The representation of Boolean algebras in the spotlight of a proof checker

Companion proof-scenario

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> This pretty-printed scenario reflects an early phase in the formal development of the proof of Stone's theorem on the representation of Boolean algebras: only the algebraic version of that theorem is proved here. Moreover, this scenario takes many preliminary propositions (including the most crucial Zorn's lemma) for granted, whose formalized proofs have been published elsewhere.
> As of today, the full scenario of Stone's theorem—in its topological form—is available in keyboard-oriented format at
> `http://www2.units.it/eomodeo/StoneReprScenario.txt`.

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1 A warm-up exercise: Representation of partial orderings

Our next theory shows how to represent any non-strict partial ordering as an inclusion ordering. This is a sort of analogue, in the small, of the Stone representation theorem for Boolean algebras, whose proof will be developed soon.

THEORY $\mathsf{pord}\big(\mathsf{dd}, \mathsf{Le}(\mathsf{U}, \mathsf{V})\big)$

|| dychotomy

$\big\langle \forall x, y \mid \{x, y\} \subseteq \mathsf{dd} \rightarrow \big(\mathsf{Le}(x, y) \,\&\, \mathsf{Le}(y, x) \leftrightarrow x = y\big)\big\rangle$

|| transitivity

$\big\langle \forall x, y, z \mid \{x, y, z\} \subseteq \mathsf{dd} \rightarrow \mathsf{Le}(x, y) \,\&\, \mathsf{Le}(y, z) \rightarrow \mathsf{Le}(x, z)\big\rangle$

END pord

ENTER_THEORY pord

DEF pord_1: [standard isomorphism between a partial ordering and an inclusion ordering] $\mathsf{polso}_\Theta =_{\mathrm{Def}} \{[x, \{v \in \mathsf{dd} \mid \mathsf{Le}(v, x)\}] : x \in \mathsf{dd}\}$

-- reflexivity of partial orderings

THM pord_0. $X \in \mathsf{dd} \rightarrow \mathsf{Le}(X, X) \,\&\, \mathsf{polso}_\Theta \restriction X = \{v \in \mathsf{dd} \mid \mathsf{Le}(v, X)\}$. PROOF:

Suppose_not(x_0) $\Rightarrow$ AUTO

Suppose $\Rightarrow$ $\mathsf{polso}_\Theta \restriction x_0 \neq \{v \in \mathsf{dd} \mid \mathsf{Le}(v, x_0)\}$

ELEM $\Rightarrow$ *Stat1*: $[x_0, \{v \in \mathsf{dd} \mid \mathsf{Le}(v, x_0)\}]^{[1]} = x_0 \,\&\, [x_0, \{v \in \mathsf{dd} \mid \mathsf{Le}(v, x_0)\}]^{[2]} = \{v \in \mathsf{dd} \mid \mathsf{Le}(v, x_0)\}$

$\big\langle \{[x, \{v \in \mathsf{dd} \mid \mathsf{Le}(v, x)\}] : x \in \mathsf{dd}\}, [x_0, \{v \in \mathsf{dd} \mid \mathsf{Le}(v, x_0)\}]\big\rangle \hookrightarrow T74$ (*Stat1*$\star$) $\Rightarrow$

$[x_0, \{v \in \mathsf{dd} \mid \mathsf{Le}(v, x_0)\}] \in \{[x, \{v \in \mathsf{dd} \mid \mathsf{Le}(v, x)\}] : x \in \mathsf{dd}\} \rightarrow$

$\{[x, \{v \in \mathsf{dd} \mid \mathsf{Le}(v, x)\}] : x \in \mathsf{dd}\} \restriction [x_0, \{v \in \mathsf{dd} \mid \mathsf{Le}(v, x_0)\}]^{[1]} = \{v \in \mathsf{dd} \mid \mathsf{Le}(v, x_0)\}$

Use_def(polso_Θ) $\Rightarrow$ $\mathsf{polso}_\Theta = \{[x, \{v \in \mathsf{dd} \mid \mathsf{Le}(v, x)\}] : x \in \mathsf{dd}\}$

EQUAL $\langle$*Stat1*$\rangle$ $\Rightarrow$ $[x_0, \{v \in \mathsf{dd} \mid \mathsf{Le}(v, x_0)\}] \in \{[x, \{v \in \mathsf{dd} \mid \mathsf{Le}(v, x)\}] : x \in \mathsf{dd}\} \rightarrow \mathsf{polso}_\Theta \restriction x_0 = \{v \in \mathsf{dd} \mid \mathsf{Le}(v, x_0)\}$

Suppose $\Rightarrow$ *Stat2*: $[x_0, \{v \in \mathsf{dd} \mid \mathsf{Le}(v, x_0)\}] \notin \{[x, \{v \in \mathsf{dd} \mid \mathsf{Le}(v, x)\}] : x \in \mathsf{dd}\}$

$\langle x_0 \rangle \hookrightarrow$ *Stat2*$(\star)$ $\Rightarrow$ false; Discharge $\Rightarrow$ false

Discharge $\Rightarrow$ AUTO

Assump $\Rightarrow$ *Stat3*: $\big\langle \forall x, y \mid \{x, y\} \subseteq \mathsf{dd} \rightarrow \big(\mathsf{Le}(x, y) \,\&\, \mathsf{Le}(y, x) \leftrightarrow x = y\big)\big\rangle$

$\langle x_0, x_0 \rangle \hookrightarrow$ *Stat3* $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- order homomorphism property

THM pord_1. $\{X, Y\} \subseteq \mathsf{dd} \rightarrow \big(\mathsf{Le}(X, Y) \leftrightarrow \mathsf{polso}_\Theta \restriction X \subseteq \mathsf{polso}_\Theta \restriction Y\big)$. PROOF:

Suppose_not(x_0, y_0) $\Rightarrow$ AUTO

$\langle x_0\rangle\hookrightarrow T\text{pord}_0 \Rightarrow$ *Stat1*: $\text{polso}_\Theta\restriction x_0 = \{v \in dd \mid Le(v, x_0)\}$
$\langle y_0\rangle\hookrightarrow T\text{pord}_0 \Rightarrow$ $\text{polso}_\Theta\restriction y_0 = \{v \in dd \mid Le(v, y_0)\}$
Suppose $\Rightarrow$ *Stat2*: $\text{polso}_\Theta\restriction x_0 \not\subseteq \text{polso}_\Theta\restriction y_0$
$\langle z_0\rangle\hookrightarrow$ *Stat2(Stat1⋆)* $\Rightarrow$ *Stat3*: $z_0 \in \{v : v \in dd \mid Le(v, x_0)\}\ \&\ z_0 \notin \{v : v \in dd \mid Le(v, y_0)\}$
$\langle z_1, z_1\rangle\hookrightarrow$ *Stat3(Stat3⋆)* $\Rightarrow$ $z_1 \in dd\ \&\ Le(z_1, x_0)\ \&\ \neg Le(z_1, y_0)$
Assump $\Rightarrow$ *Stat4*: $\langle \forall x, y, z \mid \{x, y, z\} \subseteq dd \to Le(x, y)\ \&\ Le(y, z) \to Le(x, z)\rangle$
$\langle z_1, x_0, y_0\rangle\hookrightarrow$ *Stat4(⋆)* $\Rightarrow$ false; Discharge $\Rightarrow$ $\neg Le(x_0, y_0)\ \&\ \{v : v \in dd \mid Le(v, x_0)\} \subseteq \{v : v \in dd \mid Le(v, y_0)\}$
Suppose $\Rightarrow$ *Stat6*: $x_0 \notin \{v \in dd \mid Le(v, x_0)\}$
$\langle\ \rangle\hookrightarrow$ *Stat6(⋆)* $\Rightarrow$ $x_0 \in dd\ \&\ \neg Le(x_0, x_0)$
$\langle x_0\rangle\hookrightarrow T\text{pord}_0$ *(Stat6⋆)* $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat7*: $x_0 \in \{v \in dd \mid Le(v, y_0)\}\ \&\ \neg Le(x_0, y_0)$
$\langle\ \rangle\hookrightarrow$ *Stat7(Stat7⋆)* $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- injectivity of the homomorphism

THM pord_2. $1\text{–}1(\text{polso}_\Theta)\ \&\ \mathbf{domain}(\text{polso}_\Theta) = dd$. PROOF:
Suppose_not() $\Rightarrow$ AUTO

> Arguing by contradiction, let us assume that the claim is false. Then, since by its very definition polso_Θ is a single-valued map and has the domain indicated in the claim, there must be distinct pairs p, q in polso_Θ whose second components coincide.

Use_def(polso_Θ) $\Rightarrow$ $Svm(\text{polso}_\Theta)\ \&\ \mathbf{domain}(\text{polso}_\Theta) = dd$
Use_def(1–1) $\Rightarrow$ *Stat1*: $\neg\langle \forall p \in \text{polso}_\Theta, q \in \text{polso}_\Theta \mid p^{[2]} = q^{[2]} \to p = q\rangle$
$\langle p, q\rangle\hookrightarrow$ *Stat1(Stat1⋆)* $\Rightarrow$ *Stat2*: $p, q \in \text{polso}_\Theta\ \&\ p^{[2]} = q^{[2]}\ \&\ p \neq q$

> It follows from the definition of polso_Θ that if x_0, x_1 are the respective first components of p, q, then the corresponding images $\{v \in dd \mid Le(v, x_0)\} = \text{polso}_\Theta\restriction x_0$ and $\{v \in dd \mid Le(v, x_1)\} = \text{polso}_\Theta\restriction x_1$ coincide.

Use_def(polso_Θ) $\Rightarrow$ *Stat3*: $p, q \in \{[x, \{v \in dd \mid Le(v, x)\}] : x \in dd\}$
$\langle x_0, x_1\rangle\hookrightarrow$ *Stat3(Stat3⋆)* $\Rightarrow$ *Stat4*: $x_0, x_1 \in dd\ \&\ p = [x_0, \{v \in dd \mid Le(v, x_0)\}]\ \&\ q = [x_1, \{v \in dd \mid Le(v, x_1)\}]$
(*Stat4⋆*)ELEM $\Rightarrow$ $[x_0, \{v \in dd \mid Le(v, x_0)\}]^{[1]} = x_0\ \&\ [x_0, \{v \in dd \mid Le(v, x_0)\}]^{[2]} = \{v \in dd \mid Le(v, x_0)\}\ \&$
$[x_1, \{v \in dd \mid Le(v, x_1)\}]^{[1]} = x_1\ \&\ [x_1, \{v \in dd \mid Le(v, x_1)\}]^{[2]} = \{v \in dd \mid Le(v, x_1)\}$
EQUAL $\langle$*Stat2*$\rangle$ $\Rightarrow$ $\{v \in dd \mid Le(v, x_0)\} = \{v \in dd \mid Le(v, x_1)\}$
Suppose $\Rightarrow$ $x_0 = x_1$
EQUAL $\langle$*Stat2*$\rangle$ $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO
$\langle x_0\rangle\hookrightarrow T\text{pord}_0 \Rightarrow$ AUTO
$\langle x_1\rangle\hookrightarrow T\text{pord}_0 \Rightarrow$ AUTO

This implies, by the preceding Theorem pord_1, that $\mathsf{Le}(x_0, x_1)$ and $\mathsf{Le}(x_1, x_0)$ hold together, and hence that $x_0 = x_1$, conflicting with $p \neq q$, and hence leading to the desired contradiction.

$\langle x_0, x_1 \rangle \hookrightarrow T\mathsf{pord}_1$ $(Stat4\star) \Rightarrow$ $\mathsf{Le}(x_0, x_1)$
$\langle x_1, x_0 \rangle \hookrightarrow T\mathsf{pord}_1$ $(Stat4\star) \Rightarrow$ $\mathsf{Le}(x_1, x_0)$
Assump $\Rightarrow$ $Stat9 : \langle \forall x, y \mid \{x, y\} \subseteq \mathsf{dd} \rightarrow (\mathsf{Le}(x, y) \,\&\, \mathsf{Le}(y, x) \leftrightarrow x = y)\rangle$
$\langle x_0, x_1 \rangle \hookrightarrow Stat9(Stat4\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

ENTER_THEORY Set_theory

DISPLAY pord

THEORY $\mathsf{pord}(\mathsf{dd}, \mathsf{Le}(U, V))$
$\langle \forall x, y \mid \{x, y\} \subseteq \mathsf{dd} \rightarrow (\mathsf{Le}(x, y) \,\&\, \mathsf{Le}(y, x) \leftrightarrow x = y)\rangle$
$\langle \forall x, y, z \mid \{x, y, z\} \subseteq \mathsf{dd} \rightarrow \mathsf{Le}(x, y) \,\&\, \mathsf{Le}(y, z) \rightarrow \mathsf{Le}(x, z)\rangle$
$\Rightarrow (\mathsf{polso}_\Theta)$
$\mathsf{polso}_\Theta = \{[x, \{v \in \mathsf{dd} \mid \mathsf{Le}(v, x)\}] : x \in \mathsf{dd}\}$
$\langle \forall x \mid x \in \mathsf{dd} \rightarrow \mathsf{Le}(x, x) \,\&\, \mathsf{polso}_\Theta \restriction x = \{v \in \mathsf{dd} \mid \mathsf{Le}(v, x)\}\rangle$
$\langle \forall x, y \mid \{x, y\} \subseteq \mathsf{dd} \rightarrow (\mathsf{Le}(x, y) \leftrightarrow \mathsf{polso}_\Theta \restriction x \subseteq \mathsf{polso}_\Theta \restriction y)\rangle$
$1\text{–}1(\mathsf{polso}_\Theta) \,\&\, \mathbf{domain}(\mathsf{polso}_\Theta) = \mathsf{dd}$
END pord

2 Boolean rings

Preliminary to the theory of Boolean algebras, our next theory introduces an abstract algebraic structure which differs from Boolean algebra only because it may lack a multiplicative unit. A ring is said to be *Boolean* when each X belonging to its domain of support is self-inverse relative to addition and idempotent relative to multiplication:

$$\mathsf{X} + \mathsf{X} = \emptyset\,,$$

$$\mathsf{X} * \mathsf{X} = \mathsf{X}\,.$$

These two laws confer such a richness of structure to the algebraic variety of Boolean rings to cause even embarrassment, to say it with Halmos. For example, the first of them makes it superfluous to postulate the commutativity of addition (since it implies it); the second enables one to easily prove that multiplication is commutative. When a multiplicative unit 1 is available, it is customary to say that the Boolean ring is a *Boolean algebra*. The richness in structure of Boolean rings emerges also from the observation that the relation

$$\mathsf{leq}(\mathsf{X},\mathsf{Y}) =_{\mathrm{Def}} \mathsf{X} * \mathsf{Y} = \mathsf{X}$$

is a partial ordering, in which every pair X,Y of elements admits greatest lower bound $\mathsf{glb}(\mathsf{X},\mathsf{Y}) =_{\mathrm{Def}} \mathsf{X} * \mathsf{Y}$ and least upper bound $\mathsf{lub}(\mathsf{X},\mathsf{Y}) =_{\mathrm{Def}} \mathsf{X} * \mathsf{Y} + \mathsf{X} + \mathsf{Y}$. In this ordering the additive unit 0 acts as the minimum and — when available — 1 acts as maximum. Historically, Boolean algebras were first classed as lattices satisfying peculiar features (namely, being distributive and complemented). Emphasis was, in the original approach, placed on the operations $\mathsf{glb}, \mathsf{lub}$; the algebraic kinship with numerical rings was noticed later. From the algebraic viewpoint — the one which we will favor in the ongoing — the complementation operation is $\overline{\mathsf{X}} =_{\mathrm{Def}} 1 + \mathsf{X}$.

THEORY $\mathsf{booleanRing}(\mathsf{bb}, \mathsf{U}\cdot\mathsf{V}, \mathsf{U}\div\mathsf{V})$

non-vacuity assumption

$\mathsf{bb} \neq \emptyset$

closure properties

$\langle \forall x, y \mid \{x, y\} \subseteq bb \rightarrow x \cdot y \in bb \rangle$
$\langle \forall x, y \mid \{x, y\} \subseteq bb \rightarrow x \div y \in bb \rangle$

‖ associativity laws

$\langle \forall x, y, z \mid \{x, y, z\} \subseteq bb \rightarrow x \cdot (y \cdot z) = (x \cdot y) \cdot z \rangle$
$\langle \forall x, y, z \mid \{x, y, z\} \subseteq bb \rightarrow x \div (y \div z) = (x \div y) \div z \rangle$

‖ distributivity law

$\langle \forall x, y, z \mid \{x, y, z\} \subseteq bb \rightarrow (x \div y) \cdot z = z \cdot y \div z \cdot x \rangle$

‖ additive zero

$\langle \forall x, y \mid \{x, y\} \subseteq bb \rightarrow x \div x = y \div y \rangle$

‖ self-annichilation law

$\langle \forall x, y \mid \{x, y\} \subseteq bb \rightarrow x \div (y \div x) = y \rangle$

‖ idempotency of multiplication

$\langle \forall x \mid x \in bb \rightarrow x \cdot x = x \rangle$
END booleanRing

ENTER_THEORY booleanRing

DEF booleanRing$_0$: [additive zero] $\quad zz_\ominus \quad =_{\text{Def}} \quad \mathbf{arb}(bb) \div \mathbf{arb}(bb)$

-- additive zero law
THM booleanRing$_1$. $(X \in bb \rightarrow X \div X = zz_\ominus \;\&\; X \div zz_\ominus = X \;\&\; zz_\ominus \div X = X) \;\&\; zz_\ominus \in bb$. PROOF:
Suppose_not$(x_0) \Rightarrow$ AUTO

‖ For, assuming the contrary to hold for some x_0 in bb, one would have either $x_0 \div x_0 \neq \mathbf{arb}(bb) \div \mathbf{arb}(bb)$, conflicting with an assumptions of this THEORY, or $x_0 \div x_0 \div x_0 \neq x_0$, conflicting with another assumption.

Assump $\Rightarrow$ *Stat1* : $\langle \forall x, y \mid \{x, y\} \subseteq bb \rightarrow x \div y \in bb \rangle \;\&\;$ *Stat2* : $\langle \forall x, y \mid \{x, y\} \subseteq bb \rightarrow x \div x = y \div y \rangle \;\&\; bb \neq \emptyset$
Use_def$(zz_\ominus) \Rightarrow$ $zz_\ominus = \mathbf{arb}(bb) \div \mathbf{arb}(bb)$
$\langle \mathbf{arb}(bb), \mathbf{arb}(bb), x_0, \mathbf{arb}(bb) \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ $zz_\ominus \in bb \;\&\; x_0 \div x_0 = zz_\ominus$

EQUAL $\Rightarrow$ $x_0 \div zz_\Theta = x_0 \div (x_0 \div x_0)$ & $zz_\Theta \div x_0 = x_0 \div x_0 \div x_0$
Assump $\Rightarrow$ *Stat3*: $\langle \forall x, y \mid \{x, y\} \subseteq bb \rightarrow x \div (y \div x) = y \rangle$ & $\langle \forall x, y, z \mid \{x, y, z\} \subseteq bb \rightarrow x \div (y \div z) = (x \div y) \div z \rangle$
$\langle x_0, x_0, x_0, x_0, x_0 \rangle \hookrightarrow$ *Stat3* $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- commutativity of addition

THM booleanRing$_2$. $\{X, Y\} \subseteq bb \rightarrow X \div Y = Y \div X$. PROOF:
Suppose_not$(x_0, y_0) \Rightarrow$ AUTO
Assump $\Rightarrow$ *Stat1*: $\langle \forall x, y \mid \{x, y\} \subseteq bb \rightarrow x \div (y \div x) = y \rangle$ & *Stat2*: $\langle \forall x, y \mid \{x, y\} \subseteq bb \rightarrow x \div y \in bb \rangle$
$\langle y_0, x_0, y_0, x_0 \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ $y_0 \div (x_0 \div y_0) = x_0$ & $y_0 \div x_0 \in bb$
$\langle x_0, y_0 \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ $x_0 \div (y_0 \div x_0) = y_0$
$\langle y_0 \div x_0, y_0 \div x_0 \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ $y_0 \div x_0 \div \big(y_0 \div x_0 \div (y_0 \div x_0)\big) = y_0 \div x_0$

Writing $+, x, y$ instead of $\div, x_0, y_0$ for brevity, note that $y+(x+y)+(x+(y+x)) = y+x+y+(x+(y+x))$, $y+x+y+(x+(y+x)) = y+x+\big(y+(x+(y+x))\big)$, and $y+x+\big(y+(x+(y+x))\big) = y+x+(y+x+(y+x))$ by associativity, and therefore $y+(x+y)+(x+(y+x)) = y+x+(y+x+(y+x))$ where, as already seen in this proof, the left-hand side equals $x+y$ and the right-hand side equals $y+x$.

Suppose $\Rightarrow$ $y_0 \div (x_0 \div y_0) \div \big(x_0 \div (y_0 \div x_0)\big) \neq y_0 \div x_0 \div \big(y_0 \div x_0 \div (y_0 \div x_0)\big)$
Assump $\Rightarrow$ *Stat3*: $\langle \forall x, y, z \mid \{x, y, z\} \subseteq bb \rightarrow x \div (y \div z) = (x \div y) \div z \rangle$
$\langle y_0, x_0, y_0 \rangle \hookrightarrow$ *Stat3* $\Rightarrow$ $y_0 \div (x_0 \div y_0) = (y_0 \div x_0) \div y_0$
$\langle x_0, y_0 \div x_0 \rangle \hookrightarrow$ *Stat2* $\Rightarrow$ $x_0 \div (y_0 \div x_0) \in bb$
$\langle y_0 \div x_0, y_0, x_0 \div (y_0 \div x_0) \rangle \hookrightarrow$ *Stat3* $\Rightarrow$ $y_0 \div x_0 \div \big(y_0 \div \big(x_0 \div (y_0 \div x_0)\big)\big) = (y_0 \div x_0 \div y_0) \div \big(x_0 \div (y_0 \div x_0)\big)$
$\langle y_0, x_0, y_0 \div x_0 \rangle \hookrightarrow$ *Stat3* $\Rightarrow$ $y_0 \div (x_0 \div (y_0 \div x_0)) = (y_0 \div x_0) \div (y_0 \div x_0)$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ $y_0 \div (x_0 \div y_0) \div \big(x_0 \div (y_0 \div x_0)\big) = y_0 \div x_0 \div \big(y_0 \div x_0 \div (y_0 \div x_0)\big)$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- commutativity of multiplication

THM booleanRing$_3$. $\{X, Y\} \subseteq bb \rightarrow X \cdot Y = Y \cdot X$. PROOF:
Suppose_not$(x_0, y_0) \Rightarrow$ AUTO

Writing $+, *, x, y, 0$ instead of $\div, \cdot, x_0, y_0, zz_\Theta$ for brevity, we observe that $x+y = (x+y)*(x+y)$, $(x+y)*(x+y) = (x+y)*y+(x+y)*x$, $(x+y)*y+(x+y)*x = y*y+y*x+x*y+x*x$, $y*y+y*x+x*y+x*x = y+y*x+x*y+x$, and hence (by adding $x+y$ to both sides of the resulting equality) $0 = y*x+x*y$. Therefore, as one sees by adding $x*y$ to both sides, $x*y = y*x$. The formal details of this proof follow.

Assump $\Rightarrow$ *Stat1* : $\langle \forall x, y \mid \{x, y\} \subseteq bb \to x \div y \in bb \rangle$ & *Stat2* : $\langle \forall x \mid x \in bb \to x \cdot x = x \rangle$ & *Stat3* : $\langle \forall x, y, z \mid \{x, y, z\} \subseteq bb \to (x \div y) \cdot z = z \cdot y \div z \cdot x \rangle$

$\langle x_0, y_0, x_0 \div y_0, x_0, y_0, x_0 \div y_0 \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ $x_0 \div y_0 \in bb$ & $(x_0 \div y_0) \cdot (x_0 \div y_0) = x_0 \div y_0$ & $x_0 \div y_0 = (x_0 \div y_0) \cdot y_0 \div (x_0 \div y_0) \cdot x_0$

$\langle y_0, x_0, y_0, y_0 \rangle \hookrightarrow$ *Stat2* $\Rightarrow$ $y_0 \cdot y_0 = y_0$ & $(x_0 \div y_0) \cdot y_0 = y_0 \cdot y_0 \div y_0 \cdot x_0$

$\langle x_0, x_0, y_0, x_0 \rangle \hookrightarrow$ *Stat2* $\Rightarrow$ $x_0 \cdot x_0 = x_0$ & $(x_0 \div y_0) \cdot x_0 = x_0 \cdot y_0 \div x_0 \cdot x_0$

$\langle x_0 \div y_0 \rangle \hookrightarrow T$booleanRing$_1$ $\Rightarrow$ $x_0 \div y_0 \div (x_0 \div y_0) = zz_\ominus$

EQUAL $\langle$*Stat1*$\rangle$ $\Rightarrow$ *Stat4* : $zz_\ominus = x_0 \div y_0 \div (y_0 \div y_0 \cdot x_0 \div (x_0 \cdot y_0 \div x_0))$

Assump $\Rightarrow$ *Stat5* : $\langle \forall x, y \mid \{x, y\} \subseteq bb \to x \cdot y \in bb \rangle$ & *Stat6* : $\langle \forall x, y, z \mid \{x, y, z\} \subseteq bb \to x \div (y \div z) = (x \div y) \div z \rangle$

$\langle x_0, y_0 \rangle \hookrightarrow$ *Stat5* $\Rightarrow$ $x_0 \cdot y_0 \in bb$

$\langle x_0 \cdot y_0, x_0 \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ $x_0 \cdot y_0 \div x_0 \in bb$

$\langle y_0, x_0, y_0, y_0 \cdot x_0, x_0 \cdot y_0 \div x_0 \rangle \hookrightarrow$ *Stat5* $\Rightarrow$ $y_0 \cdot x_0 \in bb$ & $y_0 \div (y_0 \cdot x_0 \div (x_0 \cdot y_0 \div x_0)) = (y_0 \div y_0 \cdot x_0) \div (x_0 \cdot y_0 \div x_0)$

EQUAL $\langle$*Stat4*$\rangle$ $\Rightarrow$ *Stat7* : $zz_\ominus = x_0 \div y_0 \div \left(y_0 \div (y_0 \cdot x_0 \div (x_0 \cdot y_0 \div x_0))\right)$

$\langle y_0 \cdot x_0, x_0 \cdot y_0 \div x_0 \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ $y_0 \cdot x_0 \div (x_0 \cdot y_0 \div x_0) \in bb$

$\langle x_0 \div y_0, y_0, y_0 \cdot x_0 \div (x_0 \cdot y_0 \div x_0) \rangle \hookrightarrow$ *Stat6* $\Rightarrow$ $x_0 \div y_0 \div \left(y_0 \div (y_0 \cdot x_0 \div (x_0 \cdot y_0 \div x_0))\right) = x_0 \div y_0 \div y_0 \div (y_0 \cdot x_0 \div (x_0 \cdot y_0 \div x_0))$

$\langle x_0 \div y_0, y_0 \rangle \hookrightarrow T$booleanRing$_2$ $\Rightarrow$ $x_0 \div y_0 \div y_0 = y_0 \div (x_0 \div y_0)$

Assump $\Rightarrow$ *Stat8* : $\langle \forall x, y \mid \{x, y\} \subseteq bb \to x \div (y \div x) = y \rangle$

$\langle y_0, x_0 \rangle \hookrightarrow$ *Stat8* $\Rightarrow$ $y_0 \div (x_0 \div y_0) = x_0$

EQUAL $\langle$*Stat7*$\rangle$ $\Rightarrow$ *Stat9* : $zz_\ominus = x_0 \div (y_0 \cdot x_0 \div (x_0 \cdot y_0 \div x_0))$

$\langle y_0 \cdot x_0, x_0 \cdot y_0, x_0 \rangle \hookrightarrow$ *Stat6* $\Rightarrow$ $y_0 \cdot x_0 \div (x_0 \cdot y_0 \div x_0) = (y_0 \cdot x_0 \div x_0 \cdot y_0) \div x_0$

EQUAL $\langle$*Stat9*$\rangle$ $\Rightarrow$ $x_0 \div (y_0 \cdot x_0 \div (x_0 \cdot y_0 \div x_0)) = x_0 \div (y_0 \cdot x_0 \div x_0 \cdot y_0 \div x_0)$

$\langle y_0 \cdot x_0, x_0 \cdot y_0 \rangle \hookrightarrow$ *Stat1(Stat5)* $\Rightarrow$ $y_0 \cdot x_0 \div x_0 \cdot y_0 \in bb$

$\langle x_0, y_0 \cdot x_0 \div x_0 \cdot y_0 \rangle \hookrightarrow$ *Stat8* $\Rightarrow$ *Stat10* : $zz_\ominus = y_0 \cdot x_0 \div x_0 \cdot y_0$

$\langle x_0 \cdot y_0, y_0 \cdot x_0 \rangle \hookrightarrow$ *Stat8* $\Rightarrow$ $x_0 \cdot y_0 \div (y_0 \cdot x_0 \div x_0 \cdot y_0) = y_0 \cdot x_0$

EQUAL $\langle$*Stat10*$\rangle$ $\Rightarrow$ $x_0 \cdot y_0 \div zz_\ominus = y_0 \cdot x_0$

$\langle x_0 \cdot y_0 \rangle \hookrightarrow T$booleanRing$_1$ $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- multiplication by null gives null

THM booleanRing$_4$. $X \in bb \to zz_\ominus \cdot X = zz_\ominus$. PROOF:

Suppose_not(x_0) $\Rightarrow$ AUTO

$\langle x_0 \rangle \hookrightarrow T$booleanRing$_1$ $\Rightarrow$ $x_0 \div x_0 = zz_\ominus$

Assump $\Rightarrow$ *Stat1* : $\langle \forall x, y, z \mid \{x, y, z\} \subseteq bb \to (x \div y) \cdot z = z \cdot y \div z \cdot x \rangle$ & $\langle \forall x, y \mid \{x, y\} \subseteq bb \to x \div y \in bb \rangle$

$\langle x_0, x_0, x_0, x_0, x_0 \rangle \hookrightarrow$ *Stat1*$(\star)$ $\Rightarrow$ $(x_0 \div x_0) \cdot x_0 = x_0 \cdot x_0 \div x_0 \cdot x_0$ & $x_0 \div x_0 \in bb$

$\langle x_0 \div x_0 \rangle \hookrightarrow T$booleanRing$_1$ $\Rightarrow$ $x_0 \div x_0 \div (x_0 \div x_0) = zz_\ominus$

EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- dychotomy of partial ordering

THM booleanRing$_5$. $\{U, V\} \subseteq bb \;\&\; U \cdot V = U \;\&\; V \cdot U = V \rightarrow U = V$. PROOF:

Suppose_not$(x_0, y_0) \Rightarrow$ AUTO

$\langle x_0, y_0 \rangle \hookrightarrow T$booleanRing$_3$ $\Rightarrow$ false; Discharge $\Rightarrow$ QED

ENTER_THEORY Set_theory

DISPLAY booleanRing

THEORY booleanRing$(bb, \cdot, \div)$

$bb \neq \emptyset$

$\langle \forall x, y \mid \{x, y\} \subseteq bb \rightarrow x \cdot y \in bb \rangle$

$\langle \forall x, y \mid \{x, y\} \subseteq bb \rightarrow x \div y \in bb \rangle$

$\langle \forall x, y, z \mid \{x, y, z\} \subseteq bb \rightarrow x \cdot (y \cdot z) = (x \cdot y) \cdot z \rangle$

$\langle \forall x, y, z \mid \{x, y, z\} \subseteq bb \rightarrow x \div (y \div z) = (x \div y) \div z \rangle$

$\langle \forall x, y, z \mid \{x, y, z\} \subseteq bb \rightarrow (x \div y) \cdot z = z \cdot y \div z \cdot x \rangle$

$\langle \forall x, y \mid \{x, y\} \subseteq bb \rightarrow x \div x = y \div y \rangle$

$\langle \forall x, y \mid \{x, y\} \subseteq bb \rightarrow x \div (y \div x) = y \rangle$

$\langle \forall x \mid x \in bb \rightarrow x \cdot x = x \rangle$

$\Rightarrow (zz_\Theta)$

$zz_\Theta = \mathbf{arb}(bb) \div \mathbf{arb}(bb)$

$\langle \forall x \mid (x \in bb \rightarrow x \div x = zz_\Theta \;\&\; x \div zz_\Theta = x \;\&\; zz_\Theta \div x = x) \;\&\; zz_\Theta \in bb \rangle$

$\langle \forall x, y \mid x, y \in bb \rightarrow x \div y = y \div x \rangle$

$\langle \forall x, y \mid x, y \in bb \rightarrow x \cdot y = y \cdot x \rangle$

$\langle \forall x \mid x \in bb \rightarrow zz_\Theta \cdot x = zz_\Theta \rangle$

$\langle \forall u, v \mid \{u, v\} \subseteq bb \;\&\; u \cdot v = u \;\&\; v \cdot u = v \rightarrow u = v \rangle$

END booleanRing

3 Fields of sets

We will soon see an example of a Boolean ring whose support is a family of sets and whose operations are intersection and symmetric difference. In preparation for that, let's now introduce formally the latter operation.

