a sentence of *Begriffsschrift* by giving an analysis of it, what such sentences directly map, what they display (and indeed display iconically, though I have not tried to show that here), is not what *is* the case if they are true but what *follows* if they are true. And although Frege was not as self-conscious as Peirce was about why is this the right way to proceed, he like Peirce looked to mathematical practice, to the mathematician's use of concepts in reasoning, as his guide to this understanding of meaning. It was as a practicing mathematician that Frege developed his notation; he wanted a notation that would be not an empty formalism but a means of expressing content *as* it matters to mathematical practice.[29] The aim was not to abstract from content but to express it. The only question was what content, as it matters to mathematics, *is*. The answer, as Frege came explicitly to see by the early 1890s, is *Sinn*. Much as the axioms of a theory contain, as in a kernel, that is, potentially, all the theorems that can be derived from them, so a thought, the sense of a sentence, contains as in a kernel, that is, potentially, all its consequences. When we apply mathematics to the natural world we are concerned with what is the case if our mathematical claims are true. What both Peirce and Frege saw is that in mathematics itself we are concerned instead with what follows if those claims are true. We are concerned with consequences.

According to the received view, meaning is to be understood in terms of truth. The meaning of a primitive sign is its semantic value, the contribution it makes to the truth conditions of a sentence. A sentence, then, is to be read as presenting, or representing, a state of affairs, what is the case if it is true. On Brandom's inferentialist alternative, sentences are to be understood instead in terms of their inferential circumstances and consequences, in terms, that is, of what they follow from together with what follows from them. On Brandom's account, instead of taking words to have meaning in virtue of the contributions they make to the truth conditions of sentences, we are to take words to have meaning in virtue of the inferential relations they are caught up in. Not word-world relations but word-word relations are primary. The account sketched here is different again. According to Peirce and Frege, the notion of meaning that is needed to understand necessary reasoning in

[28] *Begriffsschrift*, §3, translated T. W. Bynum and included in Gottlob Frege, *Conceptual Notation and Related Articles* (Oxford: Clarendon Press, 1972).

[29] See, for instance, Frege's "On the Aim of the 'Conceptual Notation'", included in Bynum, *Conceptual Notation and Related Articles*.

mathematics concerns neither truth conditions (as on the standard view) nor inferential relations among sentences, or derivatively, words (as Brandom thinks). The conception of meaning that is needed concerns not *relations among* sentences, or words, at all but instead the *contents* of concepts as those contents are involved in inference. Such contents, we have seen, can in certain cases be exhibited in Euclidean diagrams; in other cases they are exhibited in the complex signs of Frege's *Begriffschrift*. And as I have indicated, these contents so exhibited enable inference, diagrammatic reasoning, *because* they display collections of primitives that can be variously regarded, and need so to be regarded in the course of reasoning. At least for the case of mathematical concepts, then, we can say exactly what meaning is: it is nothing more and nothing less than what is exhibited in Euclidean diagrams and in *Begriffsschrift* expressions, diagrams and expressions that directly display the senses of concept words, senses within which are contained everything necessary for a correct inference. At least as applied to the case of mathematics, Peirce's pragmatist maxim that meaning lies not in truth conditions, what is the case if a sentence is true, but instead in inferential consequences, what follows if a sentence is true, captures just this thought.