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How to go non-monotonic through context-sensitiveness

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ABSTRACT. In this paper we review some ways of producing non-monotonic logics by considering context-sensitive inferences. These approaches are all based on the notion of *control set*, a piece of logical machinery recently introduced in [3] and further developed in [7, 4, 14]. A control set informally refers to a set of contexts **S** which are supposed to prohibit the implementation of specific inferences in a proof system.

KEYWORDS: non-monotonic logic, proof theory, control sets, cut-elimination.

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1. Introduction

In this paper we survey three non-monotonic calculi, CSI, $\sigma\mathsf{CSI}$ and $\mathsf{LK}^{\mathscr{S}}$, recently introduced in [3], [7] and [14], respectively. Each one of them is obtained from well-known logical systems (CSI and $\sigma\mathsf{CSI}$ are variants of the multiplicative fragment of linear logic [10, 13], while $\mathsf{LK}^{\mathscr{S}}$ is a fragment of classical propositional logic) by stressing the notion of *control set* and the relative notion of *compatibility*. Informally speaking, a control set $\mathbf{S} = \{\Gamma_1, \ldots, \Gamma_n\}$ is a finite set of logical contexts which are supposed to block specific inferences in a proof system. A context Δ is said to be *compatible* with $\mathbf{S}$, if Δ does not include any context in $\mathbf{S}$.

In order to better appreciate how control sets constrain the dynamics of logical proofs, let's consider a couple of examples, one coming from biochemistry and the other from medical reasoning.

In biochemistry the empirical evidence supports the following fact: an *enzyme E* binds with a *substratum S* provided that an *inhibitor I* is not present in the same environment at the very moment of the reaction [2]. Let's denote with Γ the set of reactants present in the environment. We can formalize this fact by means of the following controlled axiom

$$\Gamma, E, S \vdash_{\{\{I\}\}} E \odot S$$

to be read as follows: *E and S binds so as to form the compound $E \odot S$ provided that* $\{I\} \nsubseteq \Gamma$, i.e., *the inhibitor I is not present in the environment.*

Once formalized, the concurrent enzyme inhibition phenomenon becomes a clear example of logical non-monotonicity. In classical and intuitionistic logic, the monotonicity property is guaranteed by the weakening rule

$$\frac{\Gamma \vdash A}{\Gamma, B \vdash A}\ weak \vdash$$

saying that whenever the formula A is derivable from the set of assumptions Γ, A is still derivable when Γ is enriched with any formula B. In our case, this inference is no longer valid since the following instance of the weakening rule returns an (empirically) invalid sequent:

$$\frac{E, S \vdash E \odot S}{E, S, I \vdash E \odot S}\ weak \vdash.$$

The key point to understand is that the control sets mechanism allows us to preserve soundness along derivations in such a way that the transition from $E, S \vdash_{\{\{I\}\}} E \odot S$ to $E, S, I \vdash_{\{\{I\}\}} E \odot S$ is no longer admissible since the context $\{E, S, I\}$ is incompatible with the control set $\{\{I\}\}$.

Similar situations are pervasive in many other empirical settings. Let's consider, for instance, the problem of algorithmically prescribing medications in medical reasoning. For instance, it is known that people suffering from dengue should not take aspirin because it can aggravate bleeding. Here is the rule of thumb: *aspirin is recommended for flu-like symptoms, unless they are caused by dengue*. In our approach, this protocol can be easily formalized by the following sequent:

$$\Gamma, \mathsf{flu\text{-}like\ symptoms} \vdash_{\{\{\mathsf{dengue}\}\}} \mathsf{aspirin}.$$

The singleton $\{\{\mathsf{dengue}\}\}$ is the control set which constraints the soundness of the sequent: from flu-like symptoms one can derive aspirin, *provided that the context* Γ *does not contain the information* dengue. Thus, when flu-like symptoms is associated with dengue, the conclusion aspirin is no longer derivable. Accordingly, the sequent below turns out to be unsound:

$$\Gamma', \mathsf{dengue},\ \mathsf{flu\text{-}like\ symptoms} \vdash_{\{\{\mathsf{dengue}\}\}} \mathsf{aspirin}.$$

The control sets-based approach clearly reminds of standard treatments of similar situations by means of default rules [15, 1]. In spite of the obvious resemblances, the control sets approach differs from standard default reasoning in two main respects. First, controlled systems allow for a satisfactory proof theoretical presentation. As we shall see Sections 2 and 3, both the calculi $\sigma\mathsf{CSI}^-$ and $\mathsf{LK}^{\mathscr{S}}$ enjoy the cut-elimination theorem and so the subformula property. Second, in our framework, non-monotonicity is just a byproduct of the fact that controlled systems are equipped with a set of proper axioms which are supposed to encode the extra-logical information coming from the specific empirical context under consideration. As a part of their specific extra-logical content, proper axioms also undertake the task of introducing in a derivation non-trivial control sets. Proper axioms are in this way thought of as the interface between logic and empirical world, and so they are subject to continue revisions and updating.

2. The CSI sequent calculus

In this and in the following sections, we take logical contexts $\Gamma, \Delta, \ldots$ as *multiset* of formulas. In order to distinguish ordinary sets from multisets, we will use curly brackets for the former and square brackets for the latter.

Definition 1 (formulas) CSI formulas are inductively defined as follows:

- set of atoms $\mathscr{A} = \{a, b, c, \ldots\}$,
- bonding language $\mathscr{F}_{\odot} ::= \mathscr{A} \mid \mathscr{F}_{\odot} \odot \mathscr{F}_{\odot}$,
- logical language $\mathscr{F} ::= \mathscr{F}_{\odot} \mid \mathscr{F} \otimes \mathscr{F}$.

Technically speaking, the bonding operator is not a logical connective, since the CSI sequent calculus does not encompass any specific rule for introducing it. This is the reason why *the bonding language is considered as uniquely formed by atomic types* and so the set of CSI atomic formulas is, indeed, given by $\mathscr{F}_{\odot}$. Types are supposed to bind two by two and so the reason why nesting an independent bonding language into the language of CSI is that of avoiding meaningless propositions like $(A \otimes B) \odot (C \otimes D)$.

Definition 2 (control sets) A control set is a finite set of finite multisets of formulas $\{\Gamma_1, \ldots, \Gamma_n\}$ such that, for all $1 \leqslant i \leqslant n$, $\Gamma_i \subset \mathscr{F}_{\odot}$. The empty set $\varnothing$ is a control set. Control sets are denoted by boldface capital letters $\mathbf{S}, \mathbf{T}, \mathbf{U}, \ldots$.

Definition 3 (contexts, subcontexts, supports) Precontexts are rooted trees recursively defined as follows:

- $\mathscr{F}$ is a set of precontexts,
- if $T_1, \ldots, T_n$ are precontexts, then $(T_1, \ldots, T_n)$ is a precontext.

A CSI context $\mathscr{C}$ is an ordered pair $\langle T, f \rangle$ such that T is a precontext and f a function which assigns a control set to each node of T other than its leaves.