DEF symm: [symmetric difference] $\mathsf{X} \triangle \mathsf{Y} \;=_{\text{Def}}\; \mathsf{X}\backslash\mathsf{Y} \cup (\mathsf{Y}\backslash\mathsf{X})$

THM 1000. $\emptyset \triangle \emptyset = \emptyset \;\&\; \mathsf{X} \triangle \mathsf{X} = \mathsf{Y} \triangle \mathsf{Y} \;\&\; \mathsf{X} \triangle \mathsf{Y} = \mathsf{X} \cup \mathsf{Y}\backslash\mathsf{X} \cap \mathsf{Y} \;\&\; \mathsf{X} \triangle \mathsf{Y} = \mathsf{Y} \triangle \mathsf{X}$. PROOF:

Suppose_not$(\mathsf{x}_0, \mathsf{y}_0) \Rightarrow$ AUTO

Use_def$(\triangle) \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- truth-table of intersection and symmetric difference

THM 1001. $\mathsf{X} \cap \emptyset = \emptyset \;\&\; \emptyset \cap \mathsf{X} = \emptyset \;\&\; 1 \cap 1 = 1 \;\&\; \mathsf{X} \triangle \emptyset = \mathsf{X} \;\&\; \emptyset \triangle \mathsf{X} = \mathsf{X} \;\&\; \mathsf{X} \triangle \mathsf{X} = \emptyset$. PROOF:

Suppose_not$(\mathsf{x}_0) \Rightarrow$ AUTO

Use_def$(\triangle) \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- unionset of 2

THM 1002. $2 = \{\emptyset, 1\} \;\&\; \bigcup 2 = 1$. PROOF:

Suppose_not $\Rightarrow$ AUTO

TELEM $\Rightarrow$ *Stat40* : $2 = \{\emptyset, \{\emptyset\}\} \;\&\; 1 = \{\emptyset\}$

$\langle \{\emptyset\}, 2 \rangle \hookrightarrow T108$ (*Stat40*$\star$) $\Rightarrow$ $\bigcup 2 = \{\emptyset\} \cup \bigcup(2 \backslash \{\{\emptyset\}\})$

$\langle \emptyset, \{\emptyset\} \rangle \hookrightarrow T108$ (*Stat41*$\star$) $\Rightarrow$ *Stat41* : $\bigcup \{\emptyset\} = \bigcup(\{\emptyset\} \setminus \{\emptyset\}) \;\&\; \bigcup \emptyset = \emptyset$

(*Stat40*, *Stat40*$\star$)ELEM $\Rightarrow$ $\{\emptyset\} \setminus \{\emptyset\} = \emptyset \;\&\; 2 \backslash \{\{\emptyset\}\} = \{\emptyset\}$

EQUAL $\langle$*Stat40*$\rangle \Rightarrow$ $\bigcup 2 = 1$

Discharge $\Rightarrow$ QED

The following is a typical example of a Boolean ring (as a matter of fact, as will turn out from the Stone theorem, the standard example to within isomorphism)

THEORY protoBoolean(dd)

$\emptyset \neq \bigcup \mathsf{dd}$

$\langle \forall \mathsf{x}, \mathsf{y} \mid \{\mathsf{x}, \mathsf{y}\} \subseteq \mathsf{dd} \rightarrow \mathsf{x} \cap \mathsf{y} \in \mathsf{dd} \rangle$

$\langle \forall \mathsf{x}, \mathsf{y} \mid \{\mathsf{x}, \mathsf{y}\} \subseteq \mathsf{dd} \rightarrow \mathsf{x} \triangle \mathsf{y} \in \mathsf{dd} \rangle$

END protoBoolean

ENTER_THEORY protoBoolean

-- non vacuity

THM protoBoolean$_1$. $\mathsf{dd} \neq \emptyset$. PROOF:

Suppose_not $\Rightarrow$ $\mathsf{dd} = \emptyset$

$T108 \Rightarrow \quad \bigcup\emptyset = \emptyset$
Assump $\Rightarrow \quad \emptyset \neq \bigcup \mathsf{dd}$
EQUAL $\Rightarrow \quad \emptyset \neq \emptyset$
Discharge $\Rightarrow$ QED

‖ Algebraic laws:

-- associativity of intersection
THM protoBoolean$_2$. $\{X, Y, Z\} \subseteq \mathsf{dd} \to X \cap (Y \cap Z) = (X \cap Y) \cap Z$. PROOF:
Suppose_not$(x_0, y_0, z_0) \Rightarrow$ AUTO
ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED
-- associativity of symmetric difference
THM protoBoolean$_3$. $\{X, Y, Z\} \subseteq \mathsf{dd} \to X \triangle (Y \triangle Z) = (X \triangle Y) \triangle Z$. PROOF:
Suppose_not$(x_0, y_0, z_0) \Rightarrow$ AUTO
Use_def$(\triangle) \Rightarrow$ false; Discharge $\Rightarrow$ QED
-- distributivity of intersection over symmetric difference
THM protoBoolean$_4$. $\{X, Y, Z\} \subseteq \mathsf{dd} \to (X \triangle Y) \cap Z = Z \cap Y \triangle Z \cap X$. PROOF:
Suppose_not$(x_0, y_0, z_0) \Rightarrow$ AUTO
Use_def$(\triangle) \Rightarrow$ false; Discharge $\Rightarrow$ QED
-- nullity of symmetric difference of a set by itself
THM protoBoolean$_5$. $\{X, Y\} \subseteq \mathsf{dd} \to X \triangle X = Y \triangle Y$. PROOF:
Suppose_not$(x_0, y_0) \Rightarrow$ AUTO
Use_def$(\triangle) \Rightarrow$ false; Discharge $\Rightarrow$ QED
-- self annichilation law for symmetric difference
THM protoBoolean$_6$. $\{X, Y\} \subseteq \mathsf{dd} \to X \triangle (Y \triangle X) = Y$. PROOF:
Suppose_not$(x_0, y_0) \Rightarrow$ AUTO
Use_def$(\triangle) \Rightarrow$ false; Discharge $\Rightarrow$ QED
-- idempotency of intersection
THM protoBoolean$_7$. $X \in \mathsf{dd} \to X \cap X = X$. PROOF:
Suppose_not$(x_0) \Rightarrow$ AUTO
Discharge $\Rightarrow$ QED
-- self-annichilation law
THM protoBoolean$_8$. $\{X, Y\} \subseteq \mathsf{dd} \to X \triangle X = \emptyset \,\&\, X \triangle X = Y \triangle Y$. PROOF:
Suppose_not$(x_0, y_0) \Rightarrow$ AUTO
$\langle x_0, y_0 \rangle \hookrightarrow T1000 \Rightarrow \quad x_0 \triangle x_0 = y_0 \triangle y_0$
$\langle x_0, \emptyset \rangle \hookrightarrow T1000 \Rightarrow$ false; Discharge $\Rightarrow$ QED
-- multiplicative unit law
THM protoBoolean$_9$. $X \in \mathsf{dd} \to \bigcup \mathsf{dd} \cap X = X$. PROOF:
Suppose_not$(x_0) \Rightarrow$ AUTO
$\langle x_0, \mathsf{dd} \rangle \hookrightarrow T108\ (\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

APPLY $\langle zz_\Theta : zz1_\Theta \rangle$ booleanRing(bb $\mapsto$ dd, $U \cdot V \mapsto U \cap V, U \div V \mapsto U \triangle V$)$\Rightarrow$

THM protoBoolean$_{10}$. $\langle \forall x, y \mid \{x, y\} \subseteq dd \rightarrow X \triangle Y = Y \triangle X \rangle$.

ENTER_THEORY Set_theory

DISPLAY protoBoolean

THEORY protoBoolean(dd)

$\emptyset \neq \bigcup dd$
$\langle \forall x, y \mid \{x, y\} \subseteq dd \rightarrow x \cap y \in dd \rangle$
$\langle \forall x, y \mid \{x, y\} \subseteq dd \rightarrow x \triangle y \in dd \rangle$

$\Rightarrow$

$dd \neq \emptyset$
$\langle \forall x \in dd, y \in dd \mid x \cdot y \in bb \rangle$
$\langle \forall x \in dd, y \in dd \mid x \div y \in bb \rangle$
$\langle \forall x \in dd, y \in dd, z \in dd \mid x \cap (y \cap z) = (x \cap y) \cap z \rangle$
$\langle \forall x \in dd, y \in dd, z \in dd \mid x \triangle (y \triangle z) = (x \triangle y) \triangle z \rangle$
$\langle \forall x \in dd, y \in dd, z \in dd \mid (x \triangle y) \cap z = z \cap y \triangle z \cap x \rangle$
$\langle \forall x \in dd, y \in dd \mid x \triangle x = y \triangle y \rangle$
$\langle \forall x \in dd, y \in dd \mid x \triangle (y \triangle x) = y \rangle$
$\langle \forall x \in dd \mid x \cap x = x \rangle$

END protoBoolean

> It will follow, by application of the following theorems, that the family of all subsets, as well as the family of all finite and cofinite subsets, of a nonnull set constitute examples of 'protoBoolean'.

THM 1003. $W \neq \emptyset \;\&\; \{X, Y\} \subseteq \mathcal{P}W \rightarrow \{X \cap Y, X \triangle Y\} \subseteq \mathcal{P}W \;\&\; \bigcup(\mathcal{P}W) \neq \emptyset$. PROOF:

Suppose_not$(w_0, x_0, y_0) \Rightarrow$ AUTO
Use_def$(\mathcal{P}) \Rightarrow$ *Stat1*: $x_0, y_0 \in \{x : x \subseteq w_0\}$
Use_def$(\triangle) \Rightarrow$ $x_0 \triangle y_0 = x_0 \backslash y_0 \cup (y_0 \backslash x_0)$
$\langle x_1, y_1 \rangle \hookrightarrow$ *Stat1*(*Stat1*$\star$) $\Rightarrow$ *Stat2*: $x_0 \cap y_0 \subseteq w_0 \;\&\; x_0 \triangle y_0 \subseteq w_0$
Suppose $\Rightarrow$ *Stat3*: $x_0 \cap y_0 \notin \{x : x \subseteq w_0\}$
$\langle x_0 \cap y_0 \rangle \hookrightarrow$ *Stat3*(*Stat2*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO
Suppose $\Rightarrow$ *Stat4*: $x_0 \triangle y_0 \notin \{x : x \subseteq w_0\}$
$\langle x_0 \triangle y_0 \rangle \hookrightarrow$ *Stat4*(*Stat2*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO
Use_def$(\mathcal{P}) \Rightarrow$ $\{x_0 \cap y_0, x_0 \triangle y_0\} \subseteq \mathcal{P}w_0$
$\langle w_0 \rangle \hookrightarrow$ *T*215 (*Stat1*$\star$) $\Rightarrow$ $w_0 \in \mathcal{P}w_0$

$\langle w_0, \mathcal{P}w_0\rangle \hookrightarrow T108\ (\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

THM 1004. $W \neq \emptyset \ \&\ D = \{s \subseteq W \mid \mathsf{Finite}(s) \vee \mathsf{Finite}(W\backslash s)\} \ \&\ \{X, Y\} \subseteq D \rightarrow \{X \cap Y, X \triangle Y\} \subseteq D \ \&\ \bigcup D \neq \emptyset$. PROOF:

Suppose_not$(w_0, d_0, x_0, y_0) \Rightarrow$ AUTO

> Observe that $w_0 \in d_0$, since $w_0\backslash w_0$ is finite. Consequently $w_0 \subseteq \bigcup d_0$ and $\bigcup d_0 \neq \emptyset$ follows from $w_0 \neq \emptyset$. Therefore the negation of the claim can be true for w_0, d_0, x_0, y_0 only because $\{x_0 \cap y_0, x_0 \triangle y_0\} \not\subseteq d_0$.

Suppose $\Rightarrow$ *Stat1* : $\bigcup d_0 = \emptyset$
ELEM $\Rightarrow$ $w_0 \not\subseteq \bigcup d_0$
$\langle w_0, d_0\rangle \hookrightarrow T108\ (\star) \Rightarrow$ *Stat2* : $w_0 \notin \{s \subseteq w_0 \mid \mathsf{Finite}(s) \vee \mathsf{Finite}(w_0\backslash s)\}$
$\langle w_0\rangle \hookrightarrow Stat2(Stat2) \Rightarrow$ false; Discharge $\Rightarrow$ *Stat2a* : $\{x_0 \cap y_0, x_0 \triangle y_0\} \not\subseteq d_0$

> Since $x_0 \cap y_0$ and $x_0 \triangle y_0$ are subsets of d_0, we have found that either $x_0 \cap y_0$ or $x_0 \triangle y_0$ must be infinite and have an infinite complement in w_0.

ELEM $\Rightarrow$ *Stat3* : $x_0, y_0 \in \{s : s \subseteq w_0 \mid \mathsf{Finite}(s) \vee \mathsf{Finite}(w_0\backslash s)\}$
$\langle x_2, y_2\rangle \hookrightarrow Stat3(Stat3\star) \Rightarrow$ *Stat8* :
$x_0 = x_2 \ \&\ x_2 \subseteq w_0 \ \&\ \mathsf{Finite}(x_2) \vee \mathsf{Finite}(w_0\backslash x_2) \ \&\ y_0 = y_2 \ \&\ y_2 \subseteq w_0 \ \&\ \mathsf{Finite}(y_2) \vee \mathsf{Finite}(w_0\backslash y_2)$
EQUAL $\langle Stat8\rangle \Rightarrow$ *Stat8a* : $x_0 \subseteq w_0 \ \&\ \mathsf{Finite}(x_0) \vee \mathsf{Finite}(w_0\backslash x_0) \ \&\ y_0 \subseteq w_0 \ \&\ \mathsf{Finite}(y_0) \vee \mathsf{Finite}(w_0\backslash y_0)$
Use_def$(\triangle) \Rightarrow$ *Stat3a* : $x_0 \triangle y_0 = x_0\backslash y_0 \cup (y_0\backslash x_0)$
$\langle x_1, y_1\rangle \hookrightarrow Stat3(Stat3\star) \Rightarrow$ *Stat4* : $x_0 \cap y_0 \subseteq w_0 \ \&\ x_0 \triangle y_0 \subseteq w_0$
Suppose $\Rightarrow$ $\mathsf{Finite}(x_0 \cap y_0) \vee \mathsf{Finite}(w_0\backslash x_0 \cap y_0) \ \&\ \mathsf{Finite}(x_0 \triangle y_0) \vee \mathsf{Finite}\big(w_0\backslash(x_0 \triangle y_0)\big)$
Suppose $\Rightarrow$ *Stat36* : $x_0 \cap y_0 \notin \{s \subseteq w_0 \mid \mathsf{Finite}(s) \vee \mathsf{Finite}(w_0\backslash s)\}$
$\langle x_0 \cap y_0\rangle \hookrightarrow Stat36(Stat4\star) \Rightarrow$ false; Discharge $\Rightarrow$ *Stat37* : $x_0 \triangle y_0 \notin \{s \subseteq w_0 \mid \mathsf{Finite}(s) \vee \mathsf{Finite}(w_0\backslash s)\}$
$\langle x_0 \triangle y_0\rangle \hookrightarrow Stat37(Stat4\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO

> Consider the four possibilities: (1) $\mathsf{Finite}(x_0) \ \&\ \mathsf{Finite}(y_0)$; (2) one of x_0, y_0 is finite, whereas the other is cofinite; (3) both are cofinite. In each case we will find a contradiction, thus being able to conclude that the desired statement holds.

Suppose $\Rightarrow$ *Stat9a* : $\neg\mathsf{Finite}(x_0) \ \&\ \neg\mathsf{Finite}(y_0)$

> As a matter of fact, in case (3) $x_0 \cap y_0$ would be cofinite and $x_0 \triangle y_0$ would be finite . . . ,

$(Stat8a, Stat9a)$ELEM $\Rightarrow$ $\mathsf{Finite}(w_0\backslash x_0 \cap y_0)$
$\langle x_0 \cup y_0, x_0\backslash y_0 \cup (y_0\backslash x_0)\rangle \hookrightarrow T189\ (Stat8a, Stat9a, Stat3a) \Rightarrow$ $\mathsf{Finite}(x_0 \triangle y_0)$

$(Stat4\star)$Discharge $\Rightarrow$ $Stat9: \mathsf{Finite}(x_0) \vee \mathsf{Finite}(y_0)$

‖ ...in case (1) both $x_0 \cap y_0$ and $x_0 \triangle y_0$ would be finite ...,

Suppose $\Rightarrow$ $Stat5: \mathsf{Finite}(x_0) \,\&\, \mathsf{Finite}(y_0)$
$\langle x_0 \cup y_0, x_0 \backslash y_0 \cup (y_0 \backslash x_0)\rangle \hookrightarrow T189\ (Stat5, Stat3a) \Rightarrow$ $\mathsf{Finite}(x_0 \triangle y_0)$
$(Stat4\star)$Discharge $\Rightarrow$ AUTO

‖ ...and in case (2), $x_0 \cap y_0$ would be finite and $x_0 \triangle y_0$ would be cofinite.

Suppose $\Rightarrow$ $Stat11: \neg\mathsf{Finite}(x_0)$
$\langle x_0 \cup y_0, x_0 \backslash y_0 \cup (y_0 \backslash x_0)\rangle \hookrightarrow T189\ (Stat11, Stat8a, Stat3a, Stat9) \Rightarrow$ $\mathsf{Finite}\big(w_0 \backslash (x_0 \triangle y_0)\big)$
$(Stat4\star)$Discharge $\Rightarrow$ AUTO
$(Stat9)$ELEM $\Rightarrow$ $Stat14: \neg\mathsf{Finite}(y_0) \,\&\, \mathsf{Finite}(x_0 \cap y_0)$
$\langle x_0 \cup y_0, x_0 \backslash y_0 \cup (y_0 \backslash x_0)\rangle \hookrightarrow T189\ (Stat14, Stat8a, Stat3a, Stat9) \Rightarrow$ $\mathsf{Finite}\big(w_0 \backslash (x_0 \triangle y_0)\big)$
$(Stat4\star)$Discharge $\Rightarrow$ QED

‖ The following is another example of a Boolean ring, slightly more specific than the one treated by the THEORY protoBoolean. For example, the ring of all finite and cofinite subsets of an infinite set does not fall under the following THEORY archeoBoolean, whereas it falls under the preceding THEORY protoBoolean.

THEORY archeoBoolean(dd)
$\emptyset \neq \bigcup \mathsf{dd}$
$\langle \forall x, y, z \mid \{x, y\} \subseteq \mathsf{dd} \,\&\, z \subseteq x \cup y \rightarrow z \in \mathsf{dd}\rangle$
END archeoBoolean

ENTER_THEORY archeoBoolean

‖ Derived closure properties:

THM archeoBoolean$_0$. $\emptyset \in \mathsf{dd}$. PROOF:
Suppose_not() $\Rightarrow$ AUTO
Suppose $\Rightarrow$ $\mathsf{dd} = \emptyset$
Assump $\Rightarrow$ $\emptyset \neq \bigcup \mathsf{dd}$
$T108$ $\Rightarrow$ $\bigcup \emptyset = \emptyset$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO
Assump $\Rightarrow$ $Stat1: \langle \forall x, y, z \mid \{x, y\} \subseteq \mathsf{dd} \,\&\, z \subseteq x \cup y \rightarrow z \in \mathsf{dd}\rangle$
$\langle \mathbf{arb}(\mathsf{dd}), \mathbf{arb}(\mathsf{dd}), \emptyset\rangle \hookrightarrow Stat1 \Rightarrow$ false; Discharge $\Rightarrow$ QED

THM archeoBoolean$_1$. $\{X, Y\} \subseteq dd \to X \cap Y \in dd$. PROOF:

Suppose_not$(x_0, y_0) \Rightarrow$ AUTO
Assump $\Rightarrow$ *Stat1* : $\langle \forall x, y, z \mid \{x, y\} \subseteq dd \,\&\, z \subseteq x \cup y \to z \in dd \rangle$
$\langle x_0, y_0, x_0 \cup y_0 \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ $x_0 \cup y_0 \in dd$
$\langle x_0 \cup y_0, x_0 \cup y_0, x_0 \cap y_0 \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ $x_0 \cap y_0 \in dd$
Discharge $\Rightarrow$ QED

THM archeoBoolean$_2$. $\{X, Y\} \subseteq dd \to X \triangle Y \in dd$. PROOF:

Suppose_not$(x_0, y_0) \Rightarrow$ AUTO
Use_def$(\triangle) \Rightarrow$ $x_0 \backslash y_0 \cup (y_0 \backslash x_0) \notin dd$
Assump $\Rightarrow$ *Stat1* : $\langle \forall x, y, z \mid \{x, y\} \subseteq dd \,\&\, z \subseteq x \cup y \to z \in dd \rangle$
$\langle x_0, y_0, x_0 \backslash y_0 \cup (y_0 \backslash x_0) \rangle \hookrightarrow$ *Stat1*$(\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

APPLY $\langle\,\rangle$ protoBoolean$(dd \mapsto dd) \Rightarrow$

THM archeoBoolean$_3$. $dd \neq \emptyset$.

ENTER_THEORY Set_theory

> The following theorem shows that there are Boolean algebras of sets which are instances of protoBoolean but are not instances of archeoBoolean. Indeed, the collection of all finite and cofinite subsets of an infinite set is not closed with respect to inclusion.

THM 1005. $\neg$Finite$(W) \,\&\, D = \{s \subseteq W \mid$ Finite$(s) \vee$ Finite$(W \backslash s)\} \to W \in D \,\&\, \langle \exists z \subseteq W \mid z \notin D \rangle$. PROOF:

Suppose_not$(w_0, d_0) \Rightarrow$ AUTO

> Consider a toggling map t of w_0 into itself. Then the set $\{\mathbf{arb}(\{x, t \restriction x\}) : x \in w_0\}$ is an infinite subset of w_0 whose complement relative to w_0 is also infinite. Hence this set does not belong to d_0.

$\langle w_0 \rangle \hookrightarrow$ *T888* $(\star) \Rightarrow$ *Stat1* : $\langle \exists t \mid$ Is_tog$(t) \,\&\, \mathbf{domain}(t) = w_0 \rangle \,\&\, \neg$Finite$(w_0)$
$\langle t \rangle \hookrightarrow$ *Stat1*(*Stat1*$\star$) $\Rightarrow$ *Stat1a* : Is_tog$(t) \,\&\, \mathbf{domain}(t) = w_0$
EQUAL $\Rightarrow$ $\neg(w_0 \in \{s \subseteq w_0 \mid$ Finite$(s) \vee$ Finite$(w_0 \backslash s)\} \,\&\, \langle \exists z \subseteq w_0 \mid z \notin \{s \subseteq w_0 \mid$ Finite$(s) \vee$ Finite$(w_0 \backslash s)\} \rangle)$
Suppose $\Rightarrow$ *Stat2* : $w_0 \notin \{s \subseteq w_0 \mid$ Finite$(s) \vee$ Finite$(w_0 \backslash s)\}$
$\langle w_0 \rangle \hookrightarrow$ *Stat2*(*Stat2*) $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat3* : $\neg \langle \exists z \subseteq w_0 \mid z \notin \{s \subseteq w_0 \mid$ Finite$(s) \vee$ Finite$(w_0 \backslash s)\} \rangle$
$\langle \{\mathbf{arb}(\{x, t \restriction x\}) : x \in w_0\} \rangle \hookrightarrow$ *Stat3*(*Stat3*$\star$) $\Rightarrow$ $\{\mathbf{arb}(\{x, t \restriction x\}) : x \in w_0\} \not\subseteq w_0 \vee$

$\{\mathbf{arb}(\{x, t|x\}) : x \in w_0\} \in \{s \subseteq w_0 \mid \text{Finite}(s) \vee \text{Finite}(w_0 \backslash s)\}$

Use_def(Is_tog) ⇒ $\text{Svm}(t) \,\&\, t^{\leftarrow} = t \,\&\, \{p \in t \mid p^{[1]} = p^{[2]}\} = \emptyset$

Suppose ⇒ *Stat4*: $\{\mathbf{arb}(\{x, t|x\}) : x \in w_0\} \not\subseteq w_0$

$\langle c \rangle \hookrightarrow$ *Stat4*(*Stat4*⋆) ⇒ *Stat5*: $c \in \{\mathbf{arb}(\{x, t|x\}) : x \in w_0\} \,\&\, c \notin w_0$

$\langle x_0 \rangle \hookrightarrow$ *Stat5*(*Stat5*) ⇒ *Stat5a*: $x_0 \in w_0 \,\&\, t|x_0 \notin w_0$

$\langle t, x_0 \rangle \hookrightarrow$ *T887c* (*Stat1a*, *Stat5a*⋆) ⇒ false; Discharge ⇒ Auto

(*Stat3*⋆)ELEM ⇒ *Stat7*:

$\{\mathbf{arb}(\{x, t|x\}) : x \in w_0\} \in \{s \subseteq w_0 \mid \text{Finite}(s) \vee \text{Finite}(w_0 \backslash s)\} \,\&$
$\{\mathbf{arb}(\{x, t|x\}) : x \in w_0\} \cup (w_0 \backslash \{\mathbf{arb}(\{x, t|x\}) : x \in w_0\}) = w_0$

$\langle\ \rangle \hookrightarrow$ *Stat7*(*Stat7*⋆) ⇒ $\text{Finite}(\{\mathbf{arb}(\{x, t|x\}) : x \in w_0\}) \vee \text{Finite}(w_0 \backslash \{\mathbf{arb}(\{x, t|x\}) : x \in w_0\})$

The reason why neither of $\{\mathbf{arb}(\{x, t|x\}) : x \in w_0\}$, $w_0 \backslash \{\mathbf{arb}(\{x, t|x\}) : x \in w_0\}$ can be finite is that these sets are in one-one correspondence with one another and their union is w0.

Suppose ⇒ *Stat8*: $\text{Finite}(\{\mathbf{arb}(\{x, t|x\}) : x \in w_0\}) \,\&\, \text{Finite}(w_0 \backslash \{\mathbf{arb}(\{x, t|x\}) : x \in w_0\})$

(*Stat8*)ELEM ⇒ $\text{Finite}\big(\{\mathbf{arb}(\{x, t|x\}) : x \in w_0\} \cup (w_0 \backslash \{\mathbf{arb}(\{x, t|x\}) : x \in w_0\})\big)$

EQUAL $\langle$*Stat1*$\rangle$ ⇒ false; Discharge ⇒ Auto

We are about to see, in fact, that the restriction of t to the former of these sets is a one-one mapping between the two. Consequently, the two sets have the same cardinality.

Loc_def ⇒ *Stat10*: $w_1 = \{\mathbf{arb}(\{x, t|x\}) : x \in w_0\} \,\&\, w_2 = w_0 \backslash \{\mathbf{arb}(\{x, t|x\}) : x \in w_0\} \,\&\, t_1 = t_{|w_1}$

Suppose ⇒ $1\text{–}1(t_1) \,\&\, \mathbf{domain}(t_1) = w_1 \,\&\, \mathbf{range}(t_1) = w_2$

$\langle t_1 \rangle \hookrightarrow$ *T191* (*Stat10*⋆) ⇒ $\text{Finite}(\mathbf{domain}(t_1)) \leftrightarrow \text{Finite}(\mathbf{range}(t_1))$

EQUAL $\langle$*Stat10*$\rangle$ ⇒ $\text{Finite}(\{\mathbf{arb}(\{x, t|x\}) : x \in w_0\}) \leftrightarrow \text{Finite}(w_0 \backslash \{\mathbf{arb}(\{x, t|x\}) : x \in w_0\})$

(*Stat7*⋆)Discharge ⇒ Auto

EQUAL $\langle$*Stat1*$\rangle$ ⇒ *Stat10a*: $\text{Svm}(t) \,\&\, \text{Svm}(t^{\leftarrow}) \,\&\, \mathbf{domain}(t) = w_0 \,\&\, \mathbf{domain}(t^{\leftarrow}) = w_0$

$\langle t \rangle \hookrightarrow$ *T121* ⇒ Auto

$\langle t, w_1 \rangle \hookrightarrow$ *T58* ⇒ Auto

$\langle t, w_1 \rangle \hookrightarrow$ *T94* ⇒ Auto

Suppose ⇒ *Stat11*: $\{\mathbf{arb}(\{x, t|x\}) : x \in w_0\} \not\subseteq w_0$

$\langle x_1 \rangle \hookrightarrow$ *Stat11*(*Stat11*⋆) ⇒ *Stat12*: $x_1 \in \{\mathbf{arb}(\{x, t|x\}) : x \in w_0\} \,\&\, x_1 \notin w_0$

$\langle x_2 \rangle \hookrightarrow$ *Stat12*(*Stat12*) ⇒ *Stat12a*: $x_2 \in w_0 \,\&\, t|x_2 \notin w_0$

$\langle t, x_2 \rangle \hookrightarrow$ *T887c* (*Stat1a*, *Stat12a*⋆) ⇒ false; Discharge ⇒ Auto

(*Stat10*⋆)ELEM ⇒ $\mathbf{domain}(t_{|w_1}) = w_1$

$\langle t, \{\mathbf{arb}(\{x, t|x\}) : x \in w_0\} \rangle \hookrightarrow$ *T123* (*Stat10*, *Stat10a*⋆) ⇒ $\mathbf{range}(t_{|\{\mathbf{arb}(\{x,t|x\}) : x \in w_0\}}) =$
$\{t|x : x \in \mathbf{domain}(t) \mid x \in \{\mathbf{arb}(\{x, t|x\}) : x \in w_0\}\}$

EQUAL $\langle$*Stat10*$\rangle$ ⇒ *Stat13*: $\{t|x : x \in w_0 \mid x \in \{\mathbf{arb}(\{x, t|x\}) : x \in w_0\}\} \neq w_2$

$\langle d\rangle \hookrightarrow$ *Stat13(Stat13⋆)* $\Rightarrow$ $d \in \{t{\restriction}x : x \in w_0 \mid x \in \{\mathbf{arb}(\{x, t{\restriction}x\}) : x \in w_0\}\} \neq d \in w_2$

Suppose $\Rightarrow$ *Stat13a* : $d \notin w_2$

(*Stat13⋆*)ELEM $\Rightarrow$ *Stat14* : $d \in \{t{\restriction}x : x \in w_0 \mid x \in \{\mathbf{arb}(\{x, t{\restriction}x\}) : x \in w_0\}\}$

$\langle x_3\rangle \hookrightarrow$ *Stat14(Stat13a, Stat10⋆)* $\Rightarrow$ *Stat15* :

$x_3 \in \{\mathbf{arb}(\{x, t{\restriction}x\}) : x \in w_0\}$ & $t{\restriction}x_3 \notin w_0 \setminus \{\mathbf{arb}(\{x, t{\restriction}x\}) : x \in w_0\}$ & $x_3 \in w_0$

$\langle t, x_3\rangle \hookrightarrow$ *T887c (Stat1a, Stat15⋆)* $\Rightarrow$ *Stat16a* : $t{\restriction}x_3 \in w_0$

$\langle t, w_0, x_3\rangle \hookrightarrow$ *T887b (Stat1a, Stat15, Stat16a⋆)* $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO

(*Stat10⋆*)ELEM $\Rightarrow$ *Stat17* :

$d \notin \{\mathbf{arb}(\{x, t{\restriction}x\}) : x \in w_0\}$ & $d \notin \{t{\restriction}x : x \in w_0 \mid x \in \{\mathbf{arb}(\{x, t{\restriction}x\}) : x \in w_0\}\}$ & $d \in w_0$

$\langle t, d\rangle \hookrightarrow$ *T887c (Stat1a, Stat17⋆)* $\Rightarrow$ $t{\restriction}d \in w_0$

$\langle t, d\rangle \hookrightarrow$ *T887 (Stat1a, Stat17⋆)* $\Rightarrow$ $t{\restriction}(t{\restriction}d) = d$

$\langle d, t{\restriction}d\rangle \hookrightarrow$ *Stat17(Stat17⋆)* $\Rightarrow$ *Stat18* :

$t{\restriction}d \notin \{\mathbf{arb}(\{x, t{\restriction}x\}) : x \in w_0\}$ & $d \neq \mathbf{arb}(\{d, t{\restriction}d\})$

$\langle d\rangle \hookrightarrow$ *Stat18(Stat17, Stat18)* $\Rightarrow$ false; Discharge $\Rightarrow$ QED

4 The theory of Boolean algebras

THEORY $\mathsf{booleanAlgebra}(\mathsf{bb}, \mathsf{U} \cdot \mathsf{V}, \mathsf{U} \div \mathsf{V}, \mathsf{ee})$

‖ non-vacuity assumptions

$\mathsf{ee} \in \mathsf{bb}$
$\mathsf{ee} \neq \mathsf{ee} \div \mathsf{ee}$

‖ closure properties

$\langle \forall \mathsf{x}, \mathsf{y} \mid \{\mathsf{x}, \mathsf{y}\} \subseteq \mathsf{bb} \to \mathsf{x} \cdot \mathsf{y} \in \mathsf{bb} \rangle$
$\langle \forall \mathsf{x}, \mathsf{y} \mid \{\mathsf{x}, \mathsf{y}\} \subseteq \mathsf{bb} \to \mathsf{x} \div \mathsf{y} \in \mathsf{bb} \rangle$

‖ associativity laws

$\langle \forall \mathsf{x}, \mathsf{y}, \mathsf{z} \mid \{\mathsf{x}, \mathsf{y}, \mathsf{z}\} \subseteq \mathsf{bb} \to \mathsf{x} \cdot (\mathsf{y} \cdot \mathsf{z}) = (\mathsf{x} \cdot \mathsf{y}) \cdot \mathsf{z} \rangle$
$\langle \forall \mathsf{x}, \mathsf{y}, \mathsf{z} \mid \{\mathsf{x}, \mathsf{y}, \mathsf{z}\} \subseteq \mathsf{bb} \to \mathsf{x} \div (\mathsf{y} \div \mathsf{z}) = (\mathsf{x} \div \mathsf{y}) \div \mathsf{z} \rangle$

‖ distributivity law

$\langle \forall \mathsf{x}, \mathsf{y}, \mathsf{z} \mid \{\mathsf{x}, \mathsf{y}, \mathsf{z}\} \subseteq \mathsf{bb} \to (\mathsf{x} \div \mathsf{y}) \cdot \mathsf{z} = \mathsf{z} \cdot \mathsf{y} \div \mathsf{z} \cdot \mathsf{x} \rangle$

‖ additive zero

$\langle \forall \mathsf{x}, \mathsf{y} \mid \{\mathsf{x}, \mathsf{y}\} \subseteq \mathsf{bb} \to \mathsf{x} \div \mathsf{x} = \mathsf{y} \div \mathsf{y} \rangle$

‖ self-annichilation law

$\langle \forall \mathsf{x}, \mathsf{y} \mid \{\mathsf{x}, \mathsf{y}\} \subseteq \mathsf{bb} \to \mathsf{x} \div (\mathsf{y} \div \mathsf{x}) = \mathsf{y} \rangle$

‖ idempotency of multiplication

$\langle \forall \mathsf{x} \mid \mathsf{x} \in \mathsf{bb} \to \mathsf{x} \cdot \mathsf{x} = \mathsf{x} \rangle$

‖ multiplicative unit

$\langle \forall \mathsf{x} \mid \mathsf{x} \in \mathsf{bb} \to \mathsf{ee} \cdot \mathsf{x} = \mathsf{x} \rangle$
END booleanAlgebra

ENTER_THEORY booleanAlgebra

APPLY $\langle zz_\ominus : zz_\ominus\rangle$ booleanRing$(bb \mapsto bb, U\cdot V \mapsto U\cdot V, U \div V \mapsto U\div V)\Rightarrow$

-- self-annichilation and commutativity laws

THM booleanAlgebra$_0$. $\langle \forall x \mid (x \in bb \to x \div x = zz_\ominus \;\&\; x \div zz_\ominus = x \;\&\; zz_\ominus \div x = x) \;\&\; zz_\ominus \in bb\rangle \;\&\; \langle \forall x, y \mid \{x,y\} \subseteq bb \to x \div y = y \div x\rangle \;\&\; \langle \forall x, y \mid \{x,y\} \subseteq bb \to x\cdot y = y \cdot x\rangle \;\&\;$
$\langle \forall x \mid x \in bb \to zz_\ominus \cdot x = zz_\ominus\rangle \;\&\;$
$\langle \forall u, v \mid \{u,v\} \subseteq bb \;\&\; u\cdot v = u \;\&\; v \cdot u = v \to u = v\rangle$.

DEF booleanAlgebra$_1$: [complement operation] $\quad cmp_\ominus(X) \;=_{Def}\; ee \div X$

-- double-complementation law

THM booleanAlgebra$_1$. $\left(X \in bb \to cmp_\ominus(X) \in bb \;\&\; cmp_\ominus(cmp_\ominus(X)) = X\right) \;\&\; cmp_\ominus(ee) = zz_\ominus \;\&\; cmp_\ominus(zz_\ominus) = ee$. PROOF:

Suppose_not$(x_0) \Rightarrow$ AUTO
TbooleanAlgebra$_0 \Rightarrow$ *Stat0*: $\langle \forall x \mid (x \in bb \to x \div x = zz_\ominus \;\&\; x \div zz_\ominus = x \;\&\; zz_\ominus \div x = x) \;\&\; zz_\ominus \in bb\rangle$
Use_def$(cmp_\ominus(ee)) \Rightarrow$ AUTO
Assump $\Rightarrow$ *Stat9*: $\langle \forall x, y \mid \{x,y\} \subseteq bb \to x \div y \in bb\rangle \;\&\; ee \in bb$
Use_def$(cmp_\ominus(x_0)) \Rightarrow$ AUTO
$\langle ee, x_0\rangle \hookrightarrow$ *Stat9(Stat9⋆)* $\Rightarrow$ $x_0 \in bb \to cmp_\ominus(x_0) \in bb$
$\langle ee\rangle \hookrightarrow$ *Stat0(Stat0⋆)* $\Rightarrow$ $cmp_\ominus(ee) = zz_\ominus \;\&\; ee \div ee = zz_\ominus$
Use_def$(cmp_\ominus(zz_\ominus)) \Rightarrow$ AUTO
EQUAL $\langle$*Stat0*$\rangle \Rightarrow$ $cmp_\ominus(zz_\ominus) = ee \div (ee \div ee)$
Assump $\Rightarrow$ *Stat0a*: $\langle \forall x, y \mid \{x,y\} \subseteq bb \to x \div (y \div x) = y\rangle$
$\langle ee, ee\rangle \hookrightarrow$ *Stat0a(⋆)* $\Rightarrow$ *Stat1a*: $x_0 \in bb \;\&\; cmp_\ominus(cmp_\ominus(x_0)) \neq x_0$
Use_def$(cmp_\ominus) \Rightarrow$ *Stat1*: $ee \div (ee \div x_0) \neq x_0$
TbooleanAlgebra$_0 \Rightarrow$ *Stat2*: $\langle \forall x, y \mid \{x,y\} \subseteq bb \to x \div y = y \div x\rangle$
Assump $\Rightarrow$ *Stat3*: $\langle \forall x, y \mid \{x,y\} \subseteq bb \to x \div (y \div x) = y\rangle \;\&\; ee \in bb$
$\langle ee, x_0\rangle \hookrightarrow$ *Stat2(Stat1a⋆)* $\Rightarrow$ $ee \div x_0 = x_0 \div ee$
EQUAL $\langle$*Stat1*$\rangle \Rightarrow$ $ee \div (x_0 \div ee) \neq x_0$
$\langle ee, x_0\rangle \hookrightarrow$ *Stat3(Stat1a⋆)* $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- additive decomposition

THM booleanAlgebra$_{1a}$. $\{X, Y\} \subseteq bb \to cmp_\ominus(X) \div X = ee \;\&\; Y\cdot X \div Y \cdot cmp_\ominus(X) = Y \;\&\; Y \cdot X \cdot (Y \cdot cmp_\ominus(X)) = zz_\ominus$. PROOF:

Suppose_not$(x_0, y_0) \Rightarrow$ *Stat0*: $x_0, y_0 \in bb \;\&\; \neg\left(cmp_\ominus(x_0) \div x_0 = ee \;\&\; y_0 \cdot x_0 \div y_0 \cdot cmp_\ominus(x_0) = y_0 \;\&\; y_0 \cdot x_0 \cdot (y_0 \cdot cmp_\ominus(x_0)) = zz_\ominus\right)$
TbooleanAlgebra$_1 \Rightarrow$ $cmp_\ominus(zz_\ominus) = ee$
TbooleanAlgebra$_0 \Rightarrow$ *Stat1*:

$\langle \forall x \mid (x \in bb \rightarrow x \div x = zz_\ominus \,\&\, x \div zz_\ominus = x \,\&\, zz_\ominus \div x = x) \,\&\, zz_\ominus \in bb\rangle \,\&\, Stat1a:$
$\langle \forall x, y \mid \{x, y\} \subseteq bb \rightarrow x \cdot y = y \cdot x\rangle \,\&\, Stat2a: \langle \forall x \mid x \in bb \rightarrow zz_\ominus \cdot x = zz_\ominus\rangle$
$\langle x_0\rangle \hookrightarrow Stat1(\star) \Rightarrow \quad x_0 \div x_0 = zz_\ominus$
Use_def $(cmp_\ominus(x_0 \div x_0)) \Rightarrow$ AUTO
EQUAL $\Rightarrow$ $Stat2: ee \div (x_0 \div x_0) = ee$
Assump $\Rightarrow$ $Stat3: \langle \forall x, y, z \mid \{x, y, z\} \subseteq bb \rightarrow x \div (y \div z) = (x \div y) \div z\rangle \,\&\, ee \in bb$
$\langle ee, x_0, x_0\rangle \hookrightarrow Stat3(\star) \Rightarrow \quad ee \div (x_0 \div x_0) = (ee \div x_0) \div x_0$
Use_def $(cmp_\ominus) \Rightarrow$ $Stat4: ee \div (x_0 \div x_0) = cmp_\ominus(x_0) \div x_0$
EQUAL $\langle Stat2, Stat4\rangle \Rightarrow \quad cmp_\ominus(x_0) \div x_0 = ee \,\&\, (cmp_\ominus(x_0) \div x_0) \cdot y_0 = ee \cdot y_0$
Assump $\Rightarrow$ $Stat5:$
$\langle \forall x, y, z \mid \{x, y, z\} \subseteq bb \rightarrow (x \div y) \cdot z = z \cdot y \div z \cdot x\rangle \,\&\, \langle \forall x \mid x \in bb \rightarrow ee \cdot x = x\rangle \,\&\, Stat5a: \langle \forall x \mid x \in bb \rightarrow x \cdot x = x\rangle$
$\langle x_0\rangle \hookrightarrow T$booleanAlgebra$_1$ $(Stat0\star) \Rightarrow \quad Stat6: cmp_\ominus(x_0) \in bb$
$\langle cmp_\ominus(x_0), x_0, y_0, y_0\rangle \hookrightarrow Stat5(Stat0) \Rightarrow \quad (cmp_\ominus(x_0) \div x_0) \cdot y_0 = y_0 \cdot x_0 \div y_0 \cdot cmp_\ominus(x_0) \,\&\, ee \cdot y_0 = y_0$
EQUAL $\langle Stat4\rangle \Rightarrow \quad y_0 \cdot x_0 \div y_0 \cdot cmp_\ominus(x_0) = y_0$
$(Stat0\star)$ELEM $\Rightarrow \quad y_0 \cdot x_0 \cdot (y_0 \cdot cmp_\ominus(x_0)) \neq zz_\ominus$
Assump $\Rightarrow$ $Stat7: \langle \forall x, y \mid \{x, y\} \subseteq bb \rightarrow x \cdot y \in bb\rangle \,\&\, Stat8: \langle \forall x, y, z \mid \{x, y, z\} \subseteq bb \rightarrow x \cdot (y \cdot z) = (x \cdot y) \cdot z\rangle$
$\langle x_0\rangle \hookrightarrow T$booleanAlgebra$_1$ $(Stat0, Stat0\star) \Rightarrow \quad Stat9: cmp_\ominus(x_0) \in bb$
$\langle y_0, x_0, y_0 \cdot x_0, y_0, cmp_\ominus(x_0)\rangle \hookrightarrow Stat7(Stat0, Stat9\star) \Rightarrow \quad y_0 \cdot x_0 \in bb \,\&$
$(y_0 \cdot x_0 \cdot y_0) \cdot cmp_\ominus(x_0) = y_0 \cdot x_0 \cdot (y_0 \cdot cmp_\ominus(x_0))$
$\langle y_0, x_0\rangle \hookrightarrow Stat1a(Stat0, Stat0\star) \Rightarrow \quad y_0 \cdot x_0 = x_0 \cdot y_0$
$\langle x_0, y_0, y_0\rangle \hookrightarrow Stat8(Stat0, Stat0\star) \Rightarrow \quad (x_0 \cdot y_0) \cdot y_0 = x_0 \cdot (y_0 \cdot y_0)$
$\langle y_0\rangle \hookrightarrow Stat5a(Stat0, Stat0\star) \Rightarrow \quad y_0 \cdot y_0 = y_0$
EQUAL $\langle Stat6\rangle \Rightarrow \quad Stat10: x_0 \cdot y_0 \cdot cmp_\ominus(x_0) \neq zz_\ominus$
$\langle cmp_\ominus(x_0), x_0 \cdot y_0\rangle \hookrightarrow Stat1a(Stat9\star) \Rightarrow \quad cmp_\ominus(x_0) \cdot (x_0 \cdot y_0) \neq zz_\ominus$
$\langle cmp_\ominus(x_0), x_0, y_0\rangle \hookrightarrow Stat8(Stat9, Stat0\star) \Rightarrow \quad Stat11: cmp_\ominus(x_0) \cdot (x_0 \cdot y_0) = (cmp_\ominus(x_0) \cdot x_0) \cdot y_0$
Use_def $(cmp_\ominus) \Rightarrow \quad cmp_\ominus(x_0) \cdot (x_0 \cdot y_0) = (ee \div x_0) \cdot x_0 \cdot y_0$
$\langle ee, x_0, x_0, x_0\rangle \hookrightarrow Stat5(Stat3, Stat0\star) \Rightarrow \quad (ee \div x_0) \cdot x_0 = x_0 \cdot x_0 \div x_0 \cdot ee \,\&\, ee \cdot x_0 = x_0$
$\langle x_0\rangle \hookrightarrow Stat5a(Stat0, Stat0\star) \Rightarrow \quad x_0 \cdot x_0 = x_0$
$\langle x_0, ee\rangle \hookrightarrow Stat1a(Stat0, Stat3\star) \Rightarrow \quad x_0 \cdot ee = ee \cdot x_0$
$\langle x_0\rangle \hookrightarrow Stat1(Stat0, Stat0\star) \Rightarrow \quad x_0 \div x_0 = zz_\ominus$
EQUAL $\langle Stat10\rangle \Rightarrow \quad Stat12: zz_\ominus \cdot y_0 \neq zz_\ominus$
$\langle y_0\rangle \hookrightarrow Stat2a(Stat0, Stat12\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- De Morgan law
THM booleanAlgebra$_{1b}$. $\{X, Y\} \subseteq bb \rightarrow cmp_\ominus(X \div Y) = X \cdot Y \div cmp_\ominus(X) \cdot cmp_\ominus(Y)$. PROOF:
Suppose_not$(x_0, y_0) \Rightarrow$ AUTO
Use_def $(cmp_\ominus) \Rightarrow$ $Stat1: ee \div (x_0 \div y_0) \neq x_0 \cdot y_0 \div (ee \div x_0) \cdot (ee \div y_0) \,\&\, x_0, y_0 \in bb$
Assump $\Rightarrow$ $Stat2:$
$\langle \forall x, y \mid \{x, y\} \subseteq bb \rightarrow x \cdot y \in bb\rangle \,\&\, Stat3:$

$\langle \forall x, y \mid \{x, y\} \subseteq bb \rightarrow x \div y \in bb \rangle$ & *Stat4* : $\langle \forall x, y, z \mid \{x, y, z\} \subseteq bb \rightarrow (x \div y) \cdot z = z \cdot y \div z \cdot x \rangle$ & $ee \in bb$

$\langle x_0, y_0, ee, y_0, ee, x_0, ee \div y_0 \rangle \hookrightarrow$ *Stat2*(*Stat1*$\star$) $\Rightarrow$ *Stat5* :
$x_0 \cdot y_0, ee \div y_0 \in bb$ & $(ee \div x_0) \cdot (ee \div y_0) = (ee \div y_0) \cdot x_0 \div (ee \div y_0) \cdot ee$

$\langle ee, y_0, x_0 \rangle \hookrightarrow$ *Stat4*(*Stat1*$\star$) $\Rightarrow$ $(ee \div y_0) \cdot x_0 = x_0 \cdot y_0 \div x_0 \cdot ee$

TbooleanAlgebra$_0$ $\Rightarrow$ *Stat7* : $\langle \forall x, y \mid \{x, y\} \subseteq bb \rightarrow x \cdot y = y \cdot x \rangle$ & *Stat7a* : $\langle \forall x, y \mid \{x, y\} \subseteq bb \rightarrow x \div y = y \div x \rangle$

$\langle ee \div y_0, ee \rangle \hookrightarrow$ *Stat7*(*Stat5*, *Stat2*$\star$) $\Rightarrow$ $(ee \div y_0) \cdot ee = ee \cdot (ee \div y_0)$

$\langle x_0, ee \rangle \hookrightarrow$ *Stat7*(*Stat1*, *Stat2*$\star$) $\Rightarrow$ $x_0 \cdot ee = ee \cdot x_0$

Assump $\Rightarrow$ *Stat8* : $\langle \forall x \mid x \in bb \rightarrow ee \cdot x = x \rangle$

$\langle ee \div y_0 \rangle \hookrightarrow$ *Stat8*(*Stat5*$\star$) $\Rightarrow$ $ee \cdot (ee \div y_0) = ee \div y_0$

$\langle x_0 \rangle \hookrightarrow$ *Stat8*(*Stat1*$\star$) $\Rightarrow$ $ee \cdot x_0 = x_0$

EQUAL $\langle$*Stat1*$\rangle$ $\Rightarrow$ *Stat9* : $ee \div (x_0 \div y_0) \neq x_0 \cdot y_0 \div (x_0 \cdot y_0 \div x_0 \div (ee \div y_0))$

$\langle x_0 \cdot y_0, x_0 \rangle \hookrightarrow$ *Stat3*(*Stat5*, *Stat1*$\star$) $\Rightarrow$ $x_0 \cdot y_0 \div x_0 \in bb$

Assump $\Rightarrow$ *Stat10* : $\langle \forall x, y, z \mid \{x, y, z\} \subseteq bb \rightarrow x \div (y \div z) = (x \div y) \div z \rangle$

$\langle x_0 \cdot y_0, x_0, ee \div y_0 \rangle \hookrightarrow$ *Stat10*(*Stat5*, *Stat1*$\star$) $\Rightarrow$ $x_0 \cdot y_0 \div x_0 \div (ee \div y_0) =$
$x_0 \cdot y_0 \div (x_0 \div (ee \div y_0))$

$\langle x_0, ee \div y_0 \rangle \hookrightarrow$ *Stat3*(*Stat1*, *Stat5*$\star$) $\Rightarrow$ *Stat10a* : $x_0 \div (ee \div y_0) \in bb$

$\langle x_0 \cdot y_0, x_0 \cdot y_0, x_0 \div (ee \div y_0) \rangle \hookrightarrow$ *Stat10*(*Stat5*, *Stat10a*$\star$) $\Rightarrow$ $x_0 \cdot y_0 \div \left(x_0 \cdot y_0 \div (x_0 \div (ee \div y_0))\right) =$
$x_0 \cdot y_0 \div x_0 \cdot y_0 \div (x_0 \div (ee \div y_0))$

EQUAL $\langle$*Stat9*$\rangle$ $\Rightarrow$ *Stat11* : $ee \div (x_0 \div y_0) \neq x_0 \cdot y_0 \div x_0 \cdot y_0 \div (x_0 \div (ee \div y_0))$

TbooleanAlgebra$_0$ $\Rightarrow$ *Stat12* : $\langle \forall x \mid (x \in bb \rightarrow x \div x = zz_\ominus$ & $x \div zz_\ominus = x$ & $zz_\ominus \div x = x)$ & $zz_\ominus \in bb \rangle$

$\langle x_0 \cdot y_0 \rangle \hookrightarrow$ *Stat12*(*Stat5*$\star$) $\Rightarrow$ $x_0 \cdot y_0 \div x_0 \cdot y_0 = zz_\ominus$

$\langle x_0, ee \div y_0 \rangle \hookrightarrow$ *Stat3*(*Stat1*, *Stat5*$\star$) $\Rightarrow$ $x_0 \div (ee \div y_0) \in bb$

$\langle x_0 \div (ee \div y_0) \rangle \hookrightarrow$ *Stat12*(*Stat12*$\star$) $\Rightarrow$ $zz_\ominus \div (x_0 \div (ee \div y_0)) = x_0 \div (ee \div y_0)$

EQUAL $\langle$*Stat11*$\rangle$ $\Rightarrow$ *Stat13* : $ee \div (x_0 \div y_0) \neq x_0 \div (ee \div y_0)$

$\langle ee, y_0 \rangle \hookrightarrow$ *Stat7a*(*Stat1*, *Stat2*$\star$) $\Rightarrow$ $ee \div y_0 = y_0 \div ee$

$\langle x_0, y_0 \rangle \hookrightarrow$ *Stat3*(*Stat1*, *Stat1*$\star$) $\Rightarrow$ *Stat14* : $x_0 \div y_0 \in bb$

$\langle ee, x_0 \div y_0 \rangle \hookrightarrow$ *Stat7a*(*Stat14*, *Stat2*$\star$) $\Rightarrow$ $ee \div (x_0 \div y_0) = x_0 \div y_0 \div ee$

Assump $\Rightarrow$ *Stat15* : $\langle \forall x, y, z \mid \{x, y, z\} \subseteq bb \rightarrow x \div (y \div z) = (x \div y) \div z \rangle$

$\langle x_0, y_0, ee \rangle \hookrightarrow$ *Stat15*(*Stat1*, *Stat2*$\star$) $\Rightarrow$ $x_0 \div (y_0 \div ee) = (x_0 \div y_0) \div ee$

EQUAL $\langle$*Stat13*$\rangle$ $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- non-triviality , 2

THM booleanAlgebra$_2$. $X \in bb \rightarrow cmp_\ominus(X) \neq X$ & $\left(X \notin \{zz_\ominus, ee\} \rightarrow cmp_\ominus(X) \in bb \setminus \{zz_\ominus, ee\}\right)$. PROOF:

Suppose_not(x_0) $\Rightarrow$ AUTO

Suppose $\Rightarrow$ $x_0 \notin \{zz_\ominus, ee\}$ & $cmp_\ominus(x_0) \notin bb \setminus \{zz_\ominus, ee\}$

$\langle x_0 \rangle \hookrightarrow T$booleanAlgebra$_1$ $(\star)$ $\Rightarrow$ $cmp_\ominus(x_0) \in \{zz_\ominus, ee\}$ & $x_0 = cmp_\ominus(cmp_\ominus(x_0))$ & $cmp_\ominus(zz_\ominus) = ee$ & $cmp_\ominus(ee) = zz_\ominus$

Suppose $\Rightarrow$ $cmp_\ominus(x_0) = zz_\ominus$

EQUAL $\Rightarrow$ $x_0 = ee$

Discharge ⇒ $\mathsf{cmp}_\ominus(x_0) = ee$
EQUAL ⇒ $x_0 = zz_\ominus$
Discharge ⇒ AUTO
Use_def (cmp$_\ominus$) ⇒ $ee \div x_0 = x_0$
EQUAL ⇒ $x_0 \div (ee \div x_0) = x_0 \div x_0$
Assump ⇒ *Stat1*: $\langle \forall x, y \mid \{x, y\} \subseteq bb \to x \div (y \div x) = y \rangle$ & $ee \in bb$ & $ee \neq ee \div ee$
$\langle x_0, ee \rangle \hookrightarrow$ *Stat1*(⋆) ⇒ $ee = x_0 \div x_0$
*T*booleanAlgebra$_0$ ⇒ *Stat2*: $\langle \forall x \mid (x \in bb \to x \div x = zz_\ominus$ & $x \div zz_\ominus = x$ & $zz_\ominus \div x = x)$ & $zz_\ominus \in bb \rangle$
$\langle x_0 \rangle \hookrightarrow$ *Stat2*(⋆) ⇒ $ee = zz_\ominus$
$\langle ee \rangle \hookrightarrow$ *Stat2*(⋆) ⇒ false; Discharge ⇒ QED

-- when the greatest lower bound is the top

THM booleanAlgebra$_3$. $\{U, V\} \subseteq bb$ & $U \cdot V = ee \to U = ee$ & $V = ee$. PROOF:
Suppose_not (x_0, y_0) ⇒ AUTO
*T*booleanAlgebra$_0$ ⇒ *Stat1*: $\langle \forall x, y \mid \{x, y\} \subseteq bb \to x \cdot y = y \cdot x \rangle$
Assump ⇒ *Stat2*:
$\langle \forall x \mid x \in bb \to ee \cdot x = x \rangle$ & *Stat3*:
$\langle \forall x \mid x \in bb \to x \cdot x = x \rangle$ & *Stat4*: $\langle \forall x, y, z \mid \{x, y, z\} \subseteq bb \to x \cdot (y \cdot z) = (x \cdot y) \cdot z \rangle$ & $ee \in bb$
$\langle x_0, ee \rangle \hookrightarrow$ *Stat1*(⋆) ⇒ $x_0 \cdot ee = ee \cdot x_0$
$\langle x_0, x_0, x_0, x_0, y_0 \rangle \hookrightarrow$ *Stat2*(⋆) ⇒ $ee \cdot x_0 = x_0$ & $x_0 \cdot x_0 = x_0$ & $x_0 \cdot (x_0 \cdot y_0) = (x_0 \cdot x_0) \cdot y_0$
$\langle y_0, y_0, x_0, y_0, y_0 \rangle \hookrightarrow$ *Stat2*(⋆) ⇒ $ee \cdot y_0 = y_0$ & $y_0 \cdot y_0 = y_0$ & $x_0 \cdot (y_0 \cdot y_0) = (x_0 \cdot y_0) \cdot y_0$
EQUAL ⇒ $x_0 = y_0$ & $x_0 \cdot x_0 = ee$
$\langle x_0 \rangle \hookrightarrow$ *Stat3*(*Stat2*⋆) ⇒ $x_0 = ee$
Discharge ⇒ QED

> The following lemma is needed for the proof of Theorem booleanAlgebra$_6$, which is not relevant to our main flow of discourse.

-- immaterial multiplication

THM booleanAlgebra$_4$. $\{U, V, X, Y\} \subseteq bb \to U \cdot \mathsf{cmp}_\ominus(X) \div V \cdot \mathsf{cmp}_\ominus(Y) = \big(U \cdot \mathsf{cmp}_\ominus(X) \div V \cdot \mathsf{cmp}_\ominus(Y)\big) \cdot \mathsf{cmp}_\ominus(X \cdot Y)$. PROOF:
Suppose_not (a, b, x, y) ⇒ *Stat30*: $a \cdot \mathsf{cmp}_\ominus(x) \div b \cdot \mathsf{cmp}_\ominus(y) \neq \big(a \cdot \mathsf{cmp}_\ominus(x) \div b \cdot \mathsf{cmp}_\ominus(y)\big) \cdot \mathsf{cmp}_\ominus(x \cdot y)$ & $\{a, b, x, y\} \subseteq bb$
$\langle y \rangle \hookrightarrow$ *T*booleanAlgebra$_1$ (*Stat30*⋆) ⇒ $\mathsf{cmp}_\ominus(y) \in bb$
Assump ⇒ *Stat3*:
$\langle \forall x \mid x \in bb \to ee \cdot x = x \rangle$ & *Stat9*:
$\langle \forall x, y \mid \{x, y\} \subseteq bb \to x \cdot y \in bb \rangle$ & *Stat26*: $\langle \forall x, y \mid \{x, y\} \subseteq bb \to x \div y \in bb \rangle$ & *Stat9a*: $\langle \forall x, y, z \mid \{x, y, z\} \subseteq bb \to x \cdot (y \cdot z) = (x \cdot y) \cdot z \rangle$ & $ee \in bb$
$\langle x, y, ee, x \cdot y \rangle \hookrightarrow$ *Stat9*(*Stat30*⋆) ⇒ *Stat33*: $ee \div x \cdot y, x \cdot y \in bb$
$\langle x \rangle \hookrightarrow$ *T*booleanAlgebra$_1$ (*Stat30*⋆) ⇒ *Stat27a*: $x, \mathsf{cmp}_\ominus(x), x \cdot y \in bb$
$\langle a, \mathsf{cmp}_\ominus(x) \rangle \hookrightarrow$ *Stat9*(*Stat30*⋆) ⇒ *Stat25a*: $a \cdot \mathsf{cmp}_\ominus(x) \in bb$
$\langle b, \mathsf{cmp}_\ominus(y) \rangle \hookrightarrow$ *Stat9*(*Stat30*⋆) ⇒ *Stat26a*: $b \cdot \mathsf{cmp}_\ominus(y) \in bb$
$\langle a \cdot \mathsf{cmp}_\ominus(x), b \cdot \mathsf{cmp}_\ominus(y) \rangle \hookrightarrow$ *Stat26*(*Stat25a*⋆) ⇒ *Stat28a*: $a \cdot \mathsf{cmp}_\ominus(x) \div b \cdot \mathsf{cmp}_\ominus(y) \in bb$

Suppose $\Rightarrow$ *Stat1a* : $\neg\langle\forall u,v,w \mid \{u,v,w\} \subseteq bb \rightarrow w\cdot cmp_\Theta(u)\cdot(v\cdot u) = zz_\Theta\rangle$
$\langle x_0, y_0, z_0\rangle \hookrightarrow$ *Stat1a* $\Rightarrow$ AUTO
Assump $\Rightarrow$ *Stat1* :
$\langle\forall x,y \mid \{x,y\} \subseteq bb \rightarrow x\cdot y \in bb\rangle$ & *Stat2* :
$\langle\forall x,y,z \mid \{x,y,z\} \subseteq bb \rightarrow x\cdot(y\cdot z) = (x\cdot y)\cdot z\rangle$ & $\langle\forall x \mid x\in bb \rightarrow ee\cdot x = x\rangle$ & *Stat13a* : $\langle\forall x,y,z \mid \{x,y,z\} \subseteq bb \rightarrow (x \div y)\cdot z = z\cdot y \div z\cdot x\rangle$ & *Stat4* :
$\langle\forall x \mid x\in bb \rightarrow x\cdot x = x\rangle$ & $ee \in bb$
$\langle x_0\rangle \hookrightarrow T$booleanAlgebra$_1$ (*Stat1a*$\star$) $\Rightarrow$ $cmp_\Theta(x_0) \in bb$
$\langle y_0, x_0, z_0, cmp_\Theta(x_0), y_0\cdot x_0\rangle \hookrightarrow$ *Stat1*(*Stat1a*$\star$) $\Rightarrow$ *Stat6* :
$z_0\cdot\left(cmp_\Theta(x_0)\cdot(y_0\cdot x_0)\right) \neq zz_\Theta$ & $y_0\cdot x_0 \in bb$
Use_def(cmp_Θ) $\Rightarrow$ *Stat7* : $z_0\cdot\left((ee \div x_0)\cdot(y_0\cdot x_0)\right) \neq zz_\Theta$
$\langle ee, x_0, y_0\cdot x_0, x_0\rangle \hookrightarrow$ *Stat13a*(*Stat1a*$\star$) $\Rightarrow$ $(ee \div x_0)\cdot(y_0\cdot x_0) = y_0\cdot x_0\cdot x_0 \div y_0\cdot x_0\cdot ee$ &
$x_0\cdot x_0 = x_0$
$\langle y_0, x_0, x_0, y_0\cdot x_0\rangle \hookrightarrow$ *Stat2*(*Stat1a*$\star$) $\Rightarrow$ $(y_0\cdot x_0)\cdot x_0 = y_0\cdot(x_0\cdot x_0)$ &
$ee\cdot(y_0\cdot x_0) = y_0\cdot x_0$
TbooleanAlgebra$_0$ $\Rightarrow$ *Stat8* :
$\langle\forall x,y \mid \{x,y\} \subseteq bb \rightarrow x\cdot y = y\cdot x\rangle$ & $\langle\forall x \mid (x\in bb \rightarrow x\div x = zz_\Theta$ & $x \div zz_\Theta = x$ & $zz_\Theta \div x = x)$ & $zz_\Theta \in bb\rangle$ &
$\langle\forall x,y \mid \{x,y\} \subseteq bb \rightarrow x\cdot y = y\cdot x\rangle$ & $\langle\forall x \mid x\in bb \rightarrow zz_\Theta\cdot x = zz_\Theta\rangle$
$\langle y_0\cdot x_0, ee, y_0\cdot x_0, z_0, zz_\Theta, z_0\rangle \hookrightarrow$ *Stat8*(*Stat1a*$\star$) $\Rightarrow$ $y_0\cdot x_0\cdot ee = ee\cdot(y_0\cdot x_0)$ & $y_0\cdot x_0 \div y_0\cdot x_0 = zz_\Theta$ &
$z_0\cdot zz_\Theta = zz_\Theta$
EQUAL $\langle$*Stat7*$\rangle$ $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat3a* : $\langle\forall u,v,w \mid \{u,v,w\} \subseteq bb \rightarrow w\cdot cmp_\Theta(u)\cdot(v\cdot u) = zz_\Theta\rangle$