The set of the subcontexts of $\mathscr{C} = \langle T, f \rangle$ is given by the set of contexts $\mathscr{C}' = \langle T', f' \rangle$ such that T' is a proper subtree of T and f' is the function obtained by restricting the domain of f to set of the T'-nodes. By $\mathscr{C}[\mathscr{D}]$ we mean that $\mathscr{D}$ occurs as a subcontext of $\mathscr{C}$. $|\mathscr{C}|$ denotes the support of $\mathscr{C}$, namely the multiset of formulas labelling the leaves of $\mathscr{C}$.

Contexts are written by stressing the usual compact notation through nested parentheses. In order to represent the additional information encoded by the function f, right parentheses come indexed with a control set.

Loosely speaking, a CSI context is nothing else but a rooted tree having the leaves single-labelled with a formula from $\mathscr{F}$ and the other nodes double-labelled with both a precontext and a control set. It is worth noting that the unrestricted tree structure of sequents allows us to generalise the classical approach to logical non-associativity based on binary trees [9, 8]. Moreover, as we shall see in a moment, the monotonicity of derivations is specifically modulated by the information associated with precontexts by the function f.

Definition 4 (immediately acting formulas) A formula A is said to be immediately acting in a context $\mathscr{C}$ if A labels a leaf of $\mathscr{C}$ which is directly connected with the root. For any context $\mathscr{C}$, the multiset $\mathtt{imac}(\mathscr{C})$ collects all the immediately acting formulas of $\mathscr{C}$.

Example 5 The tree structure of the context $\mathscr{C} = (A, (C, D, (E)_{\mathbf{U}})_{\mathbf{T}}, B)_{\mathbf{S}}$ is displayed in Figure 1. $(C, D, (E)_{\mathbf{U}})_{\mathbf{T}}$ and $(E)_{\mathbf{U}}$ are the subcontexts of $\mathscr{C}$. In order to exemplify the notion of immediately acting formula, let's notice that $\mathtt{imac}(\mathscr{C}) = \{A, B\}$.

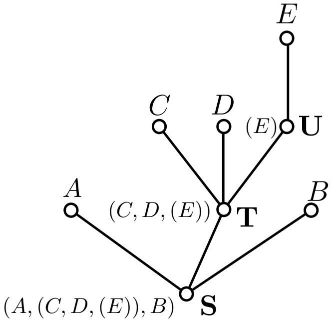

FIG. 1: *The tree-structure of the context* $\mathscr{C} = (A, (C, D, (E)_{\mathbf{U}})_{\mathbf{T}}, B)_{\mathbf{S}}$.

Inference Schemata

Axiom:

$$\frac{}{(A)_\varnothing \vdash A}\ ax.$$

Cut-rules:

$$\frac{\mathscr{C}[(\mathscr{C}_1,\ldots,\mathscr{C}_n,A)_{\mathbf{S}}] \vdash B \qquad \mathscr{D} \vdash A}{\mathscr{C}'[(\mathscr{C}_1,\ldots,\mathscr{C}_n,\mathscr{D})_{\mathbf{S}}] \vdash B}\ \textit{surgical cut}$$

$$\frac{\mathscr{C}[(\mathscr{C}_1,\ldots,\mathscr{C}_n,A)_{\mathbf{S}}] \vdash B \qquad (\mathscr{D}_1,\ldots,\mathscr{D}_m)_{\mathbf{T}} \vdash A}{\mathscr{C}'[(\mathscr{C}_1,\ldots,\mathscr{C}_n,\mathscr{D}_1,\ldots,\mathscr{D}_m)_{\mathbf{S}\cup\mathbf{T}}] \vdash B}\ \textit{deep cut}^{\dagger}$$

(†) provided that $(\mathtt{imac}(\mathscr{C}_1 \ldots,\mathscr{C}_n,\mathscr{D}_1,\ldots,\mathscr{D}_m))^\star \parallel \mathbf{S}\cup\mathbf{T}$

Structural rules:

$$\frac{\mathscr{C}[(\mathscr{D}_1,\ldots,\mathscr{D}_n,\mathscr{H},\mathscr{K})_{\mathbf{S}}] \vdash A}{\mathscr{C}'[(\mathscr{D}_1,\ldots,\mathscr{D}_n,\mathscr{K},\mathscr{H})_{\mathbf{S}}] \vdash A}\ \times\textit{change}$$

Multiplicative conjunctions:

$$\frac{\mathscr{C}[(\mathscr{D}_1,\ldots,\mathscr{D}_n,A,B)_{\mathbf{S}}] \vdash C}{\mathscr{C}'[(\mathscr{D}_1,\ldots,\mathscr{D}_n,A\otimes B)_{\mathbf{S}}] \vdash C}\ \otimes_{\mathscr{L}}$$

$$\frac{\mathscr{C} \vdash A \qquad \mathscr{D} \vdash B}{(\mathscr{C},\mathscr{D})_\varnothing \vdash A\otimes B}\ \otimes_{\mathscr{R}}$$

$$\frac{(\mathscr{C}_1,\ldots,\mathscr{C}_n)_{\mathbf{S}} \vdash A \qquad (\mathscr{D}_1,\ldots,\mathscr{D}_m)_{\mathbf{T}} \vdash B}{(\mathscr{C}_1,\ldots,\mathscr{C}_n,\mathscr{D}_1,\ldots,\mathscr{D}_m)_{\mathbf{S}\cup\mathbf{T}} \vdash A\otimes B}\ \textit{deep-}\otimes_{\mathscr{R}}^{\ddagger}$$

(‡) provided that $(\mathtt{imac}(\mathscr{C}_1 \ldots,\mathscr{C}_n,\mathscr{D}_1,\ldots,\mathscr{D}_m))^\star \parallel \mathbf{S}\cup\mathbf{T}$

Set of proper axioms

Axioms of type σ:

$$\frac{}{(E\otimes F)_{\mathbf{S}_i} \vdash E\odot F}\ \sigma_i$$

Axioms of type ρ:

$$\frac{}{(E\odot F)_\varnothing \vdash G\otimes H}\ \rho_i$$

TABLE 1: *The* CSI *sequent calculus.*

Definition 6 For any $A \in \mathscr{F}$, $A^\star$ denotes the multiset of atomic formulas occurring in A. The $\star$-operation can be straightforwardly extended to any multiset of formulas Γ as follows: $\Gamma^\star = \bigcup_{A \in \Gamma} A^\star$.

Definition 7 (compatibility) A multiset of formulas Δ is said to be compatible with a control set $\mathbf{S} = \{\Gamma_1, \ldots, \Gamma_n\}$ — in symbols, $\Delta \parallel \mathbf{S}$ — if, for all $\Gamma_i \in \mathbf{S}$, $\Gamma_i \nsubseteq \Delta^\star$.

Definition 8 (monotonicity soundness) A context $\mathscr{C}$ is said to be monotonically sound in case that, for any subcontext $\mathscr{D}$ of $\mathscr{C}$, $(\mathtt{imac}(\mathscr{D})) \parallel \mathbf{T}$ where $\mathbf{T}$ is the control set attached to the root of $\mathscr{D}$.