Following “Linear Operators” by Dunford & Schwartz, vol. 1, p. 42, ll. 8-12, we complete the proof as follows:

Use_def(cmp_Θ) $\Rightarrow$ $cmp_\Theta(x\cdot y) = ee \div x\cdot y$
EQUAL $\langle$*Stat30*$\rangle$ $\Rightarrow$ *Stat31* : $a\cdot cmp_\Theta(x) \div b\cdot cmp_\Theta(y) \neq \left(a\cdot cmp_\Theta(x) \div b\cdot cmp_\Theta(y)\right)\cdot(ee \div x\cdot y)$
Assump $\Rightarrow$ *Stat32* :
$\langle\forall u,v,w \mid \{u,v,w\} \subseteq bb \rightarrow (u \div v)\cdot w = w\cdot v \div w\cdot u\rangle$ & $ee \in bb$ & *Stat32a* :
$\langle\forall u,v,w \mid \{u,v,w\} \subseteq bb \rightarrow u \div (v \div w) = (u \div v) \div w\rangle$
$\langle a\cdot cmp_\Theta(x), b\cdot cmp_\Theta(y), ee \div x\cdot y\rangle \hookrightarrow$ *Stat32*(*Stat25a*, *Stat26a*, *Stat33*, *Stat31*$\star$) $\Rightarrow$ *Stat34* : $a\cdot cmp_\Theta(x) \div b\cdot cmp_\Theta(y) \neq$
$(ee \div x\cdot y)\cdot\left(b\cdot cmp_\Theta(y)\right) \div (ee \div x\cdot y)\cdot\left(a\cdot cmp_\Theta(x)\right)$
$\langle ee, x\cdot y, b\cdot cmp_\Theta(y)\rangle \hookrightarrow$ *Stat32*(*Stat32*, *Stat27a*, *Stat26a*$\star$) $\Rightarrow$ $(ee \div x\cdot y)\cdot\left(b\cdot cmp_\Theta(y)\right) =$
$b\cdot cmp_\Theta(y)\cdot(x\cdot y) \div b\cdot cmp_\Theta(y)\cdot ee$
$\langle ee, x\cdot y, a\cdot cmp_\Theta(x)\rangle \hookrightarrow$ *Stat32*(*Stat32*, *Stat27a*, *Stat25a*$\star$) $\Rightarrow$ $(ee \div x\cdot y)\cdot\left(a\cdot cmp_\Theta(x)\right) =$
$a\cdot cmp_\Theta(x)\cdot(x\cdot y) \div a\cdot cmp_\Theta(x)\cdot ee$
TbooleanAlgebra$_0$ $\Rightarrow$ *Stat35* : $\langle\forall x,y \mid \{x,y\} \subseteq bb \rightarrow x\cdot y = y\cdot x\rangle$ & *Stat35a* : $\langle\forall x,y \mid \{x,y\} \subseteq bb \rightarrow x \div y = y \div x\rangle$
$\langle b\cdot cmp_\Theta(y), ee\rangle \hookrightarrow$ *Stat35*(*Stat26a*, *Stat3*$\star$) $\Rightarrow$ $b\cdot cmp_\Theta(y)\cdot ee = ee\cdot\left(b\cdot cmp_\Theta(y)\right)$
$\langle a\cdot cmp_\Theta(x), ee\rangle \hookrightarrow$ *Stat35*(*Stat25a*, *Stat3*$\star$) $\Rightarrow$ $a\cdot cmp_\Theta(x)\cdot ee = ee\cdot\left(a\cdot cmp_\Theta(x)\right)$
$\langle b\cdot cmp_\Theta(y)\rangle \hookrightarrow$ *Stat3*(*Stat26a*, *Stat26a*$\star$) $\Rightarrow$ $ee\cdot\left(b\cdot cmp_\Theta(y)\right) = b\cdot cmp_\Theta(y)$
$\langle a\cdot cmp_\Theta(x)\rangle \hookrightarrow$ *Stat3*(*Stat25a*, *Stat25a*$\star$) $\Rightarrow$ $ee\cdot\left(a\cdot cmp_\Theta(x)\right) = a\cdot cmp_\Theta(x)$

EQUAL $\langle Stat34\rangle \Rightarrow$ $Stat36:\ a\cdot cmp_\Theta(x) \div b\cdot cmp_\Theta(y) \neq b\cdot cmp_\Theta(y)\cdot(x\cdot y) \div b\cdot cmp_\Theta(y) \div \left(a\cdot cmp_\Theta(x)\cdot(x\cdot y)\div a\cdot cmp_\Theta(x)\right)$

$\langle b\cdot cmp_\Theta(y), x\cdot y\rangle \hookrightarrow Stat9(Stat26a, Stat27a\star) \Rightarrow$ $Stat37:\ b\cdot cmp_\Theta(y)\cdot(x\cdot y)\in bb$

$\langle a\cdot cmp_\Theta(x), x\cdot y\rangle \hookrightarrow Stat9(Stat25a, Stat27a\star) \Rightarrow$ $Stat38:\ a\cdot cmp_\Theta(x)\cdot(x\cdot y)\in bb$

$\langle a\cdot cmp_\Theta(x)\cdot(x\cdot y), a\cdot cmp_\Theta(x)\rangle \hookrightarrow Stat26(Stat25a, Stat38\star) \Rightarrow$ $Stat39:\ a\cdot cmp_\Theta(x)\cdot(x\cdot y)\div a\cdot cmp_\Theta(x)\in bb$

$\langle b\cdot cmp_\Theta(y)\cdot(x\cdot y), b\cdot cmp_\Theta(y), a\cdot cmp_\Theta(x)\cdot(x\cdot y)\div a\cdot cmp_\Theta(x)\rangle \hookrightarrow Stat32a(Stat39, Stat37, Stat26a\star) \Rightarrow$

$b\cdot cmp_\Theta(y)\cdot(x\cdot y)\div b\cdot cmp_\Theta(y)\div\left(a\cdot cmp_\Theta(x)\cdot(x\cdot y)\div a\cdot cmp_\Theta(x)\right) =$

$b\cdot cmp_\Theta(y)\cdot(x\cdot y)\div\Big(b\cdot cmp_\Theta(y)\div\left(a\cdot cmp_\Theta(x)\cdot(x\cdot y)\div a\cdot cmp_\Theta(x)\right)\Big)$

$\langle b\cdot cmp_\Theta(y), a\cdot cmp_\Theta(x)\cdot(x\cdot y)\div a\cdot cmp_\Theta(x)\rangle \hookrightarrow Stat35a(Stat39, Stat26a\star) \Rightarrow$ $b\cdot cmp_\Theta(y)\div\left(a\cdot cmp_\Theta(x)\cdot(x\cdot y)\div a\cdot cmp_\Theta(x)\right) =$

$a\cdot cmp_\Theta(x)\cdot(x\cdot y)\div a\cdot cmp_\Theta(x)\div b\cdot cmp_\Theta(y)$

$\langle a\cdot cmp_\Theta(x)\cdot(x\cdot y), a\cdot cmp_\Theta(x), b\cdot cmp_\Theta(y)\rangle \hookrightarrow Stat32a(Stat38, Stat25a, Stat26a\star) \Rightarrow$ $a\cdot cmp_\Theta(x)\cdot(x\cdot y)\div\left(a\cdot cmp_\Theta(x)\div b\cdot cmp_\Theta(y)\right) =$

$a\cdot cmp_\Theta(x)\cdot(x\cdot y)\div a\cdot cmp_\Theta(x)\div b\cdot cmp_\Theta(y)$

$\langle b\cdot cmp_\Theta(y)\cdot(x\cdot y), a\cdot cmp_\Theta(x)\cdot(x\cdot y), a\cdot cmp_\Theta(x)\div b\cdot cmp_\Theta(y)\rangle \hookrightarrow Stat32a(Stat38, Stat37, Stat28a\star) \Rightarrow$

$b\cdot cmp_\Theta(y)\cdot(x\cdot y)\div\Big(a\cdot cmp_\Theta(x)\cdot(x\cdot y)\div\left(a\cdot cmp_\Theta(x)\div b\cdot cmp_\Theta(y)\right)\Big) =$

$b\cdot cmp_\Theta(y)\cdot(x\cdot y)\div a\cdot cmp_\Theta(x)\cdot(x\cdot y)\div\left(a\cdot cmp_\Theta(x)\div b\cdot cmp_\Theta(y)\right)$

EQUAL $\langle Stat36\rangle \Rightarrow$ $Stat40:\ a\cdot cmp_\Theta(x)\div b\cdot cmp_\Theta(y)\neq b\cdot cmp_\Theta(y)\cdot(x\cdot y)\div a\cdot cmp_\Theta(x)\cdot(x\cdot y)\div\left(a\cdot cmp_\Theta(x)\div b\cdot cmp_\Theta(y)\right)$

TbooleanAlgebra$_0$ $\Rightarrow$ $Stat41:\ \langle\forall x \mid (x\in bb \rightarrow x\div x = zz_\Theta \;\&\; x\div zz_\Theta = x \;\&\; zz_\Theta\div x = x) \;\&\; zz_\Theta\in bb\rangle$

Suppose $\Rightarrow$ $zz_\Theta = b\cdot cmp_\Theta(y)\cdot(x\cdot y)\div a\cdot cmp_\Theta(x)\cdot(x\cdot y)$

$\langle a\cdot cmp_\Theta(x)\div b\cdot cmp_\Theta(y)\rangle \hookrightarrow Stat41(Stat28a, Stat28a\star) \Rightarrow$ $zz_\Theta\div\left(a\cdot cmp_\Theta(x)\div b\cdot cmp_\Theta(y)\right) =$

$a\cdot cmp_\Theta(x)\div b\cdot cmp_\Theta(y)$

EQUAL $\langle Stat40\rangle \Rightarrow$ false; Discharge $\Rightarrow$ $Stat42:\ zz_\Theta\neq b\cdot cmp_\Theta(y)\cdot(x\cdot y)\div a\cdot cmp_\Theta(x)\cdot(x\cdot y)$

$\langle y, x, b\rangle \hookrightarrow Stat3a(Stat30, Stat30\star) \Rightarrow$ $b\cdot cmp_\Theta(y)\cdot(x\cdot y) = zz_\Theta$

$\langle x, y, a\rangle \hookrightarrow Stat3a(Stat30, Stat30\star) \Rightarrow$ $a\cdot cmp_\Theta(x)\cdot(y\cdot x) = zz_\Theta$

$\langle x, y\rangle \hookrightarrow Stat35(Stat30, Stat30\star) \Rightarrow$ $x\cdot y = y\cdot x$

$\langle zz_\Theta\rangle \hookrightarrow Stat41(Stat41, Stat41\star) \Rightarrow$ $zz_\Theta\div zz_\Theta = zz_\Theta$

EQUAL $\langle Stat42\rangle \Rightarrow$ false; Discharge $\Rightarrow$ QED

> The notion of (proper) ideal in a commutative ring with multiplicative unit is adjusted to the case of Boolean algebras as follows:

DEF booleanAlgebra$_2$: [proper ideal] $Ideal_\Theta(X) \leftrightarrow_{Def} \{x\div y :\ x\in X, y\in X\}\subseteq X \;\&\; \{x\cdot y :\ x\in bb, y\in X\}\subseteq X \;\&\; X\subseteq bb\setminus\{ee\} \;\&\; X\not\subseteq\{zz_\Theta\}$

-- closure properties of an ideal , 1

THM booleanAlgebra$_{4b}$. $Ideal_\Theta(I) \;\&\; \{X, Y\}\subseteq I \rightarrow X\div Y\in I$. PROOF:

Suppose_not$(i_0, x_0, y_0) \Rightarrow$ AUTO

Use_def$(Ideal_\Theta) \Rightarrow$ $Stat1:\ x_0\div y_0\notin\{x\div y :\ x\in i_0, y\in i_0\}$

$\langle x_0, y_0\rangle \hookrightarrow Stat1 \Rightarrow$ false; Discharge $\Rightarrow$ QED

In consequence of the definition of ideals, it turns out that the bottom zz_Θ of the Boolean algebra belongs to every ideal. Since multiplication in Boolean algebras is commutative, our ideals are bilateral: that is, they are closed not only under multiplication on the left (by any element of the domain-of-support), but also with respect to multiplication on the right. We are excluding from consideration the 'improper' ideals $\{\mathsf{zz}_\Theta\}, \mathsf{bb}$; accordingly, our ideals never contain two complementary elements (such complementary elements would bring into the ideal ee and, consequently, all elements of bb.

-- closure properties of an ideal , 2

THM booleanAlgebra$_{4a}$. $\mathsf{Ideal}_\Theta(\mathsf{I}) \rightarrow \mathsf{zz}_\Theta \in \mathsf{I}\ \&\ \big(\mathsf{X} \in \mathsf{I}\ \&\ \mathsf{Y} \in \mathsf{bb} \rightarrow \mathsf{X}\cdot\mathsf{Y}, \mathsf{Y}\cdot\mathsf{X} \in \mathsf{I}\ \&\ \mathsf{cmp}_\Theta(\mathsf{X}) \notin \mathsf{I}\big)\ \&\ \mathsf{ee} \notin \mathsf{I}$. PROOF:

Suppose_not$(\mathsf{i}_0, \mathsf{x}_0, \mathsf{y}_0) \Rightarrow$ *Stat1* : $\mathsf{Ideal}_\Theta(\mathsf{i}_0)\ \&\ \neg\Big(\mathsf{zz}_\Theta \in \mathsf{i}_0\ \&\ \big(\mathsf{x}_0 \in \mathsf{i}_0\ \&\ \mathsf{y}_0 \in \mathsf{bb} \rightarrow \mathsf{x}_0\cdot\mathsf{y}_0, \mathsf{y}_0\cdot\mathsf{x}_0 \in \mathsf{i}_0\ \&\ \mathsf{cmp}_\Theta(\mathsf{x}_0) \notin \mathsf{i}_0\big)\ \&\ \mathsf{ee} \notin \mathsf{i}_0\Big)$

Use_def$(\mathsf{Ideal}_\Theta) \Rightarrow$ *Stat2* : $\{\mathsf{x}\cdot\mathsf{y} : \mathsf{x} \in \mathsf{bb}, \mathsf{y} \in \mathsf{i}_0\} \subseteq \mathsf{i}_0\ \&\ \mathsf{i}_0 \nsubseteq \{\mathsf{zz}_\Theta\}\ \&\ \mathsf{i}_0 \subseteq \mathsf{bb}\setminus\{\mathsf{ee}\}$

TbooleanAlgebra$_0$ $\Rightarrow$ *Stat3* :

$\langle\forall \mathsf{x}, \mathsf{y} \mid \{\mathsf{x}, \mathsf{y}\} \subseteq \mathsf{bb} \rightarrow \mathsf{x}\cdot\mathsf{y} = \mathsf{y}\cdot\mathsf{x}\rangle\ \&$ *Stat3a* :

$\langle\forall \mathsf{x} \mid (\mathsf{x} \in \mathsf{bb} \rightarrow \mathsf{x} \div \mathsf{x} = \mathsf{zz}_\Theta\ \&\ \mathsf{x} \div \mathsf{zz}_\Theta = \mathsf{x}\ \&\ \mathsf{zz}_\Theta \div \mathsf{x} = \mathsf{x})\ \&\ \mathsf{zz}_\Theta \in \mathsf{bb}\rangle\ \&\ \langle\forall \mathsf{x} \mid \mathsf{x} \in \mathsf{bb} \rightarrow \mathsf{zz}_\Theta\cdot\mathsf{x} = \mathsf{zz}_\Theta\rangle$

$\langle\emptyset, \mathbf{arb}(\mathsf{i}_0\setminus\{\mathsf{zz}_\Theta\})\rangle \hookrightarrow$ *Stat3a*(*Stat2*) $\Rightarrow$ *Stat4* : $\mathsf{zz}_\Theta \in \mathsf{bb}\ \&\ \mathbf{arb}(\mathsf{i}_0\setminus\{\mathsf{zz}_\Theta\}) \in \mathsf{bb} \cap \mathsf{i}_0\ \&\ \mathsf{zz}_\Theta\cdot\mathbf{arb}(\mathsf{i}_0\setminus\{\mathsf{zz}_\Theta\}) = \mathsf{zz}_\Theta$

Suppose $\Rightarrow$ *Stat6* : $\mathsf{zz}_\Theta \notin \mathsf{i}_0$

(*Stat6*, *Stat2*$\star$)ELEM $\Rightarrow$ *Stat7* : $\mathsf{zz}_\Theta \notin \{\mathsf{x}\cdot\mathsf{y} : \mathsf{x} \in \mathsf{bb}, \mathsf{y} \in \mathsf{i}_0\}$

$\langle\mathsf{zz}_\Theta, \mathbf{arb}(\mathsf{i}_0\setminus\{\mathsf{zz}_\Theta\})\rangle \hookrightarrow$ *Stat7*(*Stat4*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO

$\langle\mathsf{x}_0, \mathsf{y}_0\rangle \hookrightarrow$ *Stat3*(*Stat1*$\star$) $\Rightarrow$ *Stat8* : $\mathsf{x}_0 \in \mathsf{i}_0\ \&\ \mathsf{y}_0 \in \mathsf{bb}\ \&\ \mathsf{x}_0\cdot\mathsf{y}_0 = \mathsf{y}_0\cdot\mathsf{x}_0$

Suppose $\Rightarrow$ *Stat11* : $\mathsf{cmp}_\Theta(\mathsf{x}_0) \in \mathsf{i}_0$

$\langle\mathsf{x}_0, \mathsf{x}_0\rangle \hookrightarrow T$booleanAlgebra$_{1a}$ (*Stat8*, *Stat2*$\star$) $\Rightarrow$ *Stat14* : $\mathsf{cmp}_\Theta(\mathsf{x}_0) \div \mathsf{x}_0 \notin \mathsf{i}_0$

$\langle\mathsf{i}_0, \mathsf{cmp}_\Theta(\mathsf{x}_0), \mathsf{x}_0\rangle \hookrightarrow T$booleanAlgebra$_{4b}$ (*Stat1*, *Stat8*, *Stat11*, *Stat14*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat10* :

$\mathsf{y}_0\cdot\mathsf{x}_0 \notin \{\mathsf{x}\cdot\mathsf{y} : \mathsf{x} \in \mathsf{bb}, \mathsf{y} \in \mathsf{i}_0\} \vee \mathsf{x}_0\cdot\mathsf{y}_0 \notin \{\mathsf{x}\cdot\mathsf{y} : \mathsf{x} \in \mathsf{bb}, \mathsf{y} \in \mathsf{i}_0\}\ \&\ \mathsf{x}_0\cdot\mathsf{y}_0 = \mathsf{y}_0\cdot\mathsf{x}_0$

$\langle\mathsf{y}_0, \mathsf{x}_0, \mathsf{y}_0, \mathsf{x}_0\rangle \hookrightarrow$ *Stat10*(*Stat2*, *Stat4*, *Stat8*, *Stat10*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ QED

As an application of Zorn's lemma, we have the following maximal ideal lemma:

-- maximal proper ideal

THM booleanAlgebra$_5$. $\mathsf{Ideal}_\Theta(\mathsf{I}) \rightarrow \langle\exists \mathsf{m} \mid \mathsf{I} \subseteq \mathsf{m}\ \&\ \langle\forall \mathsf{j} \mid \mathsf{Ideal}_\Theta(\mathsf{j})\ \&\ \mathsf{m} \subseteq \mathsf{j} \leftrightarrow \mathsf{j} = \mathsf{m}\rangle\rangle$. PROOF:

Suppose_not$(\mathsf{i}_0) \Rightarrow$ *Stat13* : $\neg\langle\exists \mathsf{m} \mid \mathsf{i}_0 \subseteq \mathsf{m}\ \&\ \langle\forall \mathsf{j} \mid \mathsf{Ideal}_\Theta(\mathsf{j})\ \&\ \mathsf{m} \subseteq \mathsf{j} \leftrightarrow \mathsf{j} = \mathsf{m}\rangle\rangle\ \&\ \mathsf{Ideal}_\Theta(\mathsf{i}_0)$

Arguing by contradiction, let us assume that i_0 is a counterexample to our claim. We will consider the family tt of all ideals which contain i_0. This is nonnull, because i_0 belongs to it; moreover, it is closed with respect to unionset formation.

Use_def$(\mathsf{Ideal}_\Theta) \Rightarrow$ *Stat0* : $\mathsf{Ideal}_\Theta(\mathsf{i}_0)\ \&\ \{\mathsf{x}\cdot\mathsf{y} : \mathsf{x} \in \mathsf{bb}, \mathsf{y} \in \mathsf{i}_0\} \subseteq \mathsf{i}_0\ \&\ \mathsf{i}_0 \subseteq \mathsf{bb}\ \&\ \mathsf{i}_0 \nsubseteq \{\mathsf{zz}_\Theta\}\ \&\ \mathsf{ee} \notin \mathsf{i}_0$

Loc_def $\Rightarrow$ *Stat2a* : $\mathsf{tt} = \{\mathsf{i} \subseteq \mathsf{bb}\setminus\{\mathsf{ee}\} \mid \mathsf{Ideal}_\Theta(\mathsf{i})\ \&\ \mathsf{i}_0 \subseteq \mathsf{i}\}$

Suppose $\Rightarrow$ $\quad$ $\mathit{Stat1a}: \neg\langle\forall x \subseteq tt \mid \langle\forall u \in x, v \in x \mid u \supseteq v \vee v \supseteq u\rangle \rightarrow \langle\exists w \in tt, \forall y \in x \mid w \supseteq y\rangle\rangle$

|| To see that the conditions for applicability of Zorn's lemma are met, we argue as follows. Suppose $t_0 \subseteq tt$ is linearly ordered by inclusion but does not admit an upper bound relative to inclusion.

$\langle t_0\rangle\hookrightarrow \mathit{Stat1a}(\mathit{Stat1a}\star) \Rightarrow$ $\quad$ $\mathit{Stat2}: \neg\langle\exists w \in tt, \forall y \in t_0 \mid w \supseteq y\rangle \;\&\; t_0 \subseteq tt \;\&\; \mathit{Stat20}: \langle\forall u \in t_0, v \in t_0 \mid u \supseteq v \vee v \supseteq u\rangle$
Suppose $\Rightarrow$ $\quad$ $t_0 = \emptyset$

|| t_0 cannot be null, else i_0 would be an upper bound for it.

$\langle i_0\rangle\hookrightarrow \mathit{Stat2}(\mathit{Stat2a}\star) \Rightarrow$ $\quad$ $\mathit{Stat0a}: i_0 \notin \{i \subseteq bb\setminus\{ee\} \mid \mathrm{Ideal}_\Theta(i) \;\&\; i_0 \subseteq i\} \vee \neg\langle\forall y \in t_0 \mid i_0 \supseteq y\rangle$
$\langle i_0\rangle\hookrightarrow \mathit{Stat0a}(\mathit{Stat0}, \mathit{Stat0a}\star) \Rightarrow$ $\quad$ $\mathit{Stat3a}: \neg\langle\forall y \in t_0 \mid i_0 \supseteq y\rangle$
$\langle y_0\rangle\hookrightarrow \mathit{Stat3a}(\mathit{Stat2}\star) \Rightarrow$ $\quad$ false; $\qquad$ Discharge $\Rightarrow$ $\quad$ AUTO

|| Therefore we can draw an ideal i_2 from t_0; this, as therefore its subset i_0, is obviously included in $\bigcup t_0$.

Loc_def $\Rightarrow$ $\quad$ $i_2 = \mathbf{arb}(t_0)$
$\langle i_2, t_0\rangle\hookrightarrow T108 \Rightarrow$ $\quad$ AUTO
$(\mathit{Stat2a})$ELEM $\Rightarrow$ $\quad$ $\mathit{Stat4a}: i_2 \in \{i \subseteq bb\setminus\{ee\} \mid \mathrm{Ideal}_\Theta(i) \;\&\; i_0 \subseteq i\} \;\&\; i_2 \subseteq \bigcup t_0$
$\langle\,\rangle\hookrightarrow \mathit{Stat4a}(\mathit{Stat4a}\star) \Rightarrow$ $\quad$ $i_0 \subseteq \bigcup t_0$

|| Either $\bigcup t_0$ does not belong to tt or it does not include some member y_1 of t_0, because otherwise $\bigcup t_0$ would be an upper bound for t_0. We readily exclude the latter possibility.

$\langle\bigcup t_0\rangle\hookrightarrow \mathit{Stat2}(\mathit{Stat2}\star) \Rightarrow$ $\quad$ $\bigcup t_0 \notin tt \vee \neg\langle\forall y \in t_0 \mid \bigcup t_0 \supseteq y\rangle$
Suppose $\Rightarrow$ $\quad$ $\mathit{Stat10}: \neg\langle\forall y \in t_0 \mid \bigcup t_0 \supseteq y\rangle$
$\langle y_1\rangle\hookrightarrow \mathit{Stat10}(\mathit{Stat10}\star) \Rightarrow$ $\quad$ $y_1 \in t_0 \;\&\; \bigcup t_0 \not\supseteq y_1$
$\langle y_1, t_0\rangle\hookrightarrow T108\ (\mathit{Stat10}\star) \Rightarrow$ $\quad$ false; $\qquad$ Discharge $\Rightarrow$ $\quad$ $\mathit{Stat3}: \bigcup t_0 \notin \{i \subseteq bb\setminus\{ee\} \mid \mathrm{Ideal}_\Theta(i) \;\&\; i_0 \subseteq i\}$

|| We will exclude the other alternative as well, arguing as follows. If $\bigcup t_0$ does not belong to tt, since it clearly includes i_0 it must fail to be an ideal and the reason why $\bigcup t_0$ might fail to be an ideal cannot be that it is not included in $bb\setminus\{ee\}$.

$\langle\bigcup t_0\rangle\hookrightarrow \mathit{Stat3}(\mathit{Stat4a}\star) \Rightarrow$ $\quad$ $\bigcup t_0 \not\subseteq bb\setminus\{ee\} \vee \neg\mathrm{Ideal}_\Theta(\bigcup t_0)$
Suppose $\Rightarrow$ $\quad$ $\mathit{Stat4}: \bigcup t_0 \not\subseteq bb\setminus\{ee\}$
$\langle t_0, bb\setminus\{ee\}\rangle\hookrightarrow T274\ (\mathit{Stat3}\star) \Rightarrow$ $\quad$ $\mathit{Stat5}: \neg\langle\forall x \in t_0 \mid x \subseteq bb\setminus\{ee\}\rangle$
$\langle x_0\rangle\hookrightarrow \mathit{Stat5}(\mathit{Stat2}, \mathit{Stat2a}\star) \Rightarrow$ $\quad$ $\mathit{Stat5a}: x_0 \in \{i \subseteq bb\setminus\{ee\} \mid \mathrm{Ideal}_\Theta(i) \;\&\; i_0 \subseteq i\} \;\&\; x_0 \not\subseteq bb\setminus\{ee\}$

$\langle\,\rangle\hookrightarrow$ *Stat5a*(*Stat5a*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO
Use_def $(\mathsf{Ideal}_{\ominus}(\bigcup \mathsf{t}_0))$ $\Rightarrow$ AUTO

Notice that since t_0 is an inclusion chain in tt, its unionset is closed with respect to addition, because if we take two elements x_3, x_4 in $\bigcup t_0$, then each will belong to a set in t_0 and therefore both will belong to the larger of these sets, which will be an ideal because all sets in tt, and hence all sets in t_0, are ideals; but then the addition of x_3, x_4 will belong to this larger set, which is included in $\bigcup t_0$.

Suppose $\Rightarrow$ *Stat21* : $\{x \div y : x \in \bigcup t_0, y \in \bigcup t_0\} \not\subseteq \bigcup t_0$
$\langle p\rangle\hookrightarrow$ *Stat21*(*Stat21*$\star$) $\Rightarrow$ *Stat22* : $p \in \{x \div y : x \in \bigcup t_0, y \in \bigcup t_0\} \;\&\; p \notin \bigcup t_0$
Use_def $(\bigcup t_0)$ $\Rightarrow$ AUTO
$\langle x_3, x_4\rangle\hookrightarrow$ *Stat22*(*Stat22*$\star$) $\Rightarrow$ *Stat23* : $x_3 \in \{x : y \in t_0, x \in y\} \;\&\; x_4 \in \bigcup t_0 \;\&\; x_3 \div x_4 \notin \bigcup t_0$
$\langle y_4, t_0\rangle\hookrightarrow$ *T108* $\Rightarrow$ AUTO
$\langle y_4, x_5\rangle\hookrightarrow$ *Stat23*(*Stat22*$\star$) $\Rightarrow$ *Stat24* : $x_4 \in \{x : y \in t_0, x \in y\} \;\&\; y_4 \in t_0 \;\&\; x_3 \in y_4 \;\&\; x_3 \div x_4 \notin y_4$
$\langle y_5, t_0\rangle\hookrightarrow$ *T108* $\Rightarrow$ AUTO
$\langle y_5, x_6\rangle\hookrightarrow$ *Stat24*(*Stat23*$\star$) $\Rightarrow$ *Stat25* : $y_5 \in t_0 \;\&\; x_4 \in y_5 \;\&\; x_3 \div x_4 \notin y_5$
(*Stat2, Stat2a, Stat24, Stat25*$\star$)ELEM $\Rightarrow$ *Stat26* :
$y_4 \in \{i \subseteq bb \setminus \{ee\} \mid \mathsf{Ideal}_{\ominus}(i) \;\&\; i_0 \subseteq i\} \;\&\;$ *Stat27* : $y_5 \in \{i \subseteq bb \setminus \{ee\} \mid \mathsf{Ideal}_{\ominus}(i) \;\&\; i_0 \subseteq i\}$
$\langle y_4, y_5\rangle\hookrightarrow$ *Stat20*(*Stat24, Stat25*$\star$) $\Rightarrow$ $y_4 \supseteq y_5 \vee y_5 \supseteq y_4$
Suppose $\Rightarrow$ *Stat25a* : $y_4 \supseteq y_5$
$\langle\,\rangle\hookrightarrow$ *Stat26*(*Stat26, Stat26*$\star$) $\Rightarrow$ *Stat28* : $\mathsf{Ideal}_{\ominus}(y_4)$
Use_def $(\mathsf{Ideal}_{\ominus})$ $\Rightarrow$ *Stat29* : $x_3 \div x_4 \notin \{x \div y : x \in y_4, y \in y_4\}$
$\langle x_3, x_4\rangle\hookrightarrow$ *Stat29*(*Stat24, Stat25, Stat25a*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO
(*Stat24*$\star$)ELEM $\Rightarrow$ $x_3 \in y_5$
$\langle\,\rangle\hookrightarrow$ *Stat27*(*Stat26, Stat26*$\star$) $\Rightarrow$ *Stat30* : $\mathsf{Ideal}_{\ominus}(y_5)$
Use_def $(\mathsf{Ideal}_{\ominus})$ $\Rightarrow$ *Stat31* : $x_3 \div x_4 \notin \{x \div y : x \in y_5, y \in y_5\}$
$\langle x_3, x_4\rangle\hookrightarrow$ *Stat31*(*Stat25*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO

Notice also that since t_0 is an inclusion chain in tt, its unionset is closed under multiplication by any element of the Boolean algebra. Actually, this fact is even more obvious than the analogous fact about addition just seen, because the assumption that t_0 is a chain (viz., totally ordered by inclusion) is immaterial here. If we take an element y_2 of $\bigcup t_0$ and an element x_2 of the algebra, then y2 will belong to a set i_1 in t_0, which will be an ideal because all sets in tt, and hence all sets in t_0, are ideals; but then the result of multiplying y_2 by x_2 will belong to this set i_1, which is included in $\bigcup t_0$.

(*Stat0*$\star$)ELEM $\Rightarrow$ *Stat6* : $\{x \cdot y : x \in bb, y \in \bigcup t_0\} \not\subseteq \bigcup t_0$

$\langle d_0 \rangle \hookrightarrow Stat6(Stat6\star) \Rightarrow \quad Stat7: \ d_0 \in \{x \cdot y : \ x \in bb, y \in \bigcup t_0\} \ \& \ d_0 \notin \bigcup t_0$
$\langle x_2, y_2 \rangle \hookrightarrow Stat7(Stat7\star) \Rightarrow \quad Stat7a: \ x_2 \in bb \ \& \ y_2 \in \bigcup t_0 \ \& \ x_2 \cdot y_2 \notin \bigcup t_0$
$\text{Use\_def}(\bigcup) \Rightarrow \quad Stat8: \ y_2 \in \{x : \ y \in t_0, x \in y\}$
$\langle i_1, x_1 \rangle \hookrightarrow Stat8(Stat2, Stat2a\star) \Rightarrow \quad Stat9: \ i_1 \in \{i \subseteq bb \setminus \{ee\} \mid \text{Ideal}_\Theta(i) \ \& \ i_0 \subseteq i\} \ \& \ i_1 \in t_0 \ \& \ y_2 \in i_1$
$\langle \ \rangle \hookrightarrow Stat9(Stat9\star) \Rightarrow \quad \text{Ideal}_\Theta(i_1)$
$\langle i_1, t_0 \rangle \hookrightarrow T108 \Rightarrow \quad \text{AUTO}$
$\langle i_1, y_2, x_2 \rangle \hookrightarrow T\text{booleanAlgebra}_{4a} \ (Stat7a\star) \Rightarrow \quad \text{false}; \qquad \text{Discharge} \Rightarrow \quad \text{AUTO}$

In conclusion, by Zorn's lemma, tt contains a maximal element m. It is easily shown that this m satisfies the consequent of the claim, thus contradicting the initial assumption of this proof. Our argument by contradiction is thereby completed, and the desired claim proved.

$\langle tt \rangle \hookrightarrow T412 \Rightarrow \quad Stat11: \ \langle \exists y \in tt, \forall x \in tt \mid \neg(x \supseteq y \ \& \ x \neq y) \rangle$
$\langle m \rangle \hookrightarrow Stat11(Stat2a, Stat2a\star) \Rightarrow \quad Stat12: \ \langle \forall x \in tt \mid \neg(x \supseteq m \ \& \ x \neq m) \rangle \ \& \ Stat14: \ m \in \{i \subseteq bb \setminus \{ee\} \mid \text{Ideal}_\Theta(i) \ \& \ i_0 \subseteq i\}$
$\langle \ \rangle \hookrightarrow Stat14 \Rightarrow \quad \text{AUTO}$
$\langle m \rangle \hookrightarrow Stat13(Stat12\star) \Rightarrow \quad Stat15: \ \neg \langle \forall j \mid \text{Ideal}_\Theta(j) \ \& \ m \subseteq j \leftrightarrow j = m \rangle$
$\langle j_1 \rangle \hookrightarrow Stat15(Stat15\star) \Rightarrow \quad \text{Ideal}_\Theta(j_1) \ \& \ m \subseteq j_1 \neq j_1 = m$
$\text{Suppose} \Rightarrow \quad j_1 = m$
$\text{EQUAL} \ \langle Stat12 \rangle \Rightarrow \quad \text{false}; \qquad \text{Discharge} \Rightarrow \quad Stat16: \ j_1 \neq m \ \& \ \text{Ideal}_\Theta(j_1) \ \& \ m \subseteq j_1 \ \& \ i_0 \subseteq m$
$\text{Use\_def}(\text{Ideal}_\Theta(j_1)) \Rightarrow \quad \text{AUTO}$
$\langle j_1 \rangle \hookrightarrow Stat12(Stat16, Stat2a\star) \Rightarrow \quad Stat17: \ j_1 \notin \{i \subseteq bb \setminus \{ee\} \mid \text{Ideal}_\Theta(i) \ \& \ i_0 \subseteq i\}$
$\langle j_1 \rangle \hookrightarrow Stat17(Stat16\star) \Rightarrow \quad \text{false}; \qquad \text{Discharge} \Rightarrow \quad \text{QED}$

The following theorem provides "fuel" for applicability of the maximal ideal lemma:

-- ideal generated by complements

THM booleanAlgebra$_6$. $B \subseteq bb \setminus \{zz_\Theta\} \ \& \ \{x \cdot y : \ x \in B, y \in B\} \subseteq B \ \& \ B \not\subseteq \{ee\} \rightarrow$
$\text{Ideal}_\Theta(\{a \cdot cmp_\Theta(x) : \ a \in bb, x \in B\}) \ \& \ \{cmp_\Theta(x) : \ x \in B\} \subseteq \{a \cdot cmp_\Theta(x) : \ a \in bb, x \in B\}$. PROOF:
$\text{Suppose\_not}(b_0) \Rightarrow \quad \text{AUTO}$

Assuming that a counterexample b_0 exists, $\{a \cdot cmp_\Theta(x) : \ a \in bb, x \in b_0\}$ would not be an ideal. As a preliminary to a refutation of this, observe that this set has an element which differs from zz_Θ and from ee.

$\text{Loc\_def} \Rightarrow \quad a_0 = \mathbf{arb}(b_0 \setminus \{ee\})$
$\text{ELEM} \Rightarrow \quad Stat1: \ \{x \cdot y : \ x \in b_0, y \in b_0\} \subseteq b_0 \ \& \ a_0 \in b_0 \setminus \{ee\} \ \& \ a_0 \in bb \ \& \ b_0 \subseteq bb \setminus \{zz_\Theta\}$
$\text{Loc\_def} \Rightarrow \quad Stat2: \ i_0 = \{a \cdot cmp_\Theta(x) : \ a \in bb, x \in b_0\}$
$\text{Assump} \Rightarrow \quad Stat3: \ \langle \forall x \mid x \in bb \rightarrow ee \cdot x = x \rangle \ \& \ ee \in bb$
$\text{Suppose} \Rightarrow \quad Stat80: \ \{cmp_\Theta(x) : \ x \in b_0\} \not\subseteq \{a \cdot cmp_\Theta(x) : \ a \in bb, x \in b_0\}$

$\langle c_6 \rangle \hookrightarrow Stat80(Stat80\star) \Rightarrow \quad Stat81: \; c_6 \in \{cmp_\Theta(x): \; x \in b_0\} \;\&\; c_6 \notin \{a \cdot cmp_\Theta(x): \; a \in bb, x \in b_0\}$
$\langle cmp_\Theta(x_6) \rangle \hookrightarrow Stat3(Stat3, Stat3\star) \Rightarrow \quad (cmp_\Theta(x_6) \in bb \to ee \cdot cmp_\Theta(x_6) = cmp_\Theta(x_6)) \;\&\; ee \in bb$
$\langle x_6 \rangle \hookrightarrow T\text{booleanAlgebra}_1 \; (Stat81\star) \Rightarrow \quad x_6 \in bb \to cmp_\Theta(x_6) \in bb$
$\langle x_6, ee, x_6 \rangle \hookrightarrow Stat81(\star) \Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow \quad \neg \text{Ideal}_\Theta(\{a \cdot cmp_\Theta(x): \; a \in bb, x \in b_0\})$
Use_def $\left(\text{Ideal}_\Theta(\{a \cdot cmp_\Theta(x): \; a \in bb, x \in b_0\})\right) \Rightarrow \quad$ AUTO
EQUAL $\Rightarrow \quad \neg(\{x \div y: \; x \in i_0, y \in i_0\} \subseteq i_0 \;\&\; \{x \cdot y: \; x \in bb, y \in i_0\} \subseteq i_0 \;\&\; i_0 \subseteq bb \setminus \{ee\} \;\&\; i_0 \not\subseteq \{zz_\Theta\})$
Suppose $\Rightarrow \quad i_0 \subseteq \{zz_\Theta\}$
$\langle a_0 \rangle \hookrightarrow T\text{booleanAlgebra}_1 \; (Stat1\star) \Rightarrow \quad cmp_\Theta(a_0) \in bb \;\&\; cmp_\Theta(cmp_\Theta(a_0)) = a_0 \;\&\; cmp_\Theta(zz_\Theta) = ee$
$\langle cmp_\Theta(a_0) \rangle \hookrightarrow Stat3(Stat3\star) \Rightarrow \quad ee \cdot cmp_\Theta(a_0) = cmp_\Theta(a_0)$
Suppose $\Rightarrow \quad Stat4: \; cmp_\Theta(a_0) \notin \{a \cdot cmp_\Theta(x): \; a \in bb, x \in b_0\}$
$\langle ee, a_0 \rangle \hookrightarrow Stat4(Stat1\star) \Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow \quad$ AUTO
$(Stat2\star)$ELEM $\Rightarrow \quad zz_\Theta = cmp_\Theta(a_0)$
EQUAL $\langle Stat2, \star \rangle \Rightarrow \quad Stat5: \; ee = a_0$
$(Stat1, Stat5\star)$Discharge $\Rightarrow \quad$ AUTO
Assump $\Rightarrow \quad Stat9: \; \langle \forall x, y \mid \{x, y\} \subseteq bb \to x \cdot y \in bb \rangle \;\&\; Stat9a: \; \langle \forall x, y, z \mid \{x, y, z\} \subseteq bb \to x \cdot (y \cdot z) = (x \cdot y) \cdot z \rangle$
Suppose $\Rightarrow \quad Stat6: \; i_0 \not\subseteq bb \setminus \{ee\}$
$\langle c \rangle \hookrightarrow Stat6(Stat2\star) \Rightarrow \quad Stat7: \; c \in \{a \cdot cmp_\Theta(x): \; a \in bb, x \in b_0\} \;\&\; c \notin bb \setminus \{ee\}$
$\langle a_3, cmp_\Theta(x_3) \rangle \hookrightarrow Stat9(Stat9, Stat9\star) \Rightarrow \quad Stat14: \; a_3, cmp_\Theta(x_3) \in bb \to a_3 \cdot cmp_\Theta(x_3) \in bb$
$\langle x_3 \rangle \hookrightarrow T\text{booleanAlgebra}_1 \; (Stat1, Stat1\star) \Rightarrow \quad Stat15: \; x_3 \in bb \to cmp_\Theta(x_3) \in bb$
$\langle a_3, x_3 \rangle \hookrightarrow Stat7(Stat15, Stat14, Stat1\star) \Rightarrow \quad Stat16:$
$\quad a_3 \in bb \;\&\; x_3 \in b_0 \;\&\; x_3, cmp_\Theta(x_3) \in bb \;\&\; a_3 \cdot cmp_\Theta(x_3) = ee$
$\langle a_3, cmp_\Theta(x_3) \rangle \hookrightarrow T\text{booleanAlgebra}_3 \; (Stat16\star) \Rightarrow \quad Stat18a: \; cmp_\Theta(x_3) = ee$
$\langle x_3 \rangle \hookrightarrow T\text{booleanAlgebra}_1 \; (Stat16, Stat16\star) \Rightarrow \quad cmp_\Theta(ee) = zz_\Theta \;\&\; cmp_\Theta(cmp_\Theta(x_3)) = x_3$
EQUAL $\langle Stat18a \rangle \Rightarrow \quad Stat18: \; zz_\Theta = x_3$
$(Stat16, Stat18, Stat1\star)$Discharge $\Rightarrow \quad$ AUTO
Suppose $\Rightarrow \quad Stat10: \; \{x \cdot y: \; x \in bb, y \in i_0\} \not\subseteq i_0$
$\langle d \rangle \hookrightarrow Stat10(Stat10\star) \Rightarrow \quad Stat11: \; d \in \{x \cdot y: \; x \in bb, y \in i_0\} \;\&\; d \notin i_0$
$\langle x_1, y_1 \rangle \hookrightarrow Stat11(Stat2\star) \Rightarrow \quad Stat12:$
$\quad y_1 \in \{a \cdot cmp_\Theta(x): \; a \in bb, x \in b_0\} \;\&\; x_1 \cdot y_1 \notin \{a \cdot cmp_\Theta(x): \; a \in bb, x \in b_0\} \;\&\; x_1 \in bb$
$\langle x_1, a_1 \rangle \hookrightarrow Stat9(Stat12\star) \Rightarrow \quad a_1 \in bb \to x_1 \cdot a_1 \in bb$
$\langle a_1, x_2, x_1 \cdot a_1, x_2 \rangle \hookrightarrow Stat12(Stat12\star) \Rightarrow \quad Stat13:$
$\quad a_1 \in bb \;\&\; x_2 \in b_0 \;\&\; y_1 = a_1 \cdot cmp_\Theta(x_2) \;\&\; x_1 \cdot y_1 \neq x_1 \cdot a_1 \cdot cmp_\Theta(x_2)$
EQUAL $\langle Stat12 \rangle \Rightarrow \quad x_1 \cdot (a_1 \cdot cmp_\Theta(x_2)) \neq x_1 \cdot a_1 \cdot cmp_\Theta(x_2)$
$\langle x_2 \rangle \hookrightarrow T\text{booleanAlgebra}_1 \; (Stat1, Stat13\star) \Rightarrow \quad cmp_\Theta(x_2) \in bb$
$\langle x_1, a_1, cmp_\Theta(x_2) \rangle \hookrightarrow Stat9a(Stat12\star) \Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow \quad$ AUTO
$(Stat2\star)$ELEM $\Rightarrow \quad Stat20: \; \{x \div y: \; x \in i_0, y \in i_0\} \not\subseteq i_0$
$\langle h \rangle \hookrightarrow Stat20(Stat20\star) \Rightarrow \quad Stat21: \; h \in \{x \div y: \; x \in i_0, y \in i_0\} \;\&\; h \notin i_0$
$\langle x_4, y_4 \rangle \hookrightarrow Stat21(Stat2, Stat2) \Rightarrow \quad Stat22:$
$\quad x_4, y_4 \in \{a \cdot cmp_\Theta(x): \; a \in bb, x \in b_0\} \;\&\; x_4 \div y_4 \notin \{u \cdot cmp_\Theta(v): \; u \in bb, v \in b_0\}$

$\langle a, x, b, y, a \cdot cmp_\ominus(x) \div b \cdot cmp_\ominus(y), x \cdot y\rangle \hookrightarrow Stat22(Stat22\star) \Rightarrow$ *Stat23* :
$a \in bb \; \& \; x \in b_0 \; \& \; x_4 = a \cdot cmp_\ominus(x) \; \& \; b \in bb \; \& \; y \in b_0 \; \& \; y_4 = b \cdot cmp_\ominus(y) \; \&$
$\neg\Big(x_4 \div y_4 = \big(a \cdot cmp_\ominus(x) \div b \cdot cmp_\ominus(y)\big) \cdot cmp_\ominus(x \cdot y) \; \& \; a \cdot cmp_\ominus(x) \div b \cdot cmp_\ominus(y) \in bb \; \& \; x \cdot y \in b_0\Big)$
$(Stat23, Stat1\star)$ELEM $\Rightarrow$ *Stat23a* : $x, y \in bb$
Suppose $\Rightarrow$ *Stat24* : $x \cdot y \notin b_0$
$(Stat24, Stat1\star)$ELEM $\Rightarrow$ *Stat25* : $x \cdot y \notin \{u \cdot v : u \in b_0, v \in b_0\}$
$\langle x, y\rangle \hookrightarrow Stat25(Stat23\star) \Rightarrow$ false; Discharge $\Rightarrow$ *Stat24a* : $\neg x \cdot y \notin b_0$
$\langle x\rangle \hookrightarrow T$booleanAlgebra$_1$ $(Stat23, Stat1, Stat24a\star) \Rightarrow$ *Stat27a* : $x, cmp_\ominus(x), x \cdot y \in bb$
$\langle y\rangle \hookrightarrow T$booleanAlgebra$_1$ $(Stat23, Stat1\star) \Rightarrow$ $x, cmp_\ominus(y) \in bb$
$\langle a, cmp_\ominus(x)\rangle \hookrightarrow Stat9(Stat23\star) \Rightarrow$ *Stat25a* : $a \cdot cmp_\ominus(x) \in bb$
$\langle b, cmp_\ominus(y)\rangle \hookrightarrow Stat9(Stat23\star) \Rightarrow$ *Stat26a* : $b \cdot cmp_\ominus(y) \in bb$
Assump $\Rightarrow$ *Stat26* : $\langle \forall u, v \mid \{u, v\} \subseteq bb \rightarrow u \div v \in bb\rangle$
$\langle a \cdot cmp_\ominus(x), b \cdot cmp_\ominus(y)\rangle \hookrightarrow Stat26(Stat23a\star) \Rightarrow$ *Stat28a* : $a \cdot cmp_\ominus(x) \div b \cdot cmp_\ominus(y) \in bb$
$(Stat23\star)$ELEM $\Rightarrow$ $x_4 \div y_4 \neq \big(a \cdot cmp_\ominus(x) \div b \cdot cmp_\ominus(y)\big) \cdot cmp_\ominus(x \cdot y)$
EQUAL $\langle Stat23\rangle \Rightarrow$ *Stat30* : $a \cdot cmp_\ominus(x) \div b \cdot cmp_\ominus(y) \neq \big(a \cdot cmp_\ominus(x) \div b \cdot cmp_\ominus(y)\big) \cdot cmp_\ominus(x \cdot y)$
$\langle a, b, x, y\rangle \hookrightarrow T$booleanAlgebra$_4$ $(Stat23, Stat23a, Stat30\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

The following is an obvious corollary of the theorem on the ideal generated by complements.

-- principal ideal
THM booleanAlgebra$_{6a}$. $X \in bb \setminus \{zz_\ominus, ee\} \rightarrow Ideal_\ominus(\{a \cdot X : a \in bb\}) \; \& \; X \in \{a \cdot X : a \in bb\}$. PROOF:
Suppose_not$(x_0) \Rightarrow$ AUTO

The negation of the is that either the set i_0 of all multiples of x_0 is not an ideal or it does not have x_0 as an element. The latter possibility is excluded readily.

Assump $\Rightarrow$ *Stat3* : $\langle \forall x \mid x \in bb \rightarrow x \cdot x = x\rangle \; \&$ *Stat2a* : $\langle \forall x \mid x \in bb \rightarrow ee \cdot x = x\rangle \; \& \; ee \in bb$
Suppose $\Rightarrow$ *Stat3a* : $x_0 \notin \{a \cdot x_0 : a \in bb\}$
$\langle x_0\rangle \hookrightarrow Stat2a(\star) \Rightarrow$ $ee \cdot x_0 = x_0$
$\langle ee\rangle \hookrightarrow Stat3a(Stat3\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO

We can apply the preceding Theorem booleanAlgebra$_6$ to the singleton of $cmp_\ominus(x_0)$, after checking that this set is closed under multipication.

Loc_def $\Rightarrow$ *Stat0* : $x_1 = cmp_\ominus(x_0)$
$\langle x_0\rangle \hookrightarrow T$booleanAlgebra$_1$ $(\star) \Rightarrow$ *Stat1* : $x_0 \in bb \; \& \; x_0 \neq ee \; \& \; x_0 \neq zz_\ominus \; \& \; x_1 \in bb \; \& \; cmp_\ominus\big(cmp_\ominus(x_0)\big) = x_0 \; \& \; \neg Ideal_\ominus(\{a \cdot x_0 : a \in bb\})$
EQUAL $\langle Stat0\rangle \Rightarrow$ *Stat2* : $cmp_\ominus(x_1) = x_0$
$\langle x_0\rangle \hookrightarrow T$booleanAlgebra$_2$ $(Stat0\star) \Rightarrow$ $\{x_1\} \subseteq bb \setminus \{zz_\ominus, ee\}$

Suppose ⇒ *Stat4* : $\{x \cdot y : x \in \{x_1\}, y \in \{x_1\}\} \not\subseteq \{x_1\}$
$\langle c \rangle \hookrightarrow$ *Stat4(Stat1⋆)* ⇒ *Stat5* : $c \in \{x \cdot y : x \in \{x_1\}, y \in \{x_1\}\}$ & $c \notin \{x_1\}$
$\langle x_2, y_2 \rangle \hookrightarrow$ *Stat5(Stat5⋆)* ⇒ *Stat5a* : $x_2 = x_1$ & $y_2 = x_1$ & $x_2 \cdot y_2 \neq x_1$
$\langle x_1 \rangle \hookrightarrow$ *Stat3(Stat1, Stat1⋆)* ⇒ $x_1 \cdot x_1 = x_1$
EQUAL $\langle$*Stat5a*$\rangle$ ⇒ false; Discharge ⇒ AUTO
$\langle \{x_1\} \rangle \hookrightarrow$ *T*booleanAlgebra$_6$ *(Stat2⋆)* ⇒ *Stat6* : $\mathsf{Ideal}_\Theta(\{a \cdot cmp_\Theta(x) : a \in bb, x \in \{x_1\}\})$

We will now check that the set $\{a \cdot cmp_\Theta(x) : a \in bb, x \in \{x_1\}\}$ which we have just see to be an ideal equals $\{a \cdot x_0 : a \in bb\}$, thus completing the proof.

Suppose ⇒ *Stat7* : $\{a \cdot cmp_\Theta(x) : a \in bb, x \in \{x_1\}\} = \{a \cdot x_0 : a \in bb\}$
EQUAL $\langle$*Stat1, Stat6, Stat7*$\rangle$ ⇒ false; Discharge ⇒ *Stat8* : $\{a \cdot cmp_\Theta(x) : a \in bb, x \in \{x_1\}\} \neq \{a \cdot x_0 : a \in bb\}$
$\langle d \rangle \hookrightarrow$ *Stat8(Stat8⋆)* ⇒ *Stat9* : $d \in \{a \cdot cmp_\Theta(x) : a \in bb, x \in \{x_1\}\} \neq d \in \{a \cdot x_0 : a \in bb\}$
Suppose ⇒ *Stat10* : $d \in \{a \cdot x_0 : a \in bb\}$ & $d \notin \{a \cdot cmp_\Theta(x) : a \in bb, x \in \{x_1\}\}$
$\langle a, a, x_1 \rangle \hookrightarrow$ *Stat10(Stat10⋆)* ⇒ *Stat10a* : $a \cdot x_0 \neq a \cdot cmp_\Theta(x_1)$
EQUAL $\langle$*Stat10a, Stat2*$\rangle$ ⇒ false; Discharge ⇒ AUTO
*(Stat9⋆)*ELEM ⇒ *Stat11* : $d \in \{a \cdot cmp_\Theta(x) : a \in bb, x \in \{x_1\}\}$ & $d \notin \{a \cdot x_0 : a \in bb\}$
$\langle b, x_3, b \rangle \hookrightarrow$ *Stat11(Stat11⋆)* ⇒ *Stat12* : $b \cdot cmp_\Theta(x_3) \neq b \cdot x_0$ & $x_3 = x_1$
EQUAL $\langle$*Stat12, Stat2*$\rangle$ ⇒ false; Discharge ⇒ QED

An alternative proof of the principal ideal theorem is provided here, to avoid a dependency from Theorem booleanAlgebra$_6$, a proposition which does not really pertain much to the Stone representation theorem.

-- principal ideal

THM booleanAlgebra$_{6b}$. $X \in bb \setminus \{zz_\Theta, ee\} \to \mathsf{Ideal}_\Theta(\{a \cdot X : a \in bb\})$ & $X \in \{a \cdot X : a \in bb\}$. PROOF:
Suppose_not(x_0) ⇒ *Stat0* : $\neg(\mathsf{Ideal}_\Theta(\{a \cdot x_0 : a \in bb\})$ & $x_0 \in \{a \cdot x_0 : a \in bb\})$ & $x_0 \in bb \setminus \{zz_\Theta, ee\}$

The negation of the is that either the set i_0 of all multiples of x_0 is not an ideal or it does not have x_0 as an element. The latter possibility is excluded readily.

Use_def $(\mathsf{Ideal}_\Theta(\{a \cdot x_0 : a \in bb\}))$ ⇒ AUTO
Loc_def ⇒ *Stat1* : $i_0 = \{a \cdot x_0 : a \in bb\}$
EQUAL ⇒ $\neg(\{x \div y : x \in i_0, y \in i_0\} \subseteq i_0$ & $\{x \cdot y : x \in bb, y \in i_0\} \subseteq i_0$ & $i_0 \subseteq bb \setminus \{ee\}$ & $i_0 \not\subseteq \{zz_\Theta\}$ & $x_0 \in \{a \cdot x_0 : a \in bb\})$
Assump ⇒ *Stat2* : $\langle \forall x \mid x \in bb \to ee \cdot x = x \rangle$ & $ee \in bb$
Suppose ⇒ *Stat3* : $x_0 \notin \{a \cdot x_0 : a \in bb\}$
$\langle x_0 \rangle \hookrightarrow$ *Stat2(Stat0⋆)* ⇒ $ee \cdot x_0 = x_0$
$\langle ee \rangle \hookrightarrow$ *Stat3(Stat2⋆)* ⇒ false; Discharge ⇒ AUTO

There are 4 conjuncts in the definition of ideals, and we will now examine them one by one, to find that i_0 meets each of them. First notice that i_0 is neither empty nor the singleton of $zz_\ominus$.

$(Stat0\star)$ELEM $\Rightarrow$ $i_0 \not\subseteq \{zz_\ominus\}$

Next observe that i_0 is included in the domain-of-support deprived of the top element.

Assump $\Rightarrow$ $Stat4: \langle \forall x, y \mid \{x, y\} \subseteq bb \rightarrow x \cdot y \in bb \rangle$ & $Stat5: \langle \forall x, y, z \mid \{x, y, z\} \subseteq bb \rightarrow x \cdot (y \cdot z) = (x \cdot y) \cdot z \rangle$
Suppose $\Rightarrow$ $Stat6: i_0 \not\subseteq bb \setminus \{ee\}$
$\langle c \rangle \hookrightarrow Stat6(Stat1\star) \Rightarrow$ $Stat7: c \in \{a \cdot x_0 : a \in bb\}$ & $c \notin bb \setminus \{ee\}$
$\langle a_0, x_0 \rangle \hookrightarrow Stat4(Stat0, Stat0\star) \Rightarrow$ $Stat8: a_0 \in bb \rightarrow a_0 \cdot x_0 \in bb$
$\langle a_0 \rangle \hookrightarrow Stat7(Stat7\star) \Rightarrow$ $Stat9: a_0 \in bb$ & $a_0 \cdot x_0 = ee$
$\langle a_0, x_0 \rangle \hookrightarrow T$booleanAlgebra$_3$ $(Stat0, Stat9\star) \Rightarrow$ $Stat11: x_0 = ee$
$(Stat11, Stat0\star)$Discharge $\Rightarrow$ AUTO

Then observe that i_0 is closed under multiplication by elements of the domain-of-support.

Suppose $\Rightarrow$ $Stat20: \{x \cdot y : x \in bb, y \in i_0\} \not\subseteq i_0$
$\langle d \rangle \hookrightarrow Stat20(Stat20\star) \Rightarrow$ $Stat21: d \in \{x \cdot y : x \in bb, y \in i_0\}$ & $d \notin i_0$
$\langle a_1, y_1 \rangle \hookrightarrow Stat21(Stat1, Stat1\star) \Rightarrow$ $Stat12:$
$y_1 \in \{a \cdot x_0 : a \in bb\}$ & $a_1 \cdot y_1 \notin \{a \cdot x_0 : a \in bb\}$ & $a_1 \in bb$
$\langle a_1, a_2 \rangle \hookrightarrow Stat4(Stat12\star) \Rightarrow$ $a_2 \in bb \rightarrow a_1 \cdot a_2 \in bb$
$\langle a_2, a_1 \cdot a_2 \rangle \hookrightarrow Stat12(Stat12\star) \Rightarrow$ $Stat13: a_2 \in bb$ & $y_1 = a_2 \cdot x_0$ & $a_1 \cdot y_1 \neq a_1 \cdot a_2 \cdot x_0$
EQUAL $\langle Stat13 \rangle \Rightarrow$ $Stat14: a_1 \cdot (a_2 \cdot x_0) \neq a_1 \cdot a_2 \cdot x_0$
$\langle a_1, a_2, x_0 \rangle \hookrightarrow Stat5(Stat12, Stat13, Stat0, Stat14\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO

To end, observe that i_0 is closed under addition.

$(Stat1\star)$ELEM $\Rightarrow$ $Stat30: \{x \div y : x \in i_0, y \in i_0\} \not\subseteq i_0$
$\langle h \rangle \hookrightarrow Stat30(Stat30\star) \Rightarrow$ $Stat31: h \in \{x \div y : x \in i_0, y \in i_0\}$ & $h \notin i_0$
$\langle x_2, y_2 \rangle \hookrightarrow Stat31(Stat1, Stat1) \Rightarrow$ $Stat22:$
$x_2, y_2 \in \{a \cdot x_0 : a \in bb\}$ & $x_2 \div y_2 \notin \{u \cdot x_0 : u \in bb\}$
Assump $\Rightarrow$ $Stat23: \langle \forall x, y \mid \{x, y\} \subseteq bb \rightarrow x \div y \in bb \rangle$ & $Stat26: \langle \forall x, y, z \mid \{x, y, z\} \subseteq bb \rightarrow (x \div y) \cdot z = z \cdot y \div z \cdot x \rangle$
$\langle a, b \rangle \hookrightarrow Stat23(Stat23\star) \Rightarrow$ $Stat23a: a, b \in bb \rightarrow a \div b \in bb$
$\langle a, b, a \div b \rangle \hookrightarrow Stat22(Stat23a, Stat0\star) \Rightarrow$ $Stat24:$
$a, b \in bb$ & $x_2 = a \cdot x_0$ & $y_2 = b \cdot x_0$ & $x_2 \div y_2 \neq (a \div b) \cdot x_0$ & $x_0 \in bb$
EQUAL $\langle Stat24, \star \rangle \Rightarrow$ $Stat25: a \cdot x_0 \div b \cdot x_0 \neq (a \div b) \cdot x_0$
$\langle a, b, x_0 \rangle \hookrightarrow Stat26 \Rightarrow$ AUTO

TbooleanAlgebra$_0$ $\Rightarrow$ *Stat27* : $\langle \forall x, y \,|\, \{x, y\} \subseteq bb \to x \div y = y \div x \rangle$ & *Stat28* : $\langle \forall x, y \,|\, \{x, y\} \subseteq bb \to x \cdot y = y \cdot x \rangle$
$\langle b, x_0 \rangle \hookrightarrow$ *Stat4(Stat24, Stat24⋆)* $\Rightarrow$ $b \cdot x_0 \in bb$
$\langle a, x_0 \rangle \hookrightarrow$ *Stat4(Stat24, Stat24⋆)* $\Rightarrow$ $a \cdot x_0 \in bb$
$\langle x_0, b \rangle \hookrightarrow$ *Stat28(Stat24, Stat24⋆)* $\Rightarrow$ $x_0 \cdot b = b \cdot x_0$
$\langle b \cdot x_0, a \cdot x_0, x_0, a \rangle \hookrightarrow$ *Stat27(Stat24⋆)* $\Rightarrow$ $x_0 \cdot a = a \cdot x_0$ &
$b \cdot x_0 \div a \cdot x_0 = a \cdot x_0 \div b \cdot x_0$
EQUAL $\langle$*Stat24*$\rangle$ $\Rightarrow$ false; Discharge $\Rightarrow$ QED

A proof of the Stone theorem follows, formulated in such terms as to leave topological notions momentarily out of consideration.