Definition 9 (sequents, related notions) A CSI sequent is an ordered pair $\langle \mathscr{C}, A \rangle$ such that $\mathscr{C}$ is a CSI context and $A \in \mathscr{F}$. Following the usual logical notation, a sequent $\langle \mathscr{C}, A \rangle$ will be henceforth written as $\mathscr{C} \vdash A$. Formulas in $|\mathscr{C}|$ are the *premises* of the sequent and A the *conclusion*.

Definition 10 (proofs) A CSI proof is a sequence of CSI sequents such that each sequent is derivable from the sequents appearing earlier in the sequence by means of the rules displayed in Table 1.

Figures 2 and 3 illustrate the basic combinatorics on contexts respectively induced by the two right tensor rules ($\otimes_{\mathscr{R}}$ and *deep*-$\otimes_{\mathscr{R}}$) and the two cut rules (*surgical* and *deep*).

Together with the familiar inference schemata, the deductive apparatus of CSI comes equipped with a set of proper axioms encoding the information acquired in the specific empirical context we are concerning with. Unlike inference schemata which provide a general pattern of inference, proper axioms refer to *specific types*. To clarify this point, let's refer again to biochemistry. Suppose that for a certain $i \in \mathbb{N}$ the axiom σ_i expresses the fact that two molecules of hydrogen bind so as to form the compound $H_2 \equiv H \odot H$:

$$\frac{}{(H \otimes H)_{\mathbf{S}_i} \vdash H \odot H}\ \sigma_i.$$

The axiom σ_i does not mean that, for any type $A \in \mathscr{F}$, $A \otimes A \vdash A \odot A$: it just says that, for the *specific* type H, we have empirical evidence allowing us to

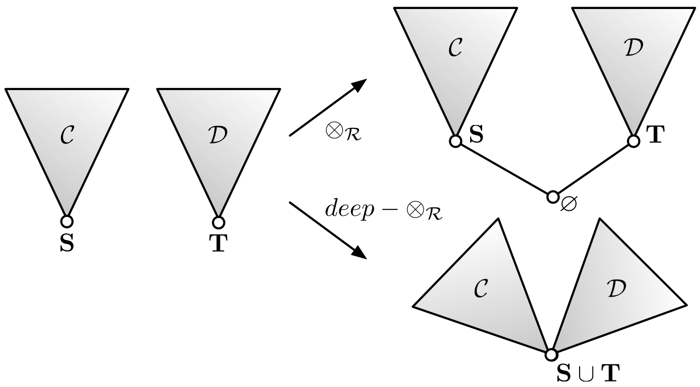

FIG. 2: *Tensor combinatorics on trees.*

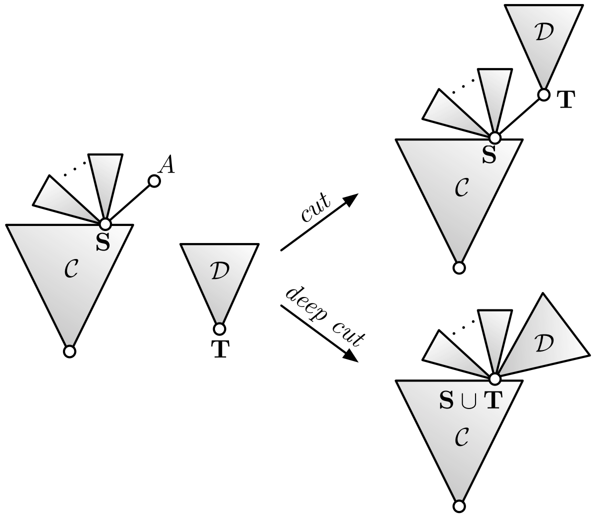

FIG. 3: *Cut combinatorics on trees.*

state $H \otimes H \vdash H \odot H$. Being our empirical information finite, CSI proper axioms are conceived as finite in number. According to [2], proper axioms are partitioned into two classes: the set of σ-axioms $\{\sigma_1, \ldots, \sigma_n\}$ and the set of ρ-axioms $\{\rho_1, \ldots, \rho_m\}$. Such a distinction can be explained by dwelling again on biochemistry: whereas σ-axioms essentially describe how molecules bind to each other, ρ-axioms carry the specific information about the possible biochemical decay of compounds.

From a more general point of view, whereas the $\otimes$-connective expresses a sort of proto-conjunction, the operator $\odot$ indicates that a proto-conjunction has been, so to speak, *activated* by means of a certain (non-logical) process, e.g. the pass of a certain lapse of time. Of course, the word 'activated' changes its meaning according to the specific empirical context under focus. It is worth noting that the need of such a distinction is a logical consequence of the fact that we are dealing with a controlled monotonicity system. Indeed, in the absence of the bonding operator, we would be in the uncomfortable situation in which two sequents like $(E \otimes F)_\varnothing \vdash E \otimes F$ and $(E \otimes F)_\mathbf{S} \vdash E \otimes F$, with $\mathbf{S} \neq \varnothing$, would be both provable, being the proof of the first one just the η-expansion of the axiom and the second one a proper axiom.

Example 11 Let's consider the concurrent enzyme inhibition phenomenon as it has been explained in the introduction, *i.e.*, the enzyme E binds with substratum S provided that the inhibitor I is not present in the environment at the moment of the reaction. This kind of empirical information can be encoded by means of the σ-axiom

$$\cfrac{}{(E \otimes S)_{\{[I],\ldots\}} \vdash E \odot S}\ \sigma_i$$

introducing the control set $\mathbf{S}_i = \{[I], \ldots\}$. In order to provide an example of a CSI deduction, we report below a proof of the sequent $((E \otimes S)_{\{[I],\ldots\}}, (I)_\varnothing)_\varnothing \vdash (E \odot S) \otimes I$:

$$\cfrac{\cfrac{\cfrac{\cfrac{}{(E)_\varnothing \vdash E}\,ax. \quad \cfrac{}{(S)_\varnothing \vdash S}\,ax.}{(E,S)_\varnothing \vdash E \otimes S}\,deep\text{-}\otimes_\mathscr{R} \quad \cfrac{}{(E \otimes S)_{\{[I],\ldots\}} \vdash E \odot S}\,\sigma_i}{(E,S)_{\{[I],\ldots\}} \vdash E \odot S}\,deep\text{-}cut \quad \cfrac{}{(I)_\varnothing \vdash I}\,ax.}{((E,S)_{\{[I],\ldots\}}, (I)_\varnothing)_\varnothing \vdash (E \odot S) \otimes I}\,\otimes_\mathscr{R}}{((E \otimes S)_{\{[I],\ldots\}}, (I)_\varnothing)_\varnothing \vdash (E \odot S) \otimes I}\,\otimes_\mathscr{L}$$

Let's notice that $(E \otimes S, I)_{\{[I],\dots\}} \vdash (E \odot S) \otimes I$ is not a CSI theorem because any derivation leading to this sequent would be unable to pass the checkpoint imposed by the control set $\mathbf{S}_i$.