DEF booleanAlgebra$_3$: [homomorphisms between this algebra and an algebra of sets] BooHom$_\Theta$(X) $\leftrightarrow_{\text{Def}}$ Svm(X) &
domain(X) = bb & $X \restriction ee = \bigcup$**range**(X) & $X \restriction ee \neq X \restriction zz_\Theta$ & $\langle \forall x \in bb, y \in bb \,|\, X \restriction (x \cdot y) = X \restriction x \cap X \restriction y$ & $X \restriction (x \div y) = X \restriction x \triangle X \restriction y \rangle$

DEF booleanAlgebra$_4$: [family of all homomorphisms between this algebra and 2] $hh_\Theta =_{\text{Def}} \{h \subseteq bb \times 2 \,|\, \text{BooHom}_\Theta(h)\}$

-- image of top under a homomorphism into 2

THM booleanAlgebra$_{6c}$. $H \in hh_\Theta \to H \restriction zz_\Theta = \emptyset$ & $H \restriction ee = 1$. PROOF:
Suppose_not(h_0) $\Rightarrow$ AUTO
Use_def(hh_Θ) $\Rightarrow$ *Stat3* : $h_0 \in \{h \subseteq bb \times 2 \,|\, \text{BooHom}_\Theta(h)\}$
Use_def$(\text{BooHom}_\Theta(h_0))$ $\Rightarrow$ AUTO
$\langle\,\rangle \hookrightarrow$ *Stat3(Stat3⋆)* $\Rightarrow$ *Stat4* :
$\langle \forall x \in bb, y \in bb \,|\, h_0 \restriction (x \cdot y) = h_0 \restriction x \cap h_0 \restriction y$ & $h_0 \restriction (x \div y) = h_0 \restriction x \triangle h_0 \restriction y \rangle$ & $h_0 \restriction ee \neq h_0 \restriction zz_\Theta$ &
domain(h_0) = bb & $h_0 \subseteq bb \times 2$
Assump $\Rightarrow$ *Stat0* : $ee \in bb$
$\langle ee, ee \rangle \hookrightarrow$ *Stat4(Stat0, Stat0⋆)* $\Rightarrow$ $h_0 \restriction (ee \div ee) = h_0 \restriction ee \triangle h_0 \restriction ee$
$\langle h_0 \restriction ee, \emptyset \rangle \hookrightarrow T1000$ *(Stat4⋆)* $\Rightarrow$ $h_0 \restriction (ee \div ee) = \emptyset$
TbooleanAlgebra$_0$ $\Rightarrow$ *Stat5* : $\langle \forall x \,|\, (x \in bb \to x \div x = zz_\Theta$ & $x \div zz_\Theta = x$ & $zz_\Theta \div x = x)$ & $zz_\Theta \in bb \rangle$
$\langle ee \rangle \hookrightarrow$ *Stat5(Stat0, Stat0⋆)* $\Rightarrow$ $ee \div ee = zz_\Theta$
EQUAL $\langle$*Stat4*$\rangle$ $\Rightarrow$ $h_0 \restriction zz_\Theta = \emptyset$ & $h_0 \restriction ee \neq \emptyset$
$\langle h_0, bb, 2 \rangle \hookrightarrow T141$ *(Stat4, Stat4⋆)* $\Rightarrow$ **range**$(h_0) \subseteq 2$
TELEM $\Rightarrow$ $2 = \{\emptyset, 1\}$
$\langle ee, h_0 \rangle \hookrightarrow T71$ $(\star)$ $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- nonnull image of difference under a homomorphism into 2

THM booleanAlgebra$_{6d}$. $H \in hh_\Theta$ & $\{X, Y\} \subseteq bb$ & $H \restriction (X \div X \cdot Y) = 1$ & $H \restriction Y = \emptyset \to H \restriction X = 1$. PROOF:
Suppose_not(h_0, x_0, x_1) $\Rightarrow$ AUTO

Arguing by contradiction, assume that $\mathsf{h}_0\lceil\mathsf{x}_1 = \emptyset$ and $\mathsf{h}_0\lceil\mathsf{x}_0 \neq 1$, where h_0 is a homomorphism of our Boolean algebra into 2. We will find that this is impossible, because $1 = \mathsf{h}_0\lceil\mathsf{ee}$, $\mathsf{h}_0\lceil\mathsf{ee} = \mathsf{h}_0\lceil(\mathsf{x}_0 \div \mathsf{x}_0 \cdot \mathsf{x}_1)$, $\mathsf{x}_0 \div \mathsf{x}_0 \cdot \mathsf{x}_1 = \mathsf{x}_0 \cdot \mathsf{x}_1 \div \mathsf{x}_0 \cdot \mathsf{ee}$, $\mathsf{x}_0 \cdot \mathsf{x}_1 \div \mathsf{x}_0 \cdot \mathsf{ee} = (\mathsf{ee} \div \mathsf{x}_1) \cdot \mathsf{x}_0$, and hence $\mathsf{h}_0\lceil\mathsf{ee} = \mathsf{h}_0\lceil(\mathsf{ee} \div \mathsf{x}_1) \cap \mathsf{h}_0\lceil\mathsf{x}_0$ and $1 = (1 \triangle \emptyset) \cap \mathsf{h}_0\lceil\mathsf{x}_0$, which implies that $\mathsf{h}_0\lceil\mathsf{x}_0 = 1$.

Use_def $(\mathsf{hh}_\Theta) \Rightarrow$ *Stat1*: $\mathsf{h}_0 \in \{\mathsf{h} \subseteq \mathsf{bb} \times 2 \mid \mathsf{BooHom}_\Theta(\mathsf{h})\}$
Use_def $(\mathsf{BooHom}_\Theta(\mathsf{h}_0)) \Rightarrow$ AUTO
$\langle\,\rangle \hookrightarrow$ *Stat1*(*Stat1*$\star$) $\Rightarrow$ *Stat2*:
$\langle \forall \mathsf{x} \in \mathsf{bb}, \mathsf{y} \in \mathsf{bb} \mid \mathsf{h}_0\lceil(\mathsf{x}\cdot\mathsf{y}) = \mathsf{h}_0\lceil\mathsf{x} \cap \mathsf{h}_0\lceil\mathsf{y}\ \&\ \mathsf{h}_0\lceil(\mathsf{x}\div\mathsf{y}) = \mathsf{h}_0\lceil\mathsf{x} \triangle \mathsf{h}_0\lceil\mathsf{y}\rangle\ \&\ \mathbf{domain}(\mathsf{h}_0) = \mathsf{bb}\ \&$
$\mathsf{h}_0 \subseteq \mathsf{bb} \times 2$
Assump $\Rightarrow$ *Stat3*:
$\langle \forall \mathsf{x},\mathsf{y} \mid \{\mathsf{x},\mathsf{y}\} \subseteq \mathsf{bb} \rightarrow \mathsf{x}\cdot\mathsf{y} \in \mathsf{bb}\rangle\ \&\ \langle \forall \mathsf{x} \mid \mathsf{x} \in \mathsf{bb} \rightarrow \mathsf{ee}\cdot\mathsf{x} = \mathsf{x}\rangle\ \&$ *Stat4*:
$\langle \forall \mathsf{x},\mathsf{y},\mathsf{z} \mid \{\mathsf{x},\mathsf{y},\mathsf{z}\} \subseteq \mathsf{bb} \rightarrow (\mathsf{x}\div\mathsf{y})\cdot\mathsf{z} = \mathsf{z}\cdot\mathsf{y}\div\mathsf{z}\cdot\mathsf{x}\rangle\ \&\ \langle \forall \mathsf{x},\mathsf{y} \mid \{\mathsf{x},\mathsf{y}\} \subseteq \mathsf{bb} \rightarrow \mathsf{x}\div\mathsf{y} \in \mathsf{bb}\rangle\ \&\ \mathsf{ee} \in \mathsf{bb}$
TbooleanAlgebra$_0$ (*Stat3*$\star$) $\Rightarrow$ *Stat6*: $\langle \forall \mathsf{x},\mathsf{y} \mid \{\mathsf{x},\mathsf{y}\} \subseteq \mathsf{bb} \rightarrow \mathsf{x}\div\mathsf{y} = \mathsf{y}\div\mathsf{x}\rangle\ \&\ \langle \forall \mathsf{x},\mathsf{y} \mid \{\mathsf{x},\mathsf{y}\} \subseteq \mathsf{bb} \rightarrow \mathsf{x}\cdot\mathsf{y} = \mathsf{y}\cdot\mathsf{x}\rangle$
$\langle \mathsf{ee}, \mathsf{x}_0\rangle \hookrightarrow$ *Stat3*($\star$) $\Rightarrow$ $\mathsf{ee}\cdot\mathsf{x}_0 \in \mathsf{bb}$
$\langle \mathsf{x}_0, \mathsf{x}_1, \mathsf{x}_0\rangle \hookrightarrow$ *Stat3*($\star$) $\Rightarrow$ $\mathsf{x}_0\cdot\mathsf{x}_1 \in \mathsf{bb}\ \&\ \mathsf{ee}\cdot\mathsf{x}_0 = \mathsf{x}_0$
$\langle \mathsf{ee}\cdot\mathsf{x}_0, \mathsf{x}_0\cdot\mathsf{x}_1, \mathsf{ee}, \mathsf{x}_0\rangle \hookrightarrow$ *Stat6*($\star$) $\Rightarrow$ $\mathsf{ee}\cdot\mathsf{x}_0 \div \mathsf{x}_0\cdot\mathsf{x}_1 = \mathsf{x}_0\cdot\mathsf{x}_1 \div \mathsf{ee}\cdot\mathsf{x}_0\ \&$
$\mathsf{ee}\cdot\mathsf{x}_0 = \mathsf{x}_0\cdot\mathsf{ee}$
$\langle \mathsf{x}_1, \mathsf{ee}, \mathsf{x}_0, \mathsf{x}_1, \mathsf{ee}\rangle \hookrightarrow$ *Stat4*($\star$) $\Rightarrow$ $(\mathsf{x}_1 \div \mathsf{ee})\cdot\mathsf{x}_0 = \mathsf{x}_0\cdot\mathsf{ee} \div \mathsf{x}_0\cdot\mathsf{x}_1\ \&\ \mathsf{x}_1 \div \mathsf{ee} \in \mathsf{bb}$
EQUAL $\Rightarrow$ $\mathsf{h}_0\lceil\big((\mathsf{x}_1 \div \mathsf{ee})\cdot\mathsf{x}_0\big) = 1$
$\langle \mathsf{x}_1 \div \mathsf{ee}, \mathsf{x}_0\rangle \hookrightarrow$ *Stat2*($\star$) $\Rightarrow$ $\mathsf{h}_0\lceil\big((\mathsf{x}_1 \div \mathsf{ee})\cdot\mathsf{x}_0\big) = \mathsf{h}_0\lceil(\mathsf{x}_1 \div \mathsf{ee}) \cap \mathsf{h}_0\lceil\mathsf{x}_0$
$\langle \mathsf{x}_1, \mathsf{ee}\rangle \hookrightarrow$ *Stat2*($\star$) $\Rightarrow$ $\mathsf{h}_0\lceil(\mathsf{x}_1 \div \mathsf{ee}) = \mathsf{h}_0\lceil\mathsf{x}_1 \triangle \mathsf{h}_0\lceil\mathsf{ee}$
$\langle \mathsf{h}_0\rangle \hookrightarrow T$booleanAlgebra$_{6c}$ ($\star$) $\Rightarrow$ $\mathsf{h}_0\lceil\mathsf{ee} = 1$
EQUAL $\Rightarrow$ *Stat7*: $1 = (\emptyset \triangle 1) \cap \mathsf{h}_0\lceil\mathsf{x}_0$
Use_def $(\triangle) \Rightarrow$ $1 = 1 \cap \mathsf{h}_0\lceil\mathsf{x}_0$
$\langle \mathsf{h}_0, \mathsf{bb}, 2\rangle \hookrightarrow T141$ (*Stat2*, *Stat2*$\star$) $\Rightarrow$ $\mathbf{range}(\mathsf{h}_0) \subseteq 2$
TELEM $\Rightarrow$ $2 = \{\emptyset, 1\}\ \&\ \emptyset \neq 1$
$\langle \mathsf{x}_0, \mathsf{h}_0\rangle \hookrightarrow T71$ ($\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ QED

Let us now show that for any non-top element x_0 of the domain-of-support, an ideal to which x_0 does not belong can be extended into an ideal to which the complement of x_0 belongs.

-- enlargement of an ideal
THM booleanAlgebra$_7$. $\mathsf{Ideal}_\Theta(\mathsf{I})\ \&\ \mathsf{X} \in \mathsf{bb}\ \&\ \mathsf{cmp}_\Theta(\mathsf{X}) \notin \mathsf{I} \cup \{\mathsf{ee}\} \rightarrow \langle \exists \mathsf{j} \mid \mathsf{Ideal}_\Theta(\mathsf{j})\ \&\ \mathsf{I} \cup \{\mathsf{X}\} \subseteq \mathsf{j}\rangle$. PROOF:
Suppose_not$(\mathsf{i}_0, \mathsf{x}_0) \Rightarrow$ *Stat1*: $\neg\langle \exists \mathsf{j} \mid \mathsf{Ideal}_\Theta(\mathsf{j})\ \&\ \mathsf{i}_0 \cup \{\mathsf{x}_0\} \subseteq \mathsf{j}\rangle\ \&\ \mathsf{Ideal}_\Theta(\mathsf{i}_0)\ \&\ \mathsf{x}_0 \in \mathsf{bb}\ \&\ \mathsf{cmp}_\Theta(\mathsf{x}_0) \notin \mathsf{i}_0 \cup \{\mathsf{ee}\}$

We will check that the set $\{b \cdot x_0 \div y : b \in bb, y \in i_0\}$ is an ideal having i_0 as a subset and x_0 as a member. To check that is has x_0 as a member, it suffices to observe that $zz_\Theta \in i_0$, because i_0 is an ideal, and that $ee \cdot x_0 \div zz_\Theta = x_0$. Then, to see that it includes i_0, it will suffice to observe that $zz_\Theta \cdot x_0 \div y = y$ for every y in i_0.

Assump $\Rightarrow$ $\mathit{Stat3}: \langle \forall x \mid x \in bb \rightarrow ee \cdot x = x \rangle \ \& \ ee \in bb$
TbooleanAlgebra$_0$ $\Rightarrow$ $\mathit{Stat4}: \langle \forall x \mid (x \in bb \rightarrow x \div x = zz_\Theta \ \& \ x \div zz_\Theta = x \ \& \ zz_\Theta \div x = x) \ \& \ zz_\Theta \in bb \rangle \ \& \ \mathit{Stat5}: \langle \forall x \mid x \in bb \rightarrow zz_\Theta \cdot x = zz_\Theta \rangle$
$\langle i_0 \rangle \hookrightarrow T$booleanAlgebra$_{4a}$ $(\mathit{Stat1}, \mathit{Stat1}\star)$ $\Rightarrow$ $\mathit{Stat8}: zz_\Theta \in i_0$

Let us assign the name i_1 to our perspective ideal, and check that x_0 indeed belongs to i_1, as announced above.

Loc_def $\Rightarrow$ $\mathit{Stat9}: i_1 = \{b \cdot x_0 \div y : b \in bb, y \in i_0\}$
$\langle x_0 \rangle \hookrightarrow \mathit{Stat3}(\mathit{Stat1}, \mathit{Stat3}\star)$ $\Rightarrow$ $\mathit{Stat9a}: ee \cdot x_0 = x_0 \ \& \ ee \in bb$
Suppose $\Rightarrow$ $\mathit{Stat10}: x_0 \notin \{b \cdot x_0 \div y : b \in bb, y \in i_0\}$
$\langle ee, zz_\Theta \rangle \hookrightarrow \mathit{Stat10}(\mathit{Stat8}\star)$ $\Rightarrow$ $x_0 \neq ee \cdot x_0 \div zz_\Theta$
$\langle x_0 \rangle \hookrightarrow \mathit{Stat4}(\mathit{Stat1}, \mathit{Stat1}\star)$ $\Rightarrow$ $x_0 \div zz_\Theta = x_0$
EQUAL $\langle \mathit{Stat9a} \rangle$ $\Rightarrow$ false; Discharge $\Rightarrow$ $\mathit{Stat11}: x_0 \in i_1$

Let us check next that i_0 is indeed included in i_1, as announced above. Incidentally this inclusion ensures that i_1 cannot be null or consist of zz_Θ alone.

Use_def (Ideal$_\Theta$) $\Rightarrow$ $\mathit{Stat2}: i_0 \subseteq bb \setminus \{ee\} \ \& \ i_0 \not\subseteq \{zz_\Theta\}$
Suppose $\Rightarrow$ $\mathit{Stat12}: i_0 \not\subseteq i_1$
$\langle c_1 \rangle \hookrightarrow \mathit{Stat12}(\mathit{Stat9}, \mathit{Stat9}\star)$ $\Rightarrow$ $\mathit{Stat13}: c_1 \notin \{b \cdot x_0 \div y : b \in bb, y \in i_0\} \ \& \ c_1 \in i_0$
$\langle c_1 \rangle \hookrightarrow \mathit{Stat4}(\mathit{Stat13}, \mathit{Stat2}\star)$ $\Rightarrow$ $zz_\Theta \div c_1 = c_1$
$\langle \emptyset, x_0 \rangle \hookrightarrow \mathit{Stat4}(\mathit{Stat4}, \mathit{Stat1}\star)$ $\Rightarrow$ $zz_\Theta \in bb \ \& \ zz_\Theta \cdot x_0 = zz_\Theta$
$\langle zz_\Theta, c_1 \rangle \hookrightarrow \mathit{Stat13}(\mathit{Stat13}\star)$ $\Rightarrow$ $zz_\Theta \cdot x_0 \div c_1 \neq c_1$
EQUAL $\langle \mathit{Stat13} \rangle$ $\Rightarrow$ false; Discharge $\Rightarrow$ $\mathit{Stat20}: i_0 \subseteq i_1 \ \& \ i_1 \not\subseteq \{zz_\Theta\}$

There are hence three potential reasons why i_1 could fail to be an ideal. It could fail being closed with respect to addition; it could fail being closed under multiplication by an element of the support domain; or it could fail being a subset of $bb \setminus \{ee\}$. One by one, we will exclude each of these three possibilities, beginning with additive closure.

Use_def (Ideal$_\Theta$(i_1)) $\Rightarrow$ AUTO
$\langle i_1 \rangle \hookrightarrow \mathit{Stat1}(\mathit{Stat11}\star)$ $\Rightarrow$ $\mathit{Stat21}: \neg(\{x \div y : x \in i_1, y \in i_1\} \subseteq i_1 \ \& \ \{x \cdot y : x \in bb, y \in i_1\} \subseteq i_1 \ \& \ i_1 \subseteq bb \setminus \{ee\})$
Assump $\Rightarrow$ $\mathit{Stat22}: \langle \forall x, y \mid \{x, y\} \subseteq bb \rightarrow x \cdot y \in bb \rangle \ \& \ \mathit{Stat23}: \langle \forall x, y \mid \{x, y\} \subseteq bb \rightarrow x \div y \in bb \rangle \ \&$

$Stat84a: \langle \forall x, y, z \mid \{x, y, z\} \subseteq bb \rightarrow x \div (y \div z) = (x \div y) \div z \rangle$ & $Stat33: \langle \forall x, y, z \mid \{x, y, z\} \subseteq bb \rightarrow (x \div y) \cdot z = z \cdot y \div z \cdot x \rangle$

TbooleanAlgebra$_0$ $\Rightarrow$ $Stat79: \langle \forall x, y \mid \{x, y\} \subseteq bb \rightarrow x \div y = y \div x \rangle$ & $Stat79a: \langle \forall x, y \mid \{x, y\} \subseteq bb \rightarrow x \cdot y = y \cdot x \rangle$

Suppose $\Rightarrow$ $Stat24: \{x \div y : x \in i_1, y \in i_1\} \not\subseteq i_1$

$\langle c_2 \rangle \hookrightarrow Stat24(Stat9, Stat9\star) \Rightarrow$ $Stat25: c_2 \in \{x \div y : x \in i_1, y \in i_1\}$ & $c_2 \notin \{b \cdot x_0 \div y : b \in bb, y \in i_0\}$ & $i_1 = \{b \cdot x_0 \div y : b \in bb, y \in i_0\}$

$\langle x_2, y_2 \rangle \hookrightarrow Stat25(Stat25\star) \Rightarrow$ $Stat26: x_2 \in \{b \cdot x_0 \div y : b \in bb, y \in i_0\}$ & $y_2 \in \{a \cdot x_0 \div v : a \in bb, v \in i_0\}$ & $x_2 \div y_2 \notin \{c \cdot x_0 \div w : c \in bb, w \in i_0\}$

$\langle a_2, b_2 \rangle \hookrightarrow Stat23(Stat26\star) \Rightarrow$ $a_2, b_2 \in bb \rightarrow a_2 \div b_2 \in bb$

$\langle b_2, u_2, a_2, v_2, a_2 \div b_2, v_2 \div u_2 \rangle \hookrightarrow Stat26(Stat26\star) \Rightarrow$ $Stat27:$

$b_2 \in bb$ & $u_2 \in i_0$ & $a_2 \in bb$ & $v_2 \in i_0$ & $a_2 \div b_2 \in bb$ & $x_2 = b_2 \cdot x_0 \div u_2$ & $y_2 = a_2 \cdot x_0 \div v_2$ & $x_2 \div y_2 \neq (a_2 \div b_2) \cdot x_0 \div (v_2 \div u_2) \vee v_2 \div u_2 \notin i_0$

$\langle i_0, v_2, u_2 \rangle \hookrightarrow T$booleanAlgebra$_{4b}$ $(Stat1, Stat27\star) \Rightarrow$ $Stat28: x_2 \div y_2 \neq (a_2 \div b_2) \cdot x_0 \div (v_2 \div u_2)$

EQUAL $\langle Stat27 \rangle \Rightarrow$ $Stat30: b_2 \cdot x_0 \div u_2 \div (a_2 \cdot x_0 \div v_2) \neq (a_2 \div b_2) \cdot x_0 \div (v_2 \div u_2)$

Reasoning in purely algebraic terms (to wit, exploiting the laws pertaining to commutative rings), we will now derive a contradiction from the inequality just found, so becoming able to discharge our most recent pending temporary assumption.

$(Stat27, Stat2, Stat1\star)$ELEM $\Rightarrow$ $u_2, v_2, a_2, b_2, x_0 \in bb$

$\langle a_2, x_0, a_2 \cdot x_0, v_2 \rangle \hookrightarrow Stat22(Stat30\star) \Rightarrow$ $a_2 \cdot x_0, a_2 \cdot x_0 \div v_2 \in bb$

$\langle b_2, x_0, u_2, v_2 \rangle \hookrightarrow Stat22(Stat30\star) \Rightarrow$ $b_2 \cdot x_0, u_2 \div v_2 \in bb$

$\langle b_2 \cdot x_0, u_2, a_2 \cdot x_0 \div v_2 \rangle \hookrightarrow Stat84a(Stat30\star) \Rightarrow$ $Stat31: b_2 \cdot x_0 \div \big(u_2 \div (a_2 \cdot x_0 \div v_2)\big) \neq (a_2 \div b_2) \cdot x_0 \div (v_2 \div u_2)$

$\langle a_2 \cdot x_0, v_2 \rangle \hookrightarrow Stat79(Stat30\star) \Rightarrow$ $a_2 \cdot x_0 \div v_2 = v_2 \div a_2 \cdot x_0$

$\langle u_2, v_2, a_2 \cdot x_0 \rangle \hookrightarrow Stat84a(Stat30\star) \Rightarrow$ $u_2 \div (v_2 \div a_2 \cdot x_0) = (u_2 \div v_2) \div a_2 \cdot x_0$

$\langle u_2 \div v_2, a_2 \cdot x_0 \rangle \hookrightarrow Stat79(Stat30\star) \Rightarrow$ $u_2 \div v_2 \div a_2 \cdot x_0 = a_2 \cdot x_0 \div (u_2 \div v_2)$

EQUAL $\langle Stat31 \rangle \Rightarrow$ $Stat32: b_2 \cdot x_0 \div \big(a_2 \cdot x_0 \div (u_2 \div v_2)\big) \neq (a_2 \div b_2) \cdot x_0 \div (v_2 \div u_2)$

$\langle a_2, b_2, x_0 \rangle \hookrightarrow Stat33(Stat30\star) \Rightarrow$ $(a_2 \div b_2) \cdot x_0 = x_0 \cdot b_2 \div x_0 \cdot a_2$

$\langle v_2, u_2, b_2, x_0 \rangle \hookrightarrow Stat79(Stat30\star) \Rightarrow$ $v_2 \div u_2 = u_2 \div v_2$ & $x_0 \cdot b_2 = b_2 \cdot x_0$

$\langle a_2, x_0 \rangle \hookrightarrow Stat79a(Stat30\star) \Rightarrow$ $x_0 \cdot a_2 = a_2 \cdot x_0$

$\langle b_2 \cdot x_0, a_2 \cdot x_0, u_2 \div v_2 \rangle \hookrightarrow Stat84a(Stat30\star) \Rightarrow$ $b_2 \cdot x_0 \div \big(a_2 \cdot x_0 \div (u_2 \div v_2)\big) = b_2 \cdot x_0 \div a_2 \cdot x_0 \div (u_2 \div v_2)$

EQUAL $\langle Stat32 \rangle \Rightarrow$ false; Discharge $\Rightarrow$ AUTO

Having now ascertained that i_1 is closed under addition, let us check that it is also closed with respect to multiplication by any element in the Boolean algebra.

Suppose $\Rightarrow$ $Stat52: \{x \cdot y : x \in bb, y \in i_1\} \not\subseteq i_1$

$\langle c_3 \rangle \hookrightarrow Stat52(Stat9, Stat9\star) \Rightarrow$ $Stat53: c_3 \in \{x \cdot y : x \in bb, y \in i_1\}$ & $Stat53a: c_3 \notin \{b \cdot x_0 \div y : b \in bb, y \in i_0\}$ & $i_1 = \{b \cdot x_0 \div y : b \in bb, y \in i_0\}$

$\langle b_3, y_3 \rangle \hookrightarrow Stat53(Stat53\star) \Rightarrow$ $Stat54: y_3 \in \{b \cdot x_0 \div y : b \in bb, y \in i_0\}$ & $b_3 \cdot y_3 \notin \{b \cdot x_0 \div y : b \in bb, y \in i_0\}$ & $b_3 \in bb$

$\langle b_3, a_3 \rangle \hookrightarrow Stat22(Stat54\star) \Rightarrow$ $a_3 \in bb \rightarrow b_3 \cdot a_3 \in bb$

$\langle a_3, x_3, b_3 \cdot a_3, b_3 \cdot x_3 \rangle \hookrightarrow Stat54(Stat54\star) \Rightarrow$ $Stat55: y_3 = a_3 \cdot x_0 \div x_3$ & $a_3 \in bb$ & $x_3 \in i_0$ & $b_3 \cdot a_3 \in bb$ & $b_3 \cdot y_3 \neq b_3 \cdot a_3 \cdot x_0 \div b_3 \cdot x_3 \vee b_3 \cdot x_3 \notin i_0$

$\langle i_0, x_3, b_3 \rangle \hookrightarrow T$booleanAlgebra$_{4a}$ $(Stat1, Stat54, Stat55\star) \Rightarrow$ $b_3 \cdot y_3 \neq b_3 \cdot a_3 \cdot x_0 \div b_3 \cdot x_3$

EQUAL $\langle Stat55\rangle \Rightarrow$ $Stat34: \ \mathsf{b}_3 \cdot (\mathsf{a}_3 \cdot \mathsf{x}_0 \div \mathsf{x}_3) \neq \mathsf{b}_3 \cdot \mathsf{a}_3 \cdot \mathsf{x}_0 \div \mathsf{b}_3 \cdot \mathsf{x}_3$

> Reasoning in purely algebraic terms (to wit, exploiting the laws pertaining to commutative ring), we will now derive a contradiction from the inequality just found, so becoming able to discharge our most recent pending temporary assumption.

$(Stat55, Stat54, Stat2, Stat1\star)$ELEM $\Rightarrow$ $\mathsf{a}_3, \mathsf{b}_3, \mathsf{x}_3, \mathsf{x}_0 \in \mathsf{bb}$
$\langle \mathsf{a}_3, \mathsf{x}_0, \mathsf{a}_3 \cdot \mathsf{x}_0, \mathsf{x}_3\rangle \hookrightarrow Stat22(Stat34\star) \Rightarrow$ $\mathsf{a}_3 \cdot \mathsf{x}_0, \mathsf{a}_3 \cdot \mathsf{x}_0 \div \mathsf{x}_3 \in \mathsf{bb}$
$\langle \mathsf{b}_3, \mathsf{a}_3 \cdot \mathsf{x}_0 \div \mathsf{x}_3\rangle \hookrightarrow Stat79a(Stat34\star) \Rightarrow$ $\mathsf{b}_3 \cdot (\mathsf{a}_3 \cdot \mathsf{x}_0 \div \mathsf{x}_3) = (\mathsf{a}_3 \cdot \mathsf{x}_0 \div \mathsf{x}_3) \cdot \mathsf{b}_3$
$\langle \mathsf{a}_3 \cdot \mathsf{x}_0, \mathsf{x}_3, \mathsf{b}_3\rangle \hookrightarrow Stat33(Stat34\star) \Rightarrow$ $\mathsf{b}_3 \cdot \mathsf{x}_3 \div \mathsf{b}_3 \cdot (\mathsf{a}_3 \cdot \mathsf{x}_0) \neq \mathsf{b}_3 \cdot \mathsf{a}_3 \cdot \mathsf{x}_0 \div \mathsf{b}_3 \cdot \mathsf{x}_3$
$\langle \mathsf{b}_3, \mathsf{x}_3\rangle \hookrightarrow Stat22(Stat34\star) \Rightarrow$ $\mathsf{b}_3 \cdot \mathsf{x}_3 \in \mathsf{bb}$
$\langle \mathsf{b}_3, \mathsf{a}_3 \cdot \mathsf{x}_0\rangle \hookrightarrow Stat22(Stat34\star) \Rightarrow$ $\mathsf{b}_3 \cdot (\mathsf{a}_3 \cdot \mathsf{x}_0) \in \mathsf{bb}$
$\langle \mathsf{b}_3 \cdot \mathsf{x}_3, \mathsf{b}_3 \cdot (\mathsf{a}_3 \cdot \mathsf{x}_0)\rangle \hookrightarrow Stat79(Stat34\star) \Rightarrow$ $\mathsf{b}_3 \cdot (\mathsf{a}_3 \cdot \mathsf{x}_0) \div \mathsf{b}_3 \cdot \mathsf{x}_3 \neq \mathsf{b}_3 \cdot \mathsf{a}_3 \cdot \mathsf{x}_0 \div \mathsf{b}_3 \cdot \mathsf{x}_3$
Assump $\Rightarrow$ $Stat35: \ \langle \forall \mathsf{x}, \mathsf{y}, \mathsf{z} \mid \{\mathsf{x}, \mathsf{y}, \mathsf{z}\} \subseteq \mathsf{bb} \rightarrow \mathsf{x} \cdot (\mathsf{y} \cdot \mathsf{z}) = (\mathsf{x} \cdot \mathsf{y}) \cdot \mathsf{z}\rangle$
$\langle \mathsf{b}_3, \mathsf{a}_3, \mathsf{x}_0\rangle \hookrightarrow Stat35(Stat34\star) \Rightarrow$ $\mathsf{b}_3 \cdot (\mathsf{a}_3 \cdot \mathsf{x}_0) = (\mathsf{b}_3 \cdot \mathsf{a}_3) \cdot \mathsf{x}_0$
EQUAL $\langle Stat34\rangle \Rightarrow$ false; Discharge $\Rightarrow$ AUTO

> We must still exclude the possibility that i_1 be not included in $\mathsf{bb} \setminus \{\mathsf{ee}\}$. We will first observe that each element of i_1 belongs to bb; then we will exclude that ee belongs to i_1 and will thus be done.

$(Stat21\star)$ELEM $\Rightarrow$ $Stat80: \ \mathsf{i}_1 \not\subseteq \mathsf{bb} \setminus \{\mathsf{ee}\}$
Suppose $\Rightarrow$ $Stat81: \ \mathsf{i}_1 \not\subseteq \mathsf{bb}$
$\langle \mathsf{c}_4\rangle \hookrightarrow Stat81(Stat9, Stat9\star) \Rightarrow$ $Stat82: \ \mathsf{c}_4 \in \{\mathsf{b} \cdot \mathsf{x}_0 \div \mathsf{y} : \ \mathsf{b} \in \mathsf{bb}, \mathsf{y} \in \mathsf{i}_0\} \ \& \ \mathsf{c}_4 \notin \mathsf{bb}$
$\langle \mathsf{b}_4, \mathsf{y}_4\rangle \hookrightarrow Stat82(Stat82, Stat2\star) \Rightarrow$ $Stat83: \ \mathsf{b}_4 \cdot \mathsf{x}_0 \div \mathsf{y}_4 \notin \mathsf{bb} \ \& \ \mathsf{b}_4 \in \mathsf{bb} \ \& \ \mathsf{y}_4 \in \mathsf{i}_0 \ \& \ \mathsf{y}_4 \in \mathsf{bb}$
$\langle \mathsf{b}_4, \mathsf{x}_0, \mathsf{b}_4 \cdot \mathsf{x}_0, \mathsf{y}_4\rangle \hookrightarrow Stat22(Stat1, Stat83\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO
$(Stat80\star)$ELEM $\Rightarrow$ $Stat82a: \ \mathsf{ee} \in \mathsf{i}_1$

> From the only possibility that has survived, namely $\mathsf{ee} \in \mathsf{i}_1$, we will derive a contradiction, thus completing the proof. For convenience, within these comments let us denote $\mathsf{zz}_\Theta, \mathsf{ee}, \div, \cdot, \mathsf{cmp}_\Theta, \mathsf{x}_0$ as $\emptyset, 1, \cup, \cap, -, \mathsf{x}$. Notice that if $1 \in \mathsf{i}_1$, then $1 = \mathsf{b} \cap \mathsf{x} \cup \mathsf{y}$ for suitable $\mathsf{b} \in \mathsf{bb}$ and $\mathsf{y} \in \mathsf{i}_0$; hence $\mathsf{y} = -(\mathsf{b} \cap \mathsf{x})$, $\mathsf{y} = -(\mathsf{x} \cap \mathsf{b})$. Consequently, since $-(\mathsf{x})$ can be decomposed as $-(\mathsf{x}) = -\big(\mathsf{x} \cap \mathsf{b} \cap \mathsf{x} \cap -(\mathsf{b})\big)$, and consequently, by the De Morgan law, as $-(\mathsf{x}) = \mathsf{x} \cap \mathsf{b} \cap \mathsf{x} \cap -(\mathsf{b}) \cup -(\mathsf{x} \cap \mathsf{b}) \cap -\big(\mathsf{x} \cap -(\mathsf{b})\big)$, where the operand $\mathsf{x} \cap \mathsf{b} \cap \mathsf{x} \cap -(\mathsf{b})$ is $\emptyset$ and can be eliminated, we have $-(\mathsf{x}) = \mathsf{y} \cap -\big(\mathsf{x} \cap -(\mathsf{b})\big)$. Therefore $-(\mathsf{x}) = -(\mathsf{x} \cap \mathsf{b}) \cap \mathsf{y}$ belongs to i_0, because it is multiple of a member y of i_0. However, as part of our initial hypothesis, $-(\mathsf{x}) \notin \mathsf{i}_0$, which gives us the desired contradiction.

$(Stat82a, Stat9\star)$ELEM $\Rightarrow$ $Stat83a:\ ee \in \{b \cdot x_0 \div y :\ b \in bb, y \in i_0\}$
$\langle b_5, y_5\rangle \hookrightarrow Stat83a(Stat83a\star) \Rightarrow$ $Stat84:\ b_5 \cdot x_0 \div y_5 = ee \,\&\, b_5 \in bb \,\&\, y_5 \in i_0$

‖ First we check that $y = \backslash(x \cap b)$.

Suppose $\Rightarrow$ $Stat90:\ y_5 \neq cmp_\Theta(x_0 \cdot b_5)$
$\langle b_5, x_0\rangle \hookrightarrow Stat79a(Stat84, Stat1\star) \Rightarrow$ $b_5 \cdot x_0 = x_0 \cdot b_5$
EQUAL $\langle Stat90\rangle \Rightarrow$ $Stat85:\ y_5 \neq cmp_\Theta(b_5 \cdot x_0)$
$\langle b_5, x_0\rangle \hookrightarrow Stat22(Stat84, Stat1\star) \Rightarrow$ $Stat85a:\ b_5 \cdot x_0 \in bb$
EQUAL $\langle Stat84\rangle \Rightarrow$ $ee \div (b_5 \cdot x_0 \div y_5 \div y_5) = ee \div (ee \div y_5)$
$\langle b_5 \cdot x_0, y_5, y_5\rangle \hookrightarrow Stat84a(Stat85a, Stat84, Stat2\star) \Rightarrow$ $b_5 \cdot x_0 \div (y_5 \div y_5) = (b_5 \cdot x_0 \div y_5) \div y_5$
$\langle y_5\rangle \hookrightarrow Stat4(Stat84, Stat2\star) \Rightarrow$ $y_5 \div y_5 = zz_\Theta \,\&\, zz_\Theta \in bb$
$\langle b_5 \cdot x_0\rangle \hookrightarrow Stat4(Stat84\star) \Rightarrow$ $b_5 \cdot x_0 \div zz_\Theta = b_5 \cdot x_0$
EQUAL $\langle Stat85\rangle \Rightarrow$ $Stat86:\ ee \div b_5 \cdot x_0 = ee \div (ee \div y_5)$
Use_def$(cmp_\Theta) \Rightarrow$ $Stat86a:\ cmp_\Theta(b_5 \cdot x_0) = cmp_\Theta(cmp_\Theta(y_5))$
$\langle y_5\rangle \hookrightarrow T$booleanAlgebra$_1$ $(Stat84, Stat85, Stat86a, Stat2\star) \Rightarrow$ false; Discharge $\Rightarrow$ $Stat87:\ y_5 = cmp_\Theta(x_0 \cdot b_5)$

‖ Next we check that $\backslash(x) = \backslash(x \cap b) \cap \backslash(x \cap \backslash(b))$, implying that $\backslash(x) = y \cap \backslash(x \cap \backslash(b))$ belongs to i_0.

$\langle b_5, x_0\rangle \hookrightarrow T$booleanAlgebra$_{1a}$ $(Stat84, Stat1\star) \Rightarrow$ $x_0 = x_0 \cdot b_5 \div x_0 \cdot cmp_\Theta(b_5) \,\&\, x_0 \cdot b_5 \cdot (x_0 \cdot cmp_\Theta(b_5)) = zz_\Theta$
$\langle b_5\rangle \hookrightarrow T$booleanAlgebra$_1$ $(Stat84, Stat84\star) \Rightarrow$ $Stat88:\ cmp_\Theta(b_5) \in bb$
$\langle x_0, cmp_\Theta(b_5)\rangle \hookrightarrow Stat22(Stat1, Stat88\star) \Rightarrow$ $Stat88a:\ x_0 \cdot cmp_\Theta(b_5) \in bb$
$\langle x_0, b_5, x_0 \cdot b_5, x_0 \cdot cmp_\Theta(b_5)\rangle \hookrightarrow Stat22(Stat84, Stat1, Stat88a\star) \Rightarrow$ $x_0 \cdot b_5 \in bb \,\&\, x_0 \cdot b_5 \div x_0 \cdot cmp_\Theta(b_5) \in bb$
$\langle b_5, x_0\rangle \hookrightarrow T$booleanAlgebra$_{1a}$ $(Stat84, Stat1\star) \Rightarrow$ $x_0 = x_0 \cdot b_5 \div x_0 \cdot cmp_\Theta(b_5) \,\&\, x_0 \cdot b_5 \cdot (x_0 \cdot cmp_\Theta(b_5)) = zz_\Theta$
$\langle x_0 \cdot b_5, x_0 \cdot cmp_\Theta(b_5)\rangle \hookrightarrow T$booleanAlgebra$_{1b}$ $(Stat88a\star) \Rightarrow$ $cmp_\Theta(x_0 \cdot b_5 \div x_0 \cdot cmp_\Theta(b_5)) = x_0 \cdot b_5 \cdot (x_0 \cdot cmp_\Theta(b_5)) \div cmp_\Theta(x_0 \cdot b_5) \cdot cmp_\Theta(x_0 \cdot cmp_\Theta(b_5))$
EQUAL $\langle Stat87\rangle \Rightarrow$ $cmp_\Theta(x_0) = zz_\Theta \div cmp_\Theta(x_0 \cdot b_5) \cdot cmp_\Theta(x_0 \cdot cmp_\Theta(b_5))$
$\langle x_0 \cdot b_5\rangle \hookrightarrow T$booleanAlgebra$_1$ $(Stat88a\star) \Rightarrow$ $cmp_\Theta(x_0 \cdot b_5) \in bb$
$\langle x_0 \cdot cmp_\Theta(b_5)\rangle \hookrightarrow T$booleanAlgebra$_1$ $(Stat88a, Stat88a\star) \Rightarrow$ $Stat89:\ cmp_\Theta(x_0 \cdot cmp_\Theta(b_5)) \in bb$
$\langle cmp_\Theta(x_0 \cdot b_5), cmp_\Theta(x_0 \cdot cmp_\Theta(b_5))\rangle \hookrightarrow Stat22(Stat88a\star) \Rightarrow$ $cmp_\Theta(x_0 \cdot b_5) \cdot cmp_\Theta(x_0 \cdot cmp_\Theta(b_5)) \in bb$
$\langle cmp_\Theta(x_0 \cdot b_5) \cdot cmp_\Theta(x_0 \cdot cmp_\Theta(b_5))\rangle \hookrightarrow Stat4(Stat88a\star) \Rightarrow$ $cmp_\Theta(x_0) = cmp_\Theta(x_0 \cdot b_5) \cdot cmp_\Theta(x_0 \cdot cmp_\Theta(b_5))$
$\langle i_0, y_5, cmp_\Theta(x_0 \cdot cmp_\Theta(b_5))\rangle \hookrightarrow T$booleanAlgebra$_{4a}$ $(Stat84, Stat1, Stat89\star) \Rightarrow$ $y_5 \cdot cmp_\Theta(x_0 \cdot cmp_\Theta(b_5)) \in i_0$
EQUAL $\langle Stat87\rangle \Rightarrow$ $cmp_\Theta(x_0) \in i_0$
Discharge $\Rightarrow$ QED

‖ As a corollary of the theorem on the enlargement of ideals, for any x_0 of the domain-of-support, a maximal ideal to which x_0 does not belong owns the complement of x_0 as a member.

-- maximal ideals and complementation

THM booleanAlgebra$_{7a}$. $X \notin M \,\&\, X \in bb \,\&\, \langle\forall j \mid Ideal_\Theta(j) \,\&\, M \subseteq j \leftrightarrow j = M\rangle \rightarrow cmp_\Theta(X) \in M$. PROOF:

Suppose_not$(x_0, m_0) \Rightarrow$ *Stat1* : $\langle\forall j \mid Ideal_\Theta(j) \,\&\, m_0 \subseteq j \leftrightarrow j = m_0\rangle \,\&\, x_0 \notin m_0 \,\&\, x_0 \in bb \,\&\, cmp_\Theta(x_0) \notin m_0$

If a counterexample to the present claim could be found, namely a maximal ideal m_0 and an element x_0 of the Boolean algebra such that neither x_0 nor its complement belongs to m_0, then we could enlarge m_0 into an ideal including x_0, as stated by Theorem booleanAlgebra$_7$; but this would conflict with the maximality of m_0. Before invoking Theorem booleanAlgebra$_7$, we must exclude the possibility $cmp_\Theta(x_0) = ee$, which is easy, because the complement of ee, which is zz_Θ belongs to every ideal.

$\langle m_0\rangle \hookrightarrow$ *Stat1*$(\star) \Rightarrow$ $Ideal_\Theta(m_0)$
Suppose $\Rightarrow$ $cmp_\Theta(x_0) = ee$
$\langle x_0\rangle \hookrightarrow T$booleanAlgebra$_1$ $(\star) \Rightarrow$ $cmp_\Theta(cmp_\Theta(x_0)) = x_0 \,\&\, cmp_\Theta(ee) = zz_\Theta$
EQUAL $\Rightarrow$ $x_0 = zz_\Theta$
$\langle m_0\rangle \hookrightarrow T$booleanAlgebra$_{4a}$ $(\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO

$\langle m_0, x_0\rangle \hookrightarrow T$booleanAlgebra$_7$ $\Rightarrow$ *Stat2* : $\langle\exists j \mid Ideal_\Theta(j) \,\&\, m_0 \cup \{x_0\} \subseteq j\rangle$
$\langle i_0\rangle \hookrightarrow$ *Stat2*(*Stat2*$\star$) $\Rightarrow$ *Stat3* : $Ideal_\Theta(i_0) \,\&\, m_0 \cup \{x_0\} \subseteq i_0$
$\langle i_0\rangle \hookrightarrow$ *Stat1*(*Stat3*, *Stat1*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- an obvious consequence of Boolean closure properties

THM booleanAlgebra$_{8e}$. $\{X, Y\} \subseteq bb \,\&\, H = \{[x, \textbf{if } x \in M \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : x \in bb\} \rightarrow$
$H\lceil X = \textbf{if } X \in M \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi} \,\&\, H\lceil Y = \textbf{if } Y \in M \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi} \,\&\, H\lceil(X \cdot Y) = \textbf{if } X \cdot Y \in M \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi} \,\&\, H\lceil(X \div Y) = \textbf{if } X \div Y \in M \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}$. PROOF:

Suppose_not$(x_0, y_0, h_0, m_0) \Rightarrow$ AUTO
Assump $\Rightarrow$ *Stat1* : $\langle\forall x, y \mid \{x, y\} \subseteq bb \rightarrow x \div y \in bb\rangle \,\&\, \langle\forall x, y \mid \{x, y\} \subseteq bb \rightarrow x \cdot y \in bb\rangle$
$\langle x_0, y_0, x_0, y_0\rangle \hookrightarrow$ *Stat1*$(\star) \Rightarrow$ *Stat2* :
$x_0 \div y_0, x_0 \cdot y_0, x_0, y_0 \in bb \,\&\, h_0 = \{[x, \textbf{if } x \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : x \in bb\}$
APPLY $\langle\,\rangle$ Must_be_svm$(b(x) \mapsto \textbf{if } x \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}, s \mapsto bb, u \mapsto x_0) \Rightarrow$
$x_0 \in bb \rightarrow \{[x, \textbf{if } x \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : x \in bb\}\lceil x_0 = \textbf{if } x_0 \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}$
APPLY $\langle\,\rangle$ Must_be_svm$(b(x) \mapsto \textbf{if } x \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}, s \mapsto bb, u \mapsto y_0) \Rightarrow$
$y_0 \in bb \rightarrow \{[x, \textbf{if } x \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : x \in bb\}\lceil y_0 = \textbf{if } y_0 \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}$
APPLY $\langle\,\rangle$ Must_be_svm$(b(x) \mapsto \textbf{if } x \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}, s \mapsto bb, u \mapsto x_0 \cdot y_0) \Rightarrow$
$x_0 \cdot y_0 \in bb \rightarrow \{[x, \textbf{if } x \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : x \in bb\}\lceil(x_0 \cdot y_0) = \textbf{if } x_0 \cdot y_0 \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}$
APPLY $\langle\,\rangle$ Must_be_svm$(b(x) \mapsto \textbf{if } x \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}, s \mapsto bb, u \mapsto x_0 \div y_0) \Rightarrow$
$x_0 \div y_0 \in bb \rightarrow \{[x, \textbf{if } x \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : x \in bb\}\lceil(x_0 \div y_0) = \textbf{if } x_0 \div y_0 \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}$
EQUAL $\langle$*Stat2*$\rangle \Rightarrow$ $h_0\lceil x_0 = \textbf{if } x_0 \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi} \,\&\, h_0\lceil y_0 = \textbf{if } y_0 \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi} \,\&$
$h_0\lceil(x_0 \cdot y_0) = \textbf{if } x_0 \cdot y_0 \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi} \,\&\, h_0\lceil(x_0 \div y_0) = \textbf{if } x_0 \div y_0 \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}$

Discharge $\Rightarrow$ QED

-- homomorphism naturally associated with a maximal ideal , 0

THM booleanAlgebra$_{8d}$. $\text{Ideal}_\Theta(M)$ & $\{X, Y\} \subseteq bb$ & $H = \{[x, \textbf{if } x \in M \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : x \in bb\} \rightarrow$
$H{\restriction}X = \emptyset \rightarrow H{\restriction}(X \cdot Y) = \emptyset$ & $\big(H{\restriction}Y = \emptyset \rightarrow H{\restriction}(X \div Y) = \emptyset\big)$. PROOF:

Suppose_not$(m_0, x_0, y_0, h_0) \Rightarrow$ AUTO

Assump $\Rightarrow$ *Stat1* : $\langle \forall x, y \mid \{x, y\} \subseteq bb \rightarrow x \div y \in bb\rangle$ & $\langle \forall x, y \mid \{x, y\} \subseteq bb \rightarrow x \cdot y \in bb\rangle$

$\langle x_0, y_0, x_0, y_0\rangle \hookrightarrow$ *Stat1*$(\star) \Rightarrow$ *Stat2* :
$x_0 \div y_0, x_0 \cdot y_0 \in bb$ & $\{x_0, y_0\} \subseteq bb$ & $\text{Ideal}_\Theta(m_0)$ & $\{x_0, y_0\} \subseteq bb$ & $h_0 = \{[x, \textbf{if } x \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : x \in bb\}$

$\langle x_0, y_0, h_0, m_0\rangle \hookrightarrow T$booleanAlgebra$_{8e}$ (*Stat2*$\star$) $\Rightarrow$ *Stat3* :
$h_0{\restriction}x_0 = \textbf{if } x_0 \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}$ & $h_0{\restriction}y_0 = \textbf{if } y_0 \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}$ &
$h_0{\restriction}(x_0 \cdot y_0) = \textbf{if } x_0 \cdot y_0 \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}$ & $h_0{\restriction}(x_0 \div y_0) = \textbf{if } x_0 \div y_0 \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}$

ELEM $\Rightarrow$ *Stat5* : $h_0{\restriction}x_0 = \emptyset$

TELEM $\Rightarrow$ *Stat4* : $\emptyset \neq 1$

$\langle m_0, x_0, y_0\rangle \hookrightarrow T$booleanAlgebra$_{4a}$ (*Stat2, Stat4, Stat3, Stat5*$\star$) $\Rightarrow$ $h_0{\restriction}(x_0 \cdot y_0) = \emptyset$

$\langle m_0, x_0, y_0\rangle \hookrightarrow T$booleanAlgebra$_{4b}$ (*Stat2, Stat4, Stat3, Stat5*$\star$) $\Rightarrow$ $h_0{\restriction}y_0 = \emptyset \rightarrow$
$h_0{\restriction}(x_0 \div y_0) = \emptyset$

Discharge $\Rightarrow$ QED

-- homomorphism naturally associated with a maximal ideal

THM booleanAlgebra$_8$. $\langle \forall j \mid \text{Ideal}_\Theta(j)$ & $M \subseteq j \leftrightarrow j = M\rangle \rightarrow \{[x, \textbf{if } x \in M \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : x \in bb\} \in hh_\Theta$. PROOF:

Suppose_not$(m_0) \Rightarrow$ *Stat1* : $\langle \forall j \mid \text{Ideal}_\Theta(j)$ & $m_0 \subseteq j \leftrightarrow j = m_0\rangle$ & $\{[x, \textbf{if } x \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : x \in bb\} \notin hh_\Theta$

Use_def$(hh_\Theta) \Rightarrow$ *Stat2* : $\{[x, \textbf{if } x \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : x \in bb\} \notin \{h \subseteq bb \times 2 \mid \text{BooHom}_\Theta(h)\}$

$\langle \{[x, \textbf{if } x \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : x \in bb\}\rangle \hookrightarrow$ *Stat2*(*Stat2*$\star$) $\Rightarrow$ $\{[x, \textbf{if } x \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : x \in bb\} \not\subseteq bb \times 2 \vee$
$\neg\text{BooHom}_\Theta(\{[x, \textbf{if } x \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : x \in bb\})$

Suppose $\Rightarrow$ *Stat3* : $\{[x, \textbf{if } x \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : x \in bb\} \not\subseteq bb \times 2$

Use_def$(\times) \Rightarrow$ $bb \times 2 = \{[x, y] : x \in bb, y \in 2\}$

$\langle c\rangle \hookrightarrow$ *Stat3*(*Stat3*$\star$) $\Rightarrow$ *Stat4* : $c \in \{[x, \textbf{if } x \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : x \in bb\}$ & $c \notin \{[x, y] : x \in bb, y \in 2\}$

TELEM $\Rightarrow$ $2 = \{\emptyset, 1\}$

$\langle x_1, x_1, \textbf{if } x_1 \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}\rangle \hookrightarrow$ *Stat4*(*Stat4*$\star$) $\Rightarrow$ *Stat5* : $\textbf{if } x_1 \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi} \notin \{\emptyset, 1\}$

(*Stat5*$\star$)Discharge $\Rightarrow$ *Stat6* : $\neg\text{BooHom}_\Theta(\{[x, \textbf{if } x \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : x \in bb\})$ & $\{[x, \textbf{if } x \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : x \in bb\} \subseteq bb \times 2$

Loc_def $\Rightarrow$ *Stat7* : $h_0 = \{[x, \textbf{if } x \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : x \in bb\}$

EQUAL $\langle$*Stat6*$\rangle \Rightarrow$ $\neg\text{BooHom}_\Theta(h_0)$

Use_def$(\text{BooHom}_\Theta) \Rightarrow$ *Stat8* :
$\neg\big(\text{Svm}(h_0)$ & $\mathbf{domain}(h_0) = bb$ & $h_0{\restriction}ee = \bigcup \mathbf{range}(h_0)$ & $h_0{\restriction}ee \neq h_0{\restriction}zz_\Theta$ & $\langle \forall x \in bb, y \in bb \mid h_0{\restriction}(x \cdot y) = h_0{\restriction}x \cap h_0{\restriction}y$ & $h_0{\restriction}(x \div y) = h_0{\restriction}x \triangle h_0{\restriction}y\rangle\big)$

TELEM $\Rightarrow$ $\text{Svm}(\{[x, \textbf{if } x \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : x \in bb\})$ & $\mathbf{domain}(\{[x, \textbf{if } x \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : x \in bb\}) = bb$

$\langle m_0\rangle \hookrightarrow$ *Stat1*(*Stat1*$\star$) $\Rightarrow$ *Stat10* : $\text{Ideal}_\Theta(m_0)$

Use_def$(\text{Ideal}_\Theta(m_0)) \Rightarrow$ AUTO

Assump $\Rightarrow$ $\mathit{Stat12a}: \mathsf{ee} \in \mathsf{bb}$
TbooleanAlgebra$_0$ $\Rightarrow$ $\mathit{Stat12}: \langle \forall \mathsf{x} \mid (\mathsf{x} \in \mathsf{bb} \rightarrow \mathsf{x} \div \mathsf{x} = \mathsf{zz}_\Theta \;\&\; \mathsf{x} \div \mathsf{zz}_\Theta = \mathsf{x} \;\&\; \mathsf{zz}_\Theta \div \mathsf{x} = \mathsf{x}) \;\&\; \mathsf{zz}_\Theta \in \mathsf{bb}\rangle \;\&$
$\mathit{Stat36}: \langle \forall \mathsf{x}, \mathsf{y} \mid \{\mathsf{x}, \mathsf{y}\} \subseteq \mathsf{bb} \rightarrow \mathsf{x} \cdot \mathsf{y} = \mathsf{y} \cdot \mathsf{x}\rangle \;\&\; \mathit{Stat36a}: \langle \forall \mathsf{x}, \mathsf{y} \mid \{\mathsf{x}, \mathsf{y}\} \subseteq \mathsf{bb} \rightarrow \mathsf{x} \div \mathsf{y} = \mathsf{y} \div \mathsf{x}\rangle$
$\langle \emptyset \rangle \hookrightarrow \mathit{Stat12}(\mathit{Stat12}\star) \Rightarrow$ $\mathsf{zz}_\Theta \in \mathsf{bb}$

Observe that the images of ee and zz_Θ under the function $\{[\mathsf{x}, \textbf{if } \mathsf{x} \in \mathsf{m}_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : \mathsf{x} \in \mathsf{bb}\}$ are 0 and 1.

Suppose $\Rightarrow$ $\neg(\{[\mathsf{x}, \textbf{if } \mathsf{x} \in \mathsf{m}_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : \mathsf{x} \in \mathsf{bb}\} \upharpoonright \mathsf{ee} = 1 \;\&\; \{[\mathsf{x}, \textbf{if } \mathsf{x} \in \mathsf{m}_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : \mathsf{x} \in \mathsf{bb}\} \upharpoonright \mathsf{zz}_\Theta = \emptyset)$
APPLY $\langle\, \rangle$ Must_be_svm $(\mathsf{b}(\mathsf{x}) \mapsto \textbf{if } \mathsf{x} \in \mathsf{m}_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}, \mathsf{s} \mapsto \mathsf{bb}, \mathsf{u} \mapsto \mathsf{ee}) \Rightarrow$
$\mathsf{ee} \in \mathsf{bb} \rightarrow \{[\mathsf{x}, \textbf{if } \mathsf{x} \in \mathsf{m}_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : \mathsf{x} \in \mathsf{bb}\} \upharpoonright \mathsf{ee} = \textbf{if } \mathsf{ee} \in \mathsf{m}_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}$
APPLY $\langle\, \rangle$ Must_be_svm $(\mathsf{b}(\mathsf{x}) \mapsto \textbf{if } \mathsf{x} \in \mathsf{m}_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}, \mathsf{s} \mapsto \mathsf{bb}, \mathsf{u} \mapsto \mathsf{zz}_\Theta) \Rightarrow$
$\mathsf{zz}_\Theta \in \mathsf{bb} \rightarrow \{[\mathsf{x}, \textbf{if } \mathsf{x} \in \mathsf{m}_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : \mathsf{x} \in \mathsf{bb}\} \upharpoonright \mathsf{zz}_\Theta = \textbf{if } \mathsf{zz}_\Theta \in \mathsf{m}_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}$
$\langle \mathsf{m}_0 \rangle \hookrightarrow T$booleanAlgebra$_{4a}$ $(\mathit{Stat10}, \mathit{Stat10}\star) \Rightarrow$ $\mathsf{zz}_\Theta \in \mathsf{m}_0$
$(\mathit{Stat10}\star)$Discharge $\Rightarrow$ AUTO

Therefore the images of 0 and 1 differ.

TELEM $\Rightarrow$ $\mathit{Stat10a}: \emptyset \neq 1$
$(\mathit{Stat10}\star)$ELEM $\Rightarrow$ $\{[\mathsf{x}, \textbf{if } \mathsf{x} \in \mathsf{m}_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : \mathsf{x} \in \mathsf{bb}\} \upharpoonright \mathsf{ee} \neq \{[\mathsf{x}, \textbf{if } \mathsf{x} \in \mathsf{m}_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : \mathsf{x} \in \mathsf{bb}\} \upharpoonright \mathsf{zz}_\Theta$
Suppose $\Rightarrow$ $\{[\mathsf{x}, \textbf{if } \mathsf{x} \in \mathsf{m}_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : \mathsf{x} \in \mathsf{bb}\} \upharpoonright \mathsf{ee} \neq \bigcup \textbf{range}(\{[\mathsf{x}, \textbf{if } \mathsf{x} \in \mathsf{m}_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : \mathsf{x} \in \mathsf{bb}\})$

Recall that $\bigcup 2 = 1$.

$T1002$ $\Rightarrow$ $2 = \{\emptyset, 1\} \;\&\; \bigcup 2 = 1$
$\langle \emptyset, \emptyset \rangle \hookrightarrow T108$ $\Rightarrow$ $\bigcup \emptyset = \emptyset$

Notice also that since $\textbf{range}(\{[\mathsf{x}, \textbf{if } \mathsf{x} \in \mathsf{m}_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : \mathsf{x} \in \mathsf{bb}\})$ comprises the images of ee and zz_Θ, which are 0 and 1 respectively, and since this set is included in 2, it must be 2.

$\langle \{[\mathsf{x}, \textbf{if } \mathsf{x} \in \mathsf{m}_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : \mathsf{x} \in \mathsf{bb}\}, \mathsf{bb} \times 2 \rangle \hookrightarrow T65$ $(\mathit{Stat6}, \mathit{Stat6}\star) \Rightarrow$
$\textbf{range}(\{[\mathsf{x}, \textbf{if } \mathsf{x} \in \mathsf{m}_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : \mathsf{x} \in \mathsf{bb}\}) \subseteq \textbf{range}(\mathsf{bb} \times 2)$
$\langle \mathsf{ee}, \{[\mathsf{x}, \textbf{if } \mathsf{x} \in \mathsf{m}_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : \mathsf{x} \in \mathsf{bb}\} \rangle \hookrightarrow T71$ $(\mathit{Stat8}\star) \Rightarrow$ $\{[\mathsf{x}, \textbf{if } \mathsf{x} \in \mathsf{m}_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : \mathsf{x} \in \mathsf{bb}\} \upharpoonright \mathsf{ee} \in$
$\textbf{range}(\{[\mathsf{x}, \textbf{if } \mathsf{x} \in \mathsf{m}_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : \mathsf{x} \in \mathsf{bb}\})$
$\langle \mathsf{zz}_\Theta, \{[\mathsf{x}, \textbf{if } \mathsf{x} \in \mathsf{m}_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : \mathsf{x} \in \mathsf{bb}\} \rangle \hookrightarrow T71$ $(\mathit{Stat8}\star) \Rightarrow$ $\{[\mathsf{x}, \textbf{if } \mathsf{x} \in \mathsf{m}_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : \mathsf{x} \in \mathsf{bb}\} \upharpoonright \mathsf{zz}_\Theta \in$
$\textbf{range}(\{[\mathsf{x}, \textbf{if } \mathsf{x} \in \mathsf{m}_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : \mathsf{x} \in \mathsf{bb}\})$
$\langle \mathsf{bb} \times 2, \mathsf{bb}, 2 \rangle \hookrightarrow T141$ $(\mathit{Stat10}\star) \Rightarrow$ $\textbf{range}(\{[\mathsf{x}, \textbf{if } \mathsf{x} \in \mathsf{m}_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : \mathsf{x} \in \mathsf{bb}\}) = 2$
EQUAL $\langle \mathit{Stat10a} \rangle \Rightarrow$ false; Discharge $\Rightarrow$ AUTO
EQUAL $\langle \mathit{Stat7} \rangle \Rightarrow$ $\mathsf{Svm}(\mathsf{h}_0) \;\&\; \textbf{domain}(\mathsf{h}_0) = \mathsf{bb} \;\&\; \mathsf{h}_0 \upharpoonright \mathsf{ee} = \bigcup \textbf{range}(\mathsf{h}_0) \;\&\; \mathsf{h}_0 \upharpoonright \mathsf{ee} \neq \mathsf{h}_0 \upharpoonright \mathsf{zz}_\Theta$

$(Stat8\star)\mathsf{ELEM} \Rightarrow$ $\;Stat9:\ \neg$
$\langle \forall x \in bb, y \in bb \mid h_0\lceil(x \cdot y) = h_0\lceil x \cap h_0\lceil y \;\&\; h_0\lceil(x \div y) = h_0\lceil x \triangle h_0\lceil y\rangle$
$\langle x_2, y_2\rangle \hookrightarrow Stat9(Stat9\star) \Rightarrow$ $\;Stat9a:$
$x_2, y_2 \in bb \;\&\; h_0\lceil(x_2 \cdot y_2) \neq h_0\lceil x_2 \cap h_0\lceil y_2 \;\&\; h_0\lceil(x_2 \div y_2) = h_0\lceil x_2 \triangle h_0\lceil y_2$
$\mathsf{Assump} \Rightarrow$ $\;Stat13:$
$\langle \forall x, y \mid \{x, y\} \subseteq bb \rightarrow x \div y \in bb\rangle \;\&\; \langle \forall x, y \mid \{x, y\} \subseteq bb \rightarrow x \cdot y \in bb\rangle \;\&\; Stat14a:\ \langle \forall x, y \mid \{x, y\} \subseteq bb \rightarrow x \div (y \div x) = y\rangle$
$\langle x_2, y_2, x_2, y_2\rangle \hookrightarrow Stat13(Stat9a\star) \Rightarrow$ $\;Stat13a:\ x_2 \div y_2, x_2 \cdot y_2 \in bb$

Here we undertake the verification that h_0 preserves multiplication, in the sense that the image of a Boolean product equals the intersection of the images of the operands.