3. The σCSI sequent calculus

The σCSI calculus is obtained from CSI by refining the structure of sequents. While preserving the tree-structure, σCSI *multi-level sequents* are designed in a way to encode some new information concerning the very 'history' of the proof in which they occur. The general aim is that to allow for sound 'deep' inferences, *i.e.*, inferences involving conclusions achieved in the previous stages of the proof [11]. In terms of tree-structures, deep inferences allow us to perform inferences involving the inner nodes of trees, not only the root-formulas.

As we shall see in Section 3, this syntactical refinement allows us to prove the cut eliminination theorem for σCSI$^-$, the conjunctive subsystem of σCSI.

3.1. Multi-level sequents and sequent calculus

Definition 12 (multi-level sequent, subsequent, proper subsequent) A *multi-level sequent* X is a finite rooted tree such that: (i) each leaf node is labelled with a formula from $\mathscr{F}$, (ii) each of the (inner) other nodes is labelled with a pair $\langle F, \mathbf{S} \rangle$, where $F \in \mathscr{F}$ and $\mathbf{S}$ is a control set, and (iii) X has at least two nodes. The set of the (multi-level) *subsequents* of X is formed by all the subtrees $X_1, X_2, \dots, X_n$ of X which are multi-level sequents.

Definition 13 (premises, conclusions, intermediate conclusions) Given a multi-level sequent X, its root is the *conclusion*, its leaves are the *premises* and the formulas labeling the inner nodes are called *intermediate conclusions*.

With $X, Y, Z, \dots$ and $\mathscr{X}, \mathscr{Y}, \mathscr{Z} \dots$ we respectively denote multi-level sequents and finite sequences of multi-level sequents. Multi-level sequents are usually written through nested parentheses. This notation represents as a string the information concerning the leaves according to the geometrical structure of the tree. In order to recover the lacking information about inner nodes, right parentheses are decorated with an expression of the form $\triangleright F|\mathbf{S}$, where the symbol "$\triangleright$" replaces and generalizes the ordinary turnstile $\vdash$ of sequent calculus.

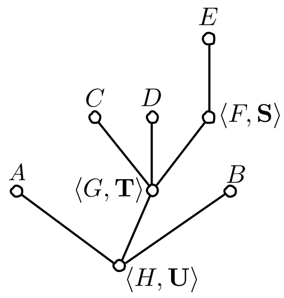

FIG. 4: *An example of multi-level sequent.*

Thus, we write $(\dots) \rhd F|\mathbf{S}$ to mean that the tree displayed on the left-hand side of the symbol $\rhd$ has the root-node labelled by $\langle F, \mathbf{S}\rangle$. Since trees are standardly considered modulo permutations of the branches emerging from the same node, any multi-level sequent X can be written as $X = (\mathscr{X}, \Gamma) \rhd C$, where all immediate subsequents are shifted to the left-hand side and single formulas remain on the right.

Example 14 In Figure 4, the structure of the multi-level sequent

$$X = (((E) \rhd F|\mathbf{S}, C, D) \rhd G|\mathbf{T}, A, B) \rhd H|\mathbf{U}$$

is geometrically represented. Notice that X itself, $((E) \rhd F|\mathbf{S}, C, D) \rhd G|\mathbf{T}$ and $(E) \rhd F|\mathbf{S}$ are the three subsequents of X.

Definition 15 (tensorial closures) If $\Gamma = F_1, \dots, F_n$, then $\Gamma_\otimes = F_1 \otimes \dots \otimes F_n$. Analogously, if $\mathscr{X} = (\mathscr{X}_1, \Gamma_1) \rhd H_1|\mathbf{T}_1, \dots, (\mathscr{X}_n, \Gamma_n) \rhd H_n|\mathbf{T}_n$, then $\mathscr{X}_\otimes = H_1 \otimes \dots \otimes H_n$.

Definition 16 (monotonicity soundness) A multi-level sequent X is said to be *monotonically sound* if, for each of its subsequents $(\mathscr{Y}, \Delta) \rhd D|\mathbf{T}$, it is $(\mathscr{Y}_\otimes \otimes \Delta_\otimes)^\star \parallel \mathbf{T}$.

Theorem 17 (monotonicity soundness) Any multi-level sequent X derivable in $\sigma\mathsf{CSI}$ is monotonically sound.

Proof The proof proceeds by induction on the length of σCSI proofs. For further details the reader can refer to [7].

The down rule '$\Downarrow$' has a clear intuitive meaning. Its premise describes the situation where the elements in $\mathscr{Y}$ and those in Δ, when glued together, produce C. Then, C is added to the elements of Γ and those of $\mathscr{X}$ so that their combination eventually leads to D. The $\Downarrow$-rule therefore assures that, if the elements in the compound $\mathscr{X}_{\otimes} \otimes \Gamma_{\otimes} \otimes \mathscr{Y}_{\otimes} \otimes \Delta_{\otimes}$ are compatible with both the control sets **S** and **T**, then D can be delivered directly by putting together, at the same time, the elements of $\mathscr{X}$ and Γ with those belonging to $\mathscr{Y}$ and Δ. *Geometrically speaking, an application of the $\Downarrow$-rule amounts to contract a branch in the specific tree under consideration.*

The rules $\Uparrow$(1) and $\Uparrow$(2) are introduced by symmetry from the $\Downarrow$-rule and can be regarded as a sort of time-interpolation device. The premise of $\Uparrow$(1) says that the elements delivered by $\mathscr{X}$, Γ, $\mathscr{Y}$ and Δ produce C. Hence the rule simply guarantees the possibility of inserting a unit-segment of time in which the elements delivered by $\mathscr{Y}$ and those in Δ just are grouped together before being added to those of $\mathscr{X}$ and Γ in a successive time. The mechanism underlying the $\Uparrow$(2) rule is analogous. *From a geometrical point of view, an application of the $\Uparrow$-rules amounts to expand a certain node into a new branch.*

Example 18 Our toy example of a controlled monotonicity derivation inspired by the concurrent enzyme inhibition phenomenon can be framed in σCSI as follows:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(E)\rhd E|\varnothing}\,ax. \quad \dfrac{}{(S)\rhd S|\varnothing}\,ax.}{((E)\rhd E|\varnothing,(S)\rhd S|\varnothing)\rhd E\otimes S|\varnothing}\otimes_{\mathscr{R}} \quad \dfrac{}{(E\otimes S)\rhd E\odot S|\{[I]\}}\,\sigma_i}{(((E)\rhd E|\varnothing,(S)\rhd S|\varnothing)\rhd E\otimes S|\varnothing)\rhd E\odot S|\{[I]\}}\,\text{cut}}{((E,(S)\rhd S|\varnothing)\rhd E\otimes S|\varnothing)\rhd E\odot S|\{[I]\}}\Downarrow}{((E,S)\rhd E\otimes S|\varnothing)\rhd E\odot S|\{[I]\}}\Downarrow \quad \dfrac{}{(I)\rhd I|\varnothing}\,ax.}{(((E,S)\rhd E\otimes S|\varnothing)\rhd E\odot S|\{[I]\},(I)\rhd I|\varnothing)\rhd(E\odot S)\otimes I|\varnothing}\otimes_{\mathscr{R}}}{(((E,S)\rhd E\otimes S|\varnothing)\rhd E\odot S|\{[I]\},I)\rhd(E\odot S)\otimes I|\varnothing}\Downarrow
$$

The tree expressed by the final multi-level sequent

$$(((E,S)\rhd E\otimes S|\varnothing)\rhd E\odot S|\{[I]\},I)\rhd(E\odot S)\otimes I|\varnothing$$

INFERENCE SCHEMATA

Logical axioms:

$$\frac{}{(A) \rhd A|\varnothing}\ ax.$$

Cut rule:

$$\frac{(\mathscr{X},\Gamma,A) \rhd C|\mathbf{S} \qquad (\mathscr{Y},\Delta) \rhd A|\mathbf{T}}{(\mathscr{X},\Gamma,(\mathscr{Y},\Delta) \rhd A|\mathbf{T}) \rhd C|\mathbf{S}}\ cut$$

Structural rules:

$$\frac{(\mathscr{X},\Gamma,(\mathscr{Y},\Delta) \rhd C|\mathbf{T}) \rhd D|\mathbf{S}}{(\mathscr{X},\Gamma,\mathscr{Y},\Delta) \rhd D|\mathbf{S}\cup\mathbf{T}}\ \Downarrow^{\dagger}$$

(†) *provided that* $(\mathscr{X}_{\otimes} \otimes \Gamma_{\otimes} \otimes \mathscr{Y}_{\otimes} \otimes \Delta_{\otimes})^{\star} \parallel \mathbf{S}\cup\mathbf{T}$.

$$\frac{(\mathscr{X},\Gamma,\mathscr{Y},\Delta) \rhd C|\mathbf{S}}{(\mathscr{X},\Gamma,(\mathscr{Y},\Delta) \rhd \mathscr{Y}_{\otimes} \otimes \Delta_{\otimes}|\varnothing) \rhd C|\mathbf{S}}\ \Uparrow(1)$$

$$\frac{(\mathscr{X},\Gamma) \rhd C|\mathbf{S}}{((\mathscr{X},\Gamma) \rhd C|\mathbf{S}) \rhd C|\varnothing}\ \Uparrow(2)$$

Multiplicative conjunction:

$$\frac{(\mathscr{X},\Gamma,A,B) \rhd C|\mathbf{S}}{(\mathscr{X},\Gamma,A \otimes B) \rhd C|\mathbf{S}}\ \otimes_{\mathscr{L}}$$

$$\frac{(\mathscr{X},\Gamma) \rhd C|\mathbf{S} \qquad (\mathscr{Y},\Delta) \rhd D|\mathbf{T}}{((\mathscr{X},\Gamma) \rhd C|\mathbf{S},(\mathscr{Y},\Delta) \rhd D|\mathbf{T}) \rhd C \otimes D|\varnothing}\ \otimes_{\mathscr{R}}$$

Arrows:

$$\frac{(\mathscr{X},\Gamma) \rhd A|\mathbf{S} \qquad (\mathscr{Y},\Delta,B) \rhd C|\mathbf{T}}{(\mathscr{Y},\Delta,(\mathscr{X},\Gamma) \rhd A|\mathbf{S},A \multimap B) \rhd C|\mathbf{T}}\ \multimap_{\mathscr{L}}$$

$$\frac{(\mathscr{X},\Gamma,A) \rhd B|\mathbf{S}}{(\mathscr{X},\Gamma) \rhd A \multimap B|\mathbf{S}\backslash A^{\star}}\ \multimap_{\mathscr{R}}$$

SET OF PROPER AXIOMS

$$\frac{}{(E \otimes F) \rhd E \odot F|\mathbf{S}_i}\ \sigma_i$$

TABLE 2: *The* σCSI *multi-level sequent calculus.*

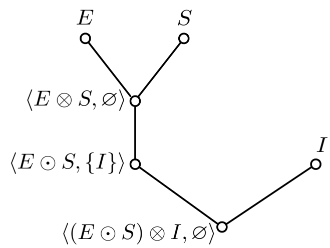

FIG. 5: *The resource tree involved in* Example 18

(cfr. Figure 5) conveys the intuitive meaning of the proof: in order to get the final compound $(E \odot S) \otimes I$ one has to make E and S interact and then add, at a later stage, the inhibitor I.

3.2. Cut-elimination and subformula property for $\sigma\mathsf{CSI}^-$

With $\sigma\mathsf{CSI}^-$ we denote the conjunctive fragment of $\sigma\mathsf{CSI}$, namely the fragment which is obtained from the rules in Table 2 by removing the arrow rules $\multimap_{\mathscr{L}}$ and $\multimap_{\mathscr{R}}$. The cut-elimination procedure just sketched in this section is limited to $\sigma\mathsf{CSI}^-$. The technical reason for this restriction is that, whereas the structural rule $\Uparrow(1)$ introduces the $\otimes$ connective, there is no structural rule introducing the arrow $\multimap$. Because of this asymmetry the cut-elimination procedure designed in this section does not extend to the whole system $\sigma\mathsf{CSI}$.

Definition 19 (logical cut, proper cut, normal proof) An application of the cut rule is said to be *proper* when: (i) it involves a proper axiom and a $\otimes_{\mathscr{R}}$-rule, and (ii) the formula $A \otimes B$ introduced by the $\otimes_{\mathscr{R}}$-rule at point (1) is the cut formula. Any other application of the cut rule is referred to as *logical*. If a $\sigma\mathsf{CSI}^-$ proof π contains no logical application of the cut rule, then π is said to be *normal*.

According to Definition 19, the general form of a proper cut is the following:

$$\dfrac{\dfrac{\begin{matrix}\pi \\ \vdots \\ (\mathscr{X},\Gamma) \rhd E|\mathbf{T}\end{matrix} \qquad \begin{matrix}\delta \\ \vdots \\ (\mathscr{Y},\Delta) \rhd F|\mathbf{U}\end{matrix}}{((\mathscr{X},\Gamma) \rhd E|\mathbf{T}, (\mathscr{Y},\Delta) \rhd F|\mathbf{U}) \rhd E \otimes F|\varnothing} \qquad \dfrac{}{(E \otimes S) \rhd E \odot F|\mathbf{S}_i}\,\sigma_i}{(((\mathscr{X},\Gamma) \rhd E|\mathbf{T}, (\mathscr{Y},\Delta) \rhd F|\mathbf{U}) \rhd E \otimes F|\varnothing) \rhd E \odot F|\mathbf{S}_i}\,\text{cut}$$

Theorem 20 (cut-elimination) If $\sigma\mathsf{CSI}^-$ proves a multi-level sequent X, then there is a $\sigma\mathsf{CSI}^-$ proof of X in which all applications of the cut rule, if any, are proper.

Proof The proof proceeds by showing that parallel reductions reduce the complexity of the cut formula and commutation steps push cut applications upwards along proofs. The reader can find a detailed proof in [7].

Definition 21 (extended subformula ordering) The relation $\succcurlyeq$ is the preorder on $\mathscr{F}$ defined as follows:

- if A is atomic, then $A \succcurlyeq A$;
- if $F = A \otimes B$, then $F \succcurlyeq A \otimes B, A, B$;
- if $F = A \odot B$, then $F \succcurlyeq A \odot B, A \otimes B$.

Corollary 22 (subformula property) Let $\mathscr{F}_\pi$ be the set of all the formulas occurring in a $\sigma\mathsf{CSI}^-$ proof π, and $\succcurlyeq_\pi$ the restriction of $\succcurlyeq$ to $\mathscr{F}_\pi$. If π is a normal proof and $F \in \mathscr{F}_\pi$ is maximal w.r.t. $\succcurlyeq_\pi$, then F occurs in the endsequent of π.

Proof The argument proceeds on the length of the *normal* proof π. Leaving aside the cut rule, it is easy to check that any other $\sigma\mathsf{CSI}^-$ rule preserves the subformula property. As far as the cut rule is concerned, Theorem 35 guarantees that only *proper* applications of the cut rule may occur in π. Since proper applications of the cut rule just replace a formula $A \otimes B$ with $A \odot B$, by Definition 21 the subformula property turns out to be preserved.

4. Controlled calculi

4.1. Control sets, compatibility, controlled sequents

CSI and σCSI are both substructural calculi so they are able to express resource sensitiveness in the sense that a context $\Gamma \uplus [A]$ is not logically equivalent to the context, say, $\Gamma \uplus [A, A]$ displaying the resource A one more time. In resource-sensitive calculi, the multiplicity of elements matters and this is the reason why we had to consider contexts as *multisets* of formulas and control set as *sets of multisets* of formulas.

In this section, we review some basic results concerning $\mathsf{LK}^{\mathscr{S}}$, a controlled version of the sequent calculus for classical propositional logic LK introduced in [14]. As is well known, classical logic is resource-insensitive [10]. For this reason, it will be more appropriate to view contexts $\Gamma, \Delta, \ldots$ as *sets* of formulas and, accordingly, control sets $\mathbf{S}, \mathbf{T}, \ldots$ as *sets of sets of contexts*. This is not the only difference with our previous definition of control set. Actually, control sets for classical logic no longer need to be restricted to sets of atoms: they just need to be completed w.r.t. conjunction and disjunction according to the following definition.

Definition 23 (control set) A *control set* is a set of sets of logical formulas, set-theoretically completed under conjunction and disjunction as follows:

- $\Gamma \cup \{A \wedge B\} \in \mathbf{S} \Rightarrow \Gamma \cup \{A\} \cup \{B\} \in \mathbf{S}$,
- $\Gamma \cup \{A \vee B\} \in \mathbf{S} \Rightarrow \Gamma \cup \{A\} \in \mathbf{S}$ and $\Gamma \cup \{B\} \in \mathbf{S}$.

We say that $\mathbf{C}_\Gamma$ is the smallest control set $\mathbf{S}$ such that $\Gamma \in \mathbf{S}$. If Γ is the empty context (i.e. $\Gamma = \varnothing$), then we pose $\mathbf{C}_\Gamma = \varnothing$.

Remark 24 According to Definition 23, if $A \in \Lambda \in \mathbf{C}_\Gamma$, then A is a subformula of some formula in Γ. This observation implies the *finiteness* of the control set $\mathbf{C}_\Gamma$ for any context Γ.

Example 25 We give some examples to illustrate Definition 23.

$$\mathbf{C}_{p \wedge (q \vee p)} = \{\{p \wedge (q \vee p)\}, \{p, q \vee p\}, \{p, q\}, \{p\}\}$$

$$\mathbf{C}_{p \vee q, r \wedge s} = \{\{p \vee q, r \wedge s\}, \{p \vee q, r, s\}, \{p, r \wedge s\}, \{q, r \wedge s\}, \{p, r, s\}, \{q, r, s\}\}$$

$$\mathbf{C}_{p\vee(q\wedge r)} = \{\{p\vee(q\wedge r)\},\{p\},\{q\wedge r\},\{q,r\}\}.$$

Definition 26 (compatibility) A context Γ is *compatible* with a control set $\mathbf{S}$, in symbols $\Gamma \parallel \mathbf{S}$, if, for all $\Sigma \in \mathbf{C}_\Gamma$ and all $\Lambda \in \mathbf{S}$, $\Lambda \nsubseteq \Sigma$.

Example 27 According to Definition 23 and Example 25, we have:

$$\{p\vee q, r\wedge s\} \parallel \{\{p,q,r,s\}\}$$

$$\{p\vee q, r\wedge s\} \nparallel \{\{q,r,s\}\}$$

$$\{p\vee q, r\wedge s\} \nparallel \{\{p,r,s\}\}.$$

Remark 28 For any context Γ: $\Gamma \parallel \varnothing$.

The following theorem establishes some basic facts about control sets and their relative notion of compatibility.

Theorem 29

1. If $\Gamma\cup\Delta \parallel \mathbf{S}$ and $\mathbf{T}\subseteq\mathbf{S}$, then $\Delta \parallel \mathbf{T}$.
2. $\Gamma\cup\{A\} \parallel \mathbf{S}$ iff $\Gamma\cup\{A\}\cup\{A\} \parallel \mathbf{S}$.
3. $\Gamma\cup\{A\wedge B\} \parallel \mathbf{S}$ iff $\Gamma\cup\{A\}\cup\{B\} \parallel \mathbf{S}$.
4. $\Gamma\cup\{A\vee B\} \parallel \mathbf{S}$ iff $\Gamma\cup\{A\} \parallel \mathbf{S}$ and $\Gamma\cup\{B\} \parallel \mathbf{S}$.

Proof See [14].

Definition 30 (controlled sequent, soundness) A *controlled sequent* is a standard sequent $\Gamma\vdash\Delta$ with attached:

- a control set $\mathbf{S}$,
- a context Σ called *repository*.

Controlled sequents will be expressed as follows:

$$\Sigma \mid \Gamma \vdash_{\mathbf{S}} \Delta.$$

When the repository stores no formula, we will simply omit it and write:

$$\cdot \mid \Gamma \vdash_{\mathbf{S}} \Delta.$$

The sequent $\Sigma \mid \Gamma \vdash_{\mathbf{S}} \Delta$ is said to be *sound* in case $\Sigma\cup\Gamma \parallel \mathbf{S}$.

Let us now provide an informal account of what control sets and controlled calculi are meant to be. As we said, control sets are concerns with gathering all the contexts which are supposed to block a certain derivation. According to Definition 23, 'forbidden contexts' have to be set-theoretically completed in order to constitute a control set and be thus amenable to a logical treatment. Completion under conjunction is obvious: if $\Gamma \cup \{A \wedge B\}$ represents a 'forbidden context', then $\Gamma \cup \{A\} \cup \{B\}$ should be considered 'forbidden' as well. In other words, the conjunction operator fails to return an actually new resource from A and B, but it pairs them, so that the formula $A \wedge B$ records this pairing operation. Thus:

$$\{\Gamma \cup \{A \wedge B\}, \Gamma \cup \{A\} \cup \{B\}\} \subseteq \mathbf{C}_{\Gamma \cup \{A \wedge B\}}.$$

The disjunction connective comes with an exclusive meaning. If $\Gamma \cup \{A \vee B\}$ is on a blacklist of contexts, then it seems quite reasonable to put both $\Gamma \cup \{A\}$ and $\Gamma \cup \{B\}$ on the same blacklist. Thus:

$$\{\Gamma \cup \{A \vee B\}, \Gamma \cup \{A\}, \Gamma \cup \{B\}\} \subseteq \mathbf{C}_{\Gamma \cup \{A \vee B\}}.$$

The exclusive feature of disjunction emerges from the observation that the context $\Gamma \cup \{A\} \cup \{B\}$ does not necessarily appear among the 'forbidden contexts'. This absence is unproblematic since any context containing $\{\Gamma, A, B\}$ will be blocked as it contains both the subsets $\{\Gamma, A\}$ and $\{\Gamma, B\}$. Anyway, the effect of dealing with an inclusive disjunction can be easily recovered by taking the union of the two control sets induced by $\Gamma \cup \{A \wedge B\}$ and $\Gamma \cup \{A \vee B\}$. In this way, we get:

$$\{\Gamma \cup \{A\} \cup \{B\}, \Gamma \cup \{A\}, \Gamma \cup \{B\}\} \subseteq \mathbf{C}_{\Gamma \cup \{A \vee B\}} \cup \mathbf{C}_{\Gamma \cup \{A \wedge B\}}.$$

Roughly speaking, a system of control sets consists in a set of functions that assign to each atomic axiom a control set and dictate how control sets have to be transmitted throughout inference steps. In other words, $\mathscr{S}$ grows out of the assignment of a control set $\mathscr{S}(p)$ to each atom of p, so that the corresponding axiom is:

$$\frac{}{\cdot \mid p \vdash_{\mathscr{S}(p)} p}\ ax.$$

Notice that uniquely atomic axioms, *i.e.*, axioms introducing atomic propositions, are authorized. Moreover, the general task of $\mathscr{S}$ is to indicate how to combine and transform control sets and repositories along derivations. Let $\mathscr{L}$ be a logic in a two-sided sequent calculus formulation.

The essential idea is that *each single application of the rules of $\mathscr{L}$ along derivations has to preserve, besides validity, the soundness of the proved sequent*. As an example, let us consider the following controlled version of the standard weakening rule:

$$\frac{\Sigma \mid \Gamma \vdash_{\mathbf{S}} A}{\Sigma \mid \Gamma, B \vdash_{\mathbf{S}} A}\ \textit{weak} \vdash.$$

When considered in a classical framework, this rule is clearly sound. However, this does not enable us to draw the conclusion $\Sigma \mid \Gamma, B \vdash_{\mathbf{S}} A$ from the premise $\Sigma \mid \Gamma \vdash_{\mathbf{S}} A$: the compatibility between the wider context $\Sigma \cup \Gamma \cup \{B\}$ and the control set $\mathbf{S}$ needs to be verified.

Although our present focus is on classical logic, it is worthy noting that a system of control sets $\mathscr{S}$ can be associated with any *two-sided* sequent calculus. A general characterization of the notion of system of control sets can be found in [14].

4.2. The controlled calculus $\mathsf{LK}^{\mathscr{S}}$

The system $\mathsf{LK}^{\mathscr{S}}$ is obtained by imposing to the Gentzen sequent system for classical logic what we called in [14] a *minimal system of control sets*. The way in which the system $\mathscr{S}$ transmits control sets and repositories along derivations is described in Table 1. In a nutshell:

1. in all unary inference rules, $\mathscr{S}$ transmits the *same* control set from the upper controlled sequent to the lower one with the exception of the structural rule σ (which arbitrarily 'expands' the control set);
2. in binary rules, $\mathscr{S}$ attaches to the lower sequent the *union* of the control sets assigned to the upper sequents.

Furthermore, concerning atomic axioms, we require that:

1. For all atoms p, $p \notin \bigcup \mathscr{S}(p)$ (where $\mathscr{S}(p)$ is the control set attached by $\mathscr{S}$ to the axiom introducing the atomic letter p).

2. The attached repository is always the empty context.

The rationale for the first condition will clearly emerge later (cf. Remark 33).

Definition 31 (proof, paraproof) Consider a rooted, finitely branching tree π whose nodes are sequents of $\mathsf{LK}^{\mathscr{S}}$, and such that it is recursively built up from axioms by means of the rules of $\mathsf{LK}^{\mathscr{S}}$. If each sequent in π is sound, π is said to be a *proof* of $\mathsf{LK}^{\mathscr{S}}$, otherwise π is called a *paraproof*.

Previous definition places upon a proof π of $\mathsf{LK}^{\mathscr{S}}$ two independent requirements. The first, *logical validity*, standardly establishes that each one of the deductive steps performed in π has to be accomplished in accordance with the rules of $\mathsf{LK}^{\mathscr{S}}$. The second requirement, *soundness*, tells that each single application of the rules in π must be *soundness preserving* so that each sequent occurring in π is sound with respect to the control set attached to it. The fulfilment of the latter condition is what turns a paraproof into a $\mathsf{LK}^{\mathscr{S}}$ proof.

Example 32 Let $\mathscr{S}$ be such that $\{p\} \in \mathscr{S}(q)$. Inasmuch as the sequent

$$q \mid p, p \to q \vdash_{\mathrm{S}\cup\{\{p\},\dots\}} q$$

is unsound, the following derivation constitutes a paraproof.

$$\dfrac{\dfrac{\dfrac{}{\cdot \mid p \vdash_{\mathscr{S}(p)} p}\,ax. \qquad \dfrac{}{\cdot \mid q \vdash_{\{\{p\},\dots\}} q}\,ax.}{q \mid p, p \to q \vdash_{\mathscr{S}(p)\cup\{\{p\},\dots\}} q}\,\to\vdash}{p, q \mid p \to q \vdash_{\mathscr{S}(p)\cup\{\{p\},\dots\}} p \to q}\,\vdash\to$$

Remark 33 The previous example shows that equalities (i.e. the provable equivalence of any formula with itself) are not necessarily guaranteed in controlled calculi. With a little ingenuity, however, we can at least preserve equalities involving atoms by additionally requiring that, for any atom p, $p \notin \bigcup \mathscr{S}(p)$.

4.3. Cut-elimination and subformula property

Lemma 34 Any cut-free paraproof in $\mathsf{LK}^{\mathscr{S}}$ is a proof if and only if its endsequent is sound.

Axiom:

$$\frac{}{\cdot \mid p \vdash_{\mathscr{S}(p)} p}\ ax. \qquad \text{with } p \text{ atomic}$$

Cut rule:

$$\frac{\Sigma \mid \Gamma \vdash_{\mathbf{S}} A, \Delta \qquad \Sigma' \mid \Gamma', A \vdash_{\mathbf{T}} \Delta'}{\Sigma, \Sigma' \mid \Gamma, \Gamma' \vdash_{\mathbf{S} \cup \mathbf{T}} \Delta, \Delta'}\ cut$$

Structural rules:

$$\frac{\Sigma \mid \Gamma \vdash_{\mathbf{S}} \Delta}{\Sigma \mid \Gamma \vdash_{\mathbf{S} \cup \mathbf{T}} \Delta}\ \sigma \qquad \frac{\Sigma \mid \Gamma \vdash_{\mathbf{S}} \Delta}{\Sigma, A \mid \Gamma \vdash_{\mathbf{S}} \Delta}\ \rho$$

Logical rules:

$$\frac{\Sigma \mid \Gamma, A, B \vdash_{\mathbf{S}} \Delta}{\Sigma \mid \Gamma, A \wedge B \vdash_{\mathbf{S}} \Delta}\ \wedge\vdash \qquad \frac{\Sigma \mid \Gamma \vdash_{\mathbf{S}} \Delta, A \qquad \Sigma' \mid \Gamma' \vdash_{\mathbf{T}} \Delta', B}{\Sigma, \Sigma' \mid \Gamma, \Gamma' \vdash_{\mathbf{S} \cup \mathbf{T}} \Delta, \Delta', A \wedge B}\ \vdash\wedge$$

$$\frac{\Sigma \mid \Gamma, A \vdash_{\mathbf{S}} \Delta \qquad \Sigma' \mid \Gamma', B \vdash_{\mathbf{T}} \Delta'}{\Sigma, \Sigma' \mid \Gamma, \Gamma', A \vee B \vdash_{\mathbf{S} \cup \mathbf{T}} \Delta, \Delta'}\ \vee\vdash \qquad \frac{\Sigma \mid \Gamma \vdash_{\mathbf{S}} \Delta, A, B}{\Sigma \mid \Gamma \vdash_{\mathbf{S}} \Delta, A \vee B}\ \vdash\vee$$

$$\frac{\Sigma \mid \Gamma \vdash_{\mathbf{S}} A, \Delta \qquad \Sigma' \mid \Gamma', B \vdash_{\mathbf{T}} \Delta'}{\Sigma, \Sigma', B \mid \Gamma, \Gamma', A \to B \vdash_{\mathbf{S} \cup \mathbf{T}} \Delta, \Delta'}\ \to\vdash \qquad \frac{\Sigma \mid \Gamma, A \vdash_{\mathbf{S}} B, \Delta}{\Sigma, A \mid \Gamma \vdash_{\mathbf{S}} A \to B, \Delta}\ \vdash\to$$

$$\frac{\Sigma \mid \Gamma \vdash_{\mathbf{S}} A, \Delta}{\Sigma \mid \Gamma, \neg A \vdash_{\mathbf{S}} \Delta}\ \neg\vdash \qquad \frac{\Sigma \mid \Gamma, A \vdash_{\mathbf{S}} \Delta}{\Sigma, A \mid \Gamma \vdash_{\mathbf{S}} \neg A, \Delta}\ \vdash\neg$$

TABLE 3: *The controlled sequent calculus* $\mathsf{LK}^{\mathscr{S}}$

Proof Let r be any $\mathsf{LK}^{\mathscr{S}}$ rule, except the cut rule. Consider the following configuration:

$$\frac{\Sigma \mid \Gamma \vdash_{\mathbf{S}} \Delta \qquad \dots}{\Sigma' \mid \Gamma' \vdash_{\mathbf{S}'} \Delta'}\, r$$

where $\Sigma \mid \Gamma \vdash_{\mathbf{S}} \Delta$ is either the only premise of r, or one (not necessarily the first) of the two premises. It suffices to remark that, for each r, $\{\Sigma, \Gamma\} \subseteq \{\Sigma', \Gamma'\}$ and $\mathbf{S} \subseteq \mathbf{S}'$. By Theorem 29(1), we get the soundness of the premise $\Sigma \mid \Gamma \vdash_{\mathbf{S}} \Delta$ from that of the conclusion $\Sigma' \mid \Gamma' \vdash_{\mathbf{S}'} \Delta'$. In the case of left-contraction, left-conjuction and left-disjunction, combine Theorem 29(1) with Theorems 29(2), (3) and (4), respectively.

Theorem 35 *(Cut-elimination).* Any provable $\mathsf{LK}^{\mathscr{S}}$ sequent has a cut-free proof.

Proof The tricky point about cut-elimination in controlled calculi is that reductions steps do not necessarily preserve the soundness of sequents, that is, the normalization procedure may turn proofs into paraproofs. The key point here is that normal forms of a $\mathsf{LK}^{\mathscr{S}}$ proof is always a proof, though the intermediate proofs in the reduction chain might not be sound. In particular, the proof consists in the following two steps.

1. We show, at first, how the standard cut-elimination algorithm for LK can be tailored for controlled calculi. More details can be found in [14].

2. Second, we show that the normal derivation π', obtained from a $\mathsf{LK}^{\mathscr{S}}$ proof π by means of the cut-elimination algorithm outlined [14], is indeed a proof, namely each one of its sequents is sound. By hypothesis, π is a proof, so its endsequent is sound. By Lemma 34, π' is a proof as well.

Corollary 36 *(Subformula Property).* If a sequent is provable in $\mathsf{LK}^{\mathscr{S}}$, then it is provable analytically, namely by means of a derivation in which all formulas are subformulas of those occurring in the end sequent.

Proof As usual, by induction on the length of cut-free proofs.

5. Conclusions

The results surveyed in this paper have to be framed within an ongoing research programme whose overall aim is to investigate the extent to which it is possible to accommodate distinctive features of classical and non-classical — especially non-monotonic — logics within a disciplined proof-theoretical framework. In some respects, one may regard our perspective — especially general controlled systems as they have been proposed in Section 4 — as similar in spirit to that of Makinson, who is concerned with 'bridging the gap' between classical and non-monotonic logic by means of a logical 'continuous' [12]. However, Makinson's work is essentially semantical, being focused on the notion of consequence relation. Here, we propose instead some general ways of decorating sequents and proofs of well-known logical systems (including substructural calculi [6, 5]) so as to control the monotonicity of their consequence relation depending on the context.

Many intriguing questions about controlled calculi and their logical nature are still open. Let us mention just one of the most abstract and interesting. Being defined as a set of sets (or multisets if the calculus to be controlled is resource-sensitive), the notion of control set appears to involve a sort of higher order conceptualization in disguise. For this reason, it would be interesting to evaluate the possibility of reproducing the control set device by resorting to second-order calculi, so as to avoid the 'external' decorations of sequents. This would put us in the position to raise the question of whether the natural logical level for dealing with controlled monotonicity is the second-order one.

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