$\langle x_2, y_2, h_0, m_0\rangle \hookrightarrow T\mathsf{booleanAlgebra}_{8e}\ (Stat9a, Stat7\star) \Rightarrow$ $\;Stat27a:\ h_0\lceil x_2 = \textbf{if } x_2 \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi} \;\&\; h_0\lceil y_2 = \textbf{if } y_2 \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi} \;\&$
$h_0\lceil(x_2 \cdot y_2) = \textbf{if } x_2 \cdot y_2 \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi} \;\&\; h_0\lceil(x_2 \div y_2) = \textbf{if } x_2 \div y_2 \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}$

$\langle x_2, m_0\rangle \hookrightarrow T\mathsf{booleanAlgebra}_{7a}\ (Stat1, Stat9a\star) \Rightarrow$ $\;Stat91a:\ x_2 \notin m_0 \rightarrow cmp_\ominus(x_2) \in m_0$
$\langle y_2, m_0\rangle \hookrightarrow T\mathsf{booleanAlgebra}_{7a}\ (Stat1, Stat9a\star) \Rightarrow$ $\;Stat92a:\ y_2 \notin m_0 \rightarrow cmp_\ominus(y_2) \in m_0$

$\mathsf{Suppose} \Rightarrow$ $\;Stat20:\ h_0\lceil(x_2 \cdot y_2) \neq h_0\lceil x_2 \cap h_0\lceil y_2$
$\mathsf{EQUAL}\ \langle Stat27a\rangle \Rightarrow$ $\;Stat29:\ \textbf{if } x_2 \cdot y_2 \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi} \neq \textbf{if } x_2 \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi} \cap \textbf{if } y_2 \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}$
$\langle m_0, x_2, y_2\rangle \hookrightarrow T\mathsf{booleanAlgebra}_{4a}\ (Stat10, Stat9a, Stat29\star) \Rightarrow$ $\;Stat23a:\ x_2 \notin m_0$
$\langle m_0, y_2, x_2\rangle \hookrightarrow T\mathsf{booleanAlgebra}_{4a}\ (Stat10, Stat9a, Stat29, Stat92a, Stat23a\star) \Rightarrow$ $\;Stat28a:$
$y_2 \notin m_0 \;\&\; cmp_\ominus(y_2), x_2 \cdot y_2 \in m_0$
$\langle y_2, x_2\rangle \hookrightarrow T\mathsf{booleanAlgebra}_{1a}\ (Stat9a, Stat23a\star) \Rightarrow$ $\;Stat35a:\ x_2 \cdot y_2 \div x_2 \cdot cmp_\ominus(y_2) \notin m_0$
$\langle m_0, cmp_\ominus(y_2), x_2\rangle \hookrightarrow T\mathsf{booleanAlgebra}_{4a}\ (Stat10, Stat9a, Stat28a\star) \Rightarrow$ $\;Stat39a:\ x_2 \cdot cmp_\ominus(y_2) \in m_0$
$\langle m_0, x_2 \cdot y_2, x_2 \cdot cmp_\ominus(y_2)\rangle \hookrightarrow T\mathsf{booleanAlgebra}_{4b}\ (Stat10, Stat28a, Stat39a, Stat35a\star) \Rightarrow$ $\;\mathsf{false};$ $\;\mathsf{Discharge} \Rightarrow$ AUTO

Here we undertake the verification that h_0 preserves addition, in the sense that the image of a Boolean sum equals the symmetric difference of the images of the operands. This will lead to a contradiction, and hence to the desired conclusion.

$(Stat9a\star)\mathsf{ELEM} \Rightarrow$ $\;Stat49a:\ \textbf{if } x_2 \div y_2 \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi} \neq \textbf{if } x_2 \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi} \triangle \textbf{if } y_2 \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}$
$\mathsf{Use\_def}(\triangle) \Rightarrow$ $\;Stat49:\ \textbf{if } x_2 \div y_2 \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi} \neq \textbf{if } x_2 \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi} \backslash \textbf{if } y_2 \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi} \cup (\textbf{if } y_2 \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi} \backslash \textbf{if } x_2 \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi})$
$\langle m_0, x_2, y_2\rangle \hookrightarrow T\mathsf{booleanAlgebra}_{4b}\ (Stat10, Stat49\star) \Rightarrow$ $\;\neg x_2, y_2 \in m_0$
$\mathsf{Assump} \Rightarrow$ $\;Stat46:\ \langle \forall x, y, z \mid \{x, y, z\} \subseteq bb \rightarrow x \div (y \div z) = (x \div y) \div z\rangle$
$\mathsf{Suppose} \Rightarrow$ $\;Stat43a:\ x_2 \notin m_0 \;\&\; y_2 \notin m_0$
$\mathsf{Suppose} \Rightarrow$ $\;Stat44a:\ x_2 \div y_2 \notin m_0$
$\langle m_0, cmp_\ominus(x_2), cmp_\ominus(y_2)\rangle \hookrightarrow T\mathsf{booleanAlgebra}_{4b}\ (Stat10, Stat43a, Stat91a, Stat92a, Stat44a\star) \Rightarrow$ $\;x_2 \div y_2 \neq cmp_\ominus(x_2) \div cmp_\ominus(y_2)$
$\mathsf{Use\_def}(cmp_\ominus) \Rightarrow$ $\;Stat47a:\ x_2 \div y_2 \neq ee \div x_2 \div (ee \div y_2)$
$\langle ee, x_2\rangle \hookrightarrow Stat13(Stat12a, Stat9a\star) \Rightarrow$ $\;Stat46a:\ ee \div x_2 \in bb$

$\langle ee \div x_2, ee, y_2\rangle \hookrightarrow$ *Stat46(Stat12a, Stat9a, Stat46a, Stat47a⋆)* $\Rightarrow$ *Stat47*: $x_2 \div y_2 \neq ee \div x_2 \div ee \div y_2$
$\langle ee, x_2, ee\rangle \hookrightarrow$ *Stat46(Stat12a, Stat9a⋆)* $\Rightarrow$ $ee \div (x_2 \div ee) = (ee \div x_2) \div ee$
$\langle ee, x_2\rangle \hookrightarrow$ *Stat14a(Stat9a, Stat12a⋆)* $\Rightarrow$ $ee \div (x_2 \div ee) = x_2$
EQUAL $\langle$*Stat47*$\rangle \Rightarrow$ false; Discharge $\Rightarrow$ $x_2 \div y_2 \in m_0$
(*Stat49⋆*)ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO
$\langle x_2 \div y_2\rangle \hookrightarrow T$booleanAlgebra$_1$ (*Stat13a, Stat13a⋆*) $\Rightarrow$ $cmp_\Theta(cmp_\Theta(x_2 \div y_2)) = x_2 \div y_2$
Use_def $(cmp_\Theta(x_2 \div y_2)) \Rightarrow$ AUTO
Use_def $\Big(cmp_\Theta(cmp_\Theta(x_2 \div y_2))\Big) \Rightarrow$ AUTO
Suppose $\Rightarrow$ *Stat63a*: $x_2 \in m_0$ & $y_2 \notin m_0$
$\langle m_0, cmp_\Theta(y_2), x_2\rangle \hookrightarrow T$booleanAlgebra$_{4b}$ (*Stat10, Stat63a, Stat92a⋆*) $\Rightarrow$ *Stat66*: $cmp_\Theta(y_2) \div x_2 \in m_0$
$\langle m_0, cmp_\Theta(y_2) \div x_2, ee\rangle \hookrightarrow T$booleanAlgebra$_{4a}$ (*Stat10, Stat66, Stat12a⋆*) $\Rightarrow$ *Stat67*: $cmp_\Theta\big(cmp_\Theta(y_2) \div x_2\big) \notin m_0$
Use_def $(cmp_\Theta) \Rightarrow$ $ee \div (ee \div y_2 \div x_2) \notin m_0$
$\langle ee, y_2, x_2\rangle \hookrightarrow$ *Stat46(Stat12a, Stat9a⋆)* $\Rightarrow$ $ee \div (y_2 \div x_2) = (ee \div y_2) \div x_2$
$\langle x_2, y_2\rangle \hookrightarrow$ *Stat36a(Stat9a, Stat9a⋆)* $\Rightarrow$ $x_2 \div y_2 = y_2 \div x_2$
EQUAL $\langle$*Stat13a*$\rangle \Rightarrow$ *Stat68*: $x_2 \div y_2 \notin m_0$
(*Stat49, Stat63a, Stat68⋆*)ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO
(*Stat49⋆*)ELEM $\Rightarrow$ *Stat73a*: $y_2 \in m_0$ & $x_2 \notin m_0$
$\langle m_0, cmp_\Theta(x_2), y_2\rangle \hookrightarrow T$booleanAlgebra$_{4b}$ (*Stat10, Stat73a, Stat91a⋆*) $\Rightarrow$ *Stat76*: $cmp_\Theta(x_2) \div y_2 \in m_0$
$\langle m_0, cmp_\Theta(x_2) \div y_2, ee\rangle \hookrightarrow T$booleanAlgebra$_{4a}$ (*Stat10, Stat76, Stat12a⋆*) $\Rightarrow$ *Stat77*: $cmp_\Theta\big(cmp_\Theta(x_2) \div y_2\big) \notin m_0$
Use_def $(cmp_\Theta) \Rightarrow$ $ee \div (ee \div x_2 \div y_2) \notin m_0$
$\langle ee, x_2, y_2\rangle \hookrightarrow$ *Stat46(Stat12a, Stat9a⋆)* $\Rightarrow$ $ee \div (x_2 \div y_2) = (ee \div x_2) \div y_2$
Use_def $(cmp_\Theta(x_2 \div y_2)) \Rightarrow$ AUTO
Use_def $\Big(cmp_\Theta(cmp_\Theta(x_2 \div y_2))\Big) \Rightarrow$ AUTO
EQUAL $\langle$*Stat13a*$\rangle \Rightarrow$ *Stat78*: $x_2 \div y_2 \notin m_0$
(*Stat49, Stat73a, Stat78⋆*)ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- isomorphism between a doubleton Boolean algebra and 2

THM booleanAlgebra$_{8b}$. $bb \subseteq \{zz_\Theta, ee\} \rightarrow \{[zz_\Theta, \emptyset], [ee, 1]\} \in hh_\Theta$. PROOF:
Suppose_not $\Rightarrow$ *Stat1a*: $bb \subseteq \{zz_\Theta, ee\}$ & $\{[zz_\Theta, \emptyset], [ee, 1]\} \notin hh_\Theta$
Use_def $(hh_\Theta) \Rightarrow$ *Stat2*: $\{[zz_\Theta, \emptyset], [ee, 1]\} \notin \{h \subseteq bb \times 2 \mid BooHom_\Theta(h)\}$
$\langle \{[zz_\Theta, \emptyset], [ee, 1]\}\rangle \hookrightarrow$ *Stat2(Stat2⋆)* $\Rightarrow$
$\neg\big(\{[zz_\Theta, \emptyset], [ee, 1]\} \subseteq bb \times 2$ & $BooHom_\Theta(\{[zz_\Theta, \emptyset], [ee, 1]\})\big)$
Assump $\Rightarrow$ $ee \in bb$
Suppose $\Rightarrow$ $bb \neq \{zz_\Theta, ee\}$
TbooleanAlgebra$_0$ (*Stat2⋆*) $\Rightarrow$ *Stat6*: $\langle \forall x \mid (x \in bb \rightarrow x \div x = zz_\Theta$ & $x \div zz_\Theta = x$ & $zz_\Theta \div x = x)$ & $zz_\Theta \in bb\rangle$
$\langle \emptyset\rangle \hookrightarrow$ *Stat6(Stat1a⋆)* $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO
Suppose $\Rightarrow$ *Stat3*: $\{[zz_\Theta, \emptyset], [ee, 1]\} \not\subseteq bb \times 2$
Use_def $(\times) \Rightarrow$ $bb \times 2 = \{[x, y] : x \in bb, y \in 2\}$
$\langle c\rangle \hookrightarrow$ *Stat3(Stat3⋆)* $\Rightarrow$ $c \notin \{[x, y] : x \in bb, y \in 2\}$ & $c \in \{[zz_\Theta, \emptyset], [ee, 1]\}$

TELEM $\Rightarrow$ *Stat4*: $2 = \{\emptyset, 1\}$

Suppose $\Rightarrow$ *Stat5*: $[\mathsf{zz}_\Theta, \emptyset] \notin \{[x, y] : x \in \mathsf{bb}, y \in 2\}$

$\langle \mathsf{zz}_\Theta, \emptyset \rangle \hookrightarrow$ *Stat5*(*Stat2*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO

(*Stat3*$\star$)ELEM $\Rightarrow$ *Stat7*: $[\mathsf{ee}, 1] \notin \{[x, y] : x \in \mathsf{bb}, y \in 2\}$

$\langle \mathsf{ee}, 1 \rangle \hookrightarrow$ *Stat7*(*Stat2*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO

Use_def $(\mathsf{BooHom}_\Theta(\{[\mathsf{zz}_\Theta, \emptyset], [\mathsf{ee}, 1]\}))$ $\Rightarrow$ AUTO

TELEM $\Rightarrow$ *Stat8*: $\{[\mathsf{zz}_\Theta, \emptyset], [\mathsf{ee}, 1]\} = \{[\mathsf{ee}, 1], [\mathsf{zz}_\Theta, \emptyset]\}$ & $\emptyset \neq 1$ & $\mathbf{range}(\{[\mathsf{zz}_\Theta, \emptyset], [\mathsf{ee}, 1]\}) = \{\emptyset, 1\}$ & $2 = \{\emptyset, 1\}$

$\langle \emptyset \rangle \hookrightarrow T\mathsf{booleanAlgebra}_1$ (*Stat8*$\star$) $\Rightarrow$ $\mathsf{cmp}_\Theta(\mathsf{ee}) = \mathsf{zz}_\Theta$

$\langle \mathsf{ee} \rangle \hookrightarrow T\mathsf{booleanAlgebra}_2$ (*Stat2*$\star$) $\Rightarrow$ $\mathsf{zz}_\Theta \neq \mathsf{ee}$

$\langle \mathsf{zz}_\Theta, \emptyset, \mathsf{ee}, 1 \rangle \hookrightarrow T92$ (*Stat8*$\star$) $\Rightarrow$ $\mathbf{domain}(\{[\mathsf{zz}_\Theta, \emptyset], [\mathsf{ee}, 1]\}) = \{\mathsf{zz}_\Theta, \mathsf{ee}\}$ & $\mathsf{Svm}(\{[\mathsf{zz}_\Theta, \emptyset], [\mathsf{ee}, 1]\})$

$\langle \mathsf{zz}_\Theta, \mathsf{ee}, \emptyset, 1 \rangle \hookrightarrow T93$ (*Stat8*$\star$) $\Rightarrow$ $\{[\mathsf{zz}_\Theta, \emptyset], [\mathsf{ee}, 1]\} \upharpoonright \mathsf{zz}_\Theta = \emptyset$

$\langle \mathsf{ee}, \mathsf{zz}_\Theta, 1, \emptyset \rangle \hookrightarrow T93$ (*Stat8*$\star$) $\Rightarrow$ $\{[\mathsf{ee}, 1], [\mathsf{zz}_\Theta, \emptyset]\} \upharpoonright \mathsf{ee} = 1$

EQUAL $\langle$*Stat8*$\rangle$ $\Rightarrow$ $\{[\mathsf{zz}_\Theta, \emptyset], [\mathsf{ee}, 1]\} \upharpoonright \mathsf{zz}_\Theta \neq \{[\mathsf{zz}_\Theta, \emptyset], [\mathsf{ee}, 1]\} \upharpoonright \mathsf{ee}$

Suppose $\Rightarrow$ $\{[\mathsf{zz}_\Theta, \emptyset], [\mathsf{ee}, 1]\} \upharpoonright \mathsf{ee} \neq \bigcup \mathbf{range}(\{[\mathsf{zz}_\Theta, \emptyset], [\mathsf{ee}, 1]\})$

EQUAL $\Rightarrow$ $1 \neq \bigcup 2$

$T1002$ (*Stat8*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO

(*Stat2*$\star$)ELEM $\Rightarrow$ *Stat9*:

$\neg\langle \forall x \in \mathsf{bb}, y \in \mathsf{bb} \mid \{[\mathsf{zz}_\Theta, \emptyset], [\mathsf{ee}, 1]\} \upharpoonright (x \cdot y) = \{[\mathsf{zz}_\Theta, \emptyset], [\mathsf{ee}, 1]\} \upharpoonright x \cap \{[\mathsf{zz}_\Theta, \emptyset], [\mathsf{ee}, 1]\} \upharpoonright y$ & $\{[\mathsf{zz}_\Theta, \emptyset], [\mathsf{ee}, 1]\} \upharpoonright (x \div y) = \{[\mathsf{zz}_\Theta, \emptyset], [\mathsf{ee}, 1]\} \upharpoonright x \bigtriangleup \{[\mathsf{zz}_\Theta, \emptyset], [\mathsf{ee}, 1]\} \upharpoonright y \rangle$

$\langle x_1, y_1 \rangle \hookrightarrow$ *Stat9*(*Stat9*$\star$) $\Rightarrow$ *Stat9a*: $x_1, y_1 \in \mathsf{bb}$ &

$\{[\mathsf{zz}_\Theta, \emptyset], [\mathsf{ee}, 1]\} \upharpoonright (x_1 \cdot y_1) \neq \{[\mathsf{zz}_\Theta, \emptyset], [\mathsf{ee}, 1]\} \upharpoonright x_1 \cap \{[\mathsf{zz}_\Theta, \emptyset], [\mathsf{ee}, 1]\} \upharpoonright y_1$ & $\{[\mathsf{zz}_\Theta, \emptyset], [\mathsf{ee}, 1]\} \upharpoonright (x_1 \div y_1) = \{[\mathsf{zz}_\Theta, \emptyset], [\mathsf{ee}, 1]\} \upharpoonright x_1 \bigtriangleup \{[\mathsf{zz}_\Theta, \emptyset], [\mathsf{ee}, 1]\} \upharpoonright y_1$

$T\mathsf{booleanAlgebra}_0$ $\Rightarrow$ *Stat44a*: $\langle \forall x \mid (x \in \mathsf{bb} \rightarrow x \div x = \mathsf{zz}_\Theta$ & $x \div \mathsf{zz}_\Theta = x$ & $\mathsf{zz}_\Theta \div x = x)$ & $\mathsf{zz}_\Theta \in \mathsf{bb} \rangle$ &

Stat42a: $\langle \forall x, y \mid \{x, y\} \subseteq \mathsf{bb} \rightarrow x \cdot y = y \cdot x \rangle$ & $\langle \forall x \mid x \in \mathsf{bb} \rightarrow \mathsf{zz}_\Theta \cdot x = \mathsf{zz}_\Theta \rangle$ & $\langle \forall x \mid x \in \mathsf{bb} \rightarrow \mathsf{zz}_\Theta \cdot x = \mathsf{zz}_\Theta \rangle$

Suppose $\Rightarrow$ $\{[\mathsf{zz}_\Theta, \emptyset], [\mathsf{ee}, 1]\} \upharpoonright (x_1 \cdot y_1) \neq \{[\mathsf{zz}_\Theta, \emptyset], [\mathsf{ee}, 1]\} \upharpoonright x_1 \cap \{[\mathsf{zz}_\Theta, \emptyset], [\mathsf{ee}, 1]\} \upharpoonright y_1$

$\langle x_1, y_1, x_1, y_1 \rangle \hookrightarrow$ *Stat42a*(*Stat9a*$\star$) $\Rightarrow$ $x_1 \cdot y_1 = y_1 \cdot x_1$ & $\mathsf{zz}_\Theta \cdot x_1 = \mathsf{zz}_\Theta$ & $\mathsf{zz}_\Theta \cdot y_1 = \mathsf{zz}_\Theta$

Suppose $\Rightarrow$ $x_1 = \mathsf{zz}_\Theta$

EQUAL $\langle$*Stat8*$\rangle$ $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO

Suppose $\Rightarrow$ $y_1 = \mathsf{zz}_\Theta$

EQUAL $\langle$*Stat8*$\rangle$ $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO

(*Stat44a*$\star$)ELEM $\Rightarrow$ *Stat41*: $x_1 \neq \mathsf{zz}_\Theta$ & $y_1 \neq \mathsf{zz}_\Theta$

(*Stat1a*, *Stat41*, *Stat9a*$\star$)ELEM $\Rightarrow$ $x_1 = \mathsf{ee}$ & $y_1 = \mathsf{ee}$

Assump $\Rightarrow$ *Stat42*: $\langle \forall x \mid x \in \mathsf{bb} \rightarrow \mathsf{ee} \cdot x = x \rangle$

$\langle \mathsf{ee} \rangle \hookrightarrow$ *Stat42*(*Stat2*$\star$) $\Rightarrow$ $\mathsf{ee} \cdot \mathsf{ee} = \mathsf{ee}$

EQUAL $\langle$*Stat9*$\rangle$ $\Rightarrow$ $\{[\mathsf{zz}_\Theta, \emptyset], [\mathsf{ee}, 1]\} \upharpoonright \mathsf{ee} \neq \{[\mathsf{zz}_\Theta, \emptyset], [\mathsf{ee}, 1]\} \upharpoonright \mathsf{ee} \cap \{[\mathsf{zz}_\Theta, \emptyset], [\mathsf{ee}, 1]\} \upharpoonright \mathsf{ee}$

(*Stat42*$\star$)Discharge $\Rightarrow$ AUTO

(*Stat9*$\star$)ELEM $\Rightarrow$ *Stat50*: $x_1, y_1 \in \mathsf{bb}$ & $\{[\mathsf{zz}_\Theta, \emptyset], [\mathsf{ee}, 1]\} \upharpoonright (x_1 \div y_1) \neq \{[\mathsf{zz}_\Theta, \emptyset], [\mathsf{ee}, 1]\} \upharpoonright x_1 \bigtriangleup \{[\mathsf{zz}_\Theta, \emptyset], [\mathsf{ee}, 1]\} \upharpoonright y_1$

Suppose $\Rightarrow$ $x_1 = y_1$

$\langle x_1 \rangle \hookrightarrow$ *Stat44a*(*Stat50*$\star$) $\Rightarrow$ $x_1 \div x_1 = \mathsf{zz}_\Theta$

EQUAL $\langle$*Stat8*$\rangle$ $\Rightarrow$ $\emptyset \neq \{[\mathsf{zz}_\Theta, \emptyset], [\mathsf{ee}, 1]\} \upharpoonright x_1 \bigtriangleup \{[\mathsf{zz}_\Theta, \emptyset], [\mathsf{ee}, 1]\} \upharpoonright x_1$

Use_def$(\bigtriangleup)$ $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat46a*: $x_1 \neq y_1$

Suppose $\Rightarrow$ $x_1 = \mathsf{zz}_\Theta$ & $y_1 = \mathsf{ee}$

$\langle y_1 \rangle \hookrightarrow Stat44a(Stat50\star) \Rightarrow \quad zz_\Theta \div y_1 = y_1$
EQUAL $\langle Stat8 \rangle \Rightarrow \quad \{[zz_\Theta, \emptyset], [ee, 1]\} \restriction y_1 \neq \emptyset \triangle \{[zz_\Theta, \emptyset], [ee, 1]\} \restriction y_1$
Use_def $(\triangle) \Rightarrow$ false; Discharge $\Rightarrow$ $Stat55: \neg(x_1 = zz_\Theta \,\&\, y_1 = ee)$
$(Stat46a, Stat1a, Stat9a, Stat55\star)$ELEM $\Rightarrow \quad x_1 = ee \,\&\, y_1 = zz_\Theta$
$\langle x_1 \rangle \hookrightarrow Stat44a(Stat50\star) \Rightarrow \quad x_1 \div zz_\Theta = x_1$
EQUAL $\langle Stat8 \rangle \Rightarrow \quad \{[zz_\Theta, \emptyset], [ee, 1]\} \restriction x_1 \neq \{[zz_\Theta, \emptyset], [ee, 1]\} \restriction x_1 \triangle \emptyset$
Use_def $(\triangle) \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- homomorphism distinguishing an element from Boolean zero

THM booleanAlgebra$_{8a}$. $X \in bb \setminus \{zz_\Theta\} \rightarrow \{h \in hh_\Theta \mid h \restriction X = 1\} \neq \emptyset$. PROOF:
Suppose_not$(x_0) \Rightarrow$ $Stat1: \{h \in hh_\Theta \mid h \restriction x_0 = 1\} = \emptyset \,\&\, x_0 \in bb \setminus \{zz_\Theta\}$

Suppose a counterexample x_0 could exist. If $bb = \{zz_\Theta, ee\}$, then $x_0 = ee$, in which case $\{[zz_\Theta, \emptyset], [ee, 1]\}$ would be a homomorphism sending x_0 to 1. Otherwise, take a $y_0 \in bb \setminus \{zz_\Theta, ee\}$, in particular choosing $y_0 = x_0$ if $x_0 \neq ee$, consider the principal ideal i_0 generated by $cmp_\Theta(y_0)$, and a maximal ideal i_1 containing i_0. Then the homomorphism naturally associated with i_1 will send y_0 to 1; therefore it will send x_0 to 1, which leads us to the desired contradiction.

Suppose $\Rightarrow$ $Stat1a: bb \subseteq \{zz_\Theta, ee\}$
TbooleanAlgebra$_{8b}$ $(Stat1a\star) \Rightarrow \quad \{[zz_\Theta, \emptyset], [ee, 1]\} \in hh_\Theta$
$\langle \{[zz_\Theta, \emptyset], [ee, 1]\} \rangle \hookrightarrow Stat1(Stat1a\star) \Rightarrow$
$\{[zz_\Theta, \emptyset], [ee, 1]\} \restriction x_0 \neq 1$
$(Stat1\star)$ELEM $\Rightarrow \quad x_0 = ee$
EQUAL $\Rightarrow \quad \{[zz_\Theta, \emptyset], [ee, 1]\} \restriction ee \neq 1$
$\langle \emptyset \rangle \hookrightarrow T$booleanAlgebra$_1$ $(Stat1a\star) \Rightarrow \quad cmp_\Theta(ee) = zz_\Theta$
Assump $\Rightarrow \quad ee \in bb$
$\langle ee \rangle \hookrightarrow T$booleanAlgebra$_2$ $(Stat1a\star) \Rightarrow \quad zz_\Theta \neq ee$
$\langle ee, zz_\Theta, 1, \emptyset \rangle \hookrightarrow T93$ $(Stat1a\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO

The trivial case of a doubleton Boolean algebra having been discarded at this point, we now begin to develop the proof proper.

Loc_def $\Rightarrow$ $Stat10: y_0 = \textbf{if } x_0 \neq ee \textbf{ then } x_0 \textbf{ else } \mathbf{arb}(bb \setminus \{zz_\Theta, ee\}) \textbf{ fi}$
$(Stat1)$ELEM $\Rightarrow \quad y_0 \in bb \setminus \{zz_\Theta, ee\}$
$\langle y_0 \rangle \hookrightarrow T$booleanAlgebra$_2$ $(Stat10\star) \Rightarrow \quad cmp_\Theta(y_0) \in bb \setminus \{zz_\Theta, ee\}$
$\langle cmp_\Theta(y_0) \rangle \hookrightarrow T$booleanAlgebra$_{6b}$ $\Rightarrow \quad Ideal_\Theta(\{a \cdot cmp_\Theta(y_0) : a \in bb\}) \,\&\, cmp_\Theta(y_0) \in \{a \cdot cmp_\Theta(y_0) : a \in bb\}$
$\langle \{a \cdot cmp_\Theta(y_0) : a \in bb\} \rangle \hookrightarrow T$booleanAlgebra$_5$ $\Rightarrow$ $Stat11: \langle \exists m \mid \{a \cdot cmp_\Theta(y_0) : a \in bb\} \subseteq m \,\&\, \langle \forall j \mid Ideal_\Theta(j) \,\&\, m \subseteq j \leftrightarrow j = m \rangle \rangle$
$\langle m_0 \rangle \hookrightarrow Stat11(Stat10\star) \Rightarrow$ $Stat12: \langle \forall j \mid Ideal_\Theta(j) \,\&\, m_0 \subseteq j \leftrightarrow j = m_0 \rangle \,\&\, cmp_\Theta(y_0) \in m_0$
$\langle m_0 \rangle \hookrightarrow T$booleanAlgebra$_8$ $(Stat12\star) \Rightarrow$ $Stat13: \{[x, \textbf{if } x \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : x \in bb\} \in hh_\Theta$

$\langle \{[x, \textbf{if } x \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : x \in bb\} \rangle \hookrightarrow Stat1(Stat13\star) \Rightarrow \quad Stat13a: \{[x, \textbf{if } x \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : x \in bb\} \restriction x_0 \neq 1$

APPLY $\langle\rangle$ Must_be_svm $(b(x) \mapsto \textbf{if } x \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}, s \mapsto bb, u \mapsto y_0) \Rightarrow$

$y_0 \in bb \to \{[x, \textbf{if } x \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : x \in bb\} \restriction y_0 = \textbf{if } y_0 \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}$

$\langle y_0 \rangle \hookrightarrow T\text{booleanAlgebra}_1\ (Stat10\star) \Rightarrow \quad cmp_\Theta(cmp_\Theta(y_0)) = y_0$

$\langle m_0 \rangle \hookrightarrow Stat12(Stat12, Stat12\star) \Rightarrow \quad Ideal_\Theta(m_0)$

$\langle m_0, cmp_\Theta(y_0), cmp_\Theta(y_0) \rangle \hookrightarrow T\text{booleanAlgebra}_{4a}\ (Stat10\star) \Rightarrow \quad y_0 \notin m_0\ \&\ zz_\Theta \in m_0$

$(Stat10\star)$ELEM $\Rightarrow \quad \{[x, \textbf{if } x \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : x \in bb\} \restriction y_0 = 1$

Suppose $\Rightarrow \quad y_0 = x_0$

EQUAL $\langle Stat10 \rangle \Rightarrow$ false; Discharge $\Rightarrow \quad Stat20: y_0 \neq x_0$

$(Stat10, Stat20\star)$ELEM $\Rightarrow \quad x_0 = ee$

Use_def $(hh_\Theta) \Rightarrow \quad Stat14: \{[x, \textbf{if } x \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : x \in bb\} \in \{h \subseteq bb \times 2 \mid BooHom_\Theta(h)\}$

Use_def $(BooHom_\Theta(\{[x, \textbf{if } x \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : x \in bb\})) \Rightarrow$ AUTO

$\langle\rangle \hookrightarrow Stat14(Stat14\star) \Rightarrow \quad \{[x, \textbf{if } x \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : x \in bb\} \restriction ee \neq \{[x, \textbf{if } x \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : x \in bb\} \restriction zz_\Theta$

APPLY $\langle\rangle$ Must_be_svm $(b(x) \mapsto \textbf{if } x \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}, s \mapsto bb, u \mapsto zz_\Theta) \Rightarrow$

$zz_\Theta \in bb \to \{[x, \textbf{if } x \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : x \in bb\} \restriction zz_\Theta = \textbf{if } zz_\Theta \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}$

Use_def $(Ideal_\Theta(m_0)) \Rightarrow$ AUTO

$(Stat13\star)$ELEM $\Rightarrow \quad \{[x, \textbf{if } x \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : x \in bb\} \restriction zz_\Theta = \emptyset$

EQUAL $\langle Stat13 \rangle \Rightarrow \quad Stat14a: \{[x, \textbf{if } x \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : x \in bb\} \restriction x_0 \neq \emptyset$

APPLY $\langle\rangle$ Must_be_svm $(b(x) \mapsto \textbf{if } x \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}, s \mapsto bb, u \mapsto x_0) \Rightarrow$

$Stat15: x_0 \in bb \to \{[x, \textbf{if } x \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : x \in bb\} \restriction x_0 = \textbf{if } x_0 \in m_0 \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}$

$(Stat1, Stat15, Stat13a, Stat14a\star)$ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- homomorphism distinguishing an element from Boolean zero

THM booleanAlgebra$_{8c}$. $hh_\Theta \neq \emptyset$. PROOF:

Suppose_not() $\Rightarrow$ AUTO

$T\text{booleanAlgebra}_{8b} \Rightarrow \quad Stat1: \mathbf{arb}(bb \setminus \{zz_\Theta, ee\}) \in bb \setminus \{zz_\Theta, ee\}$

$\langle \mathbf{arb}(bb \setminus \{zz_\Theta, ee\}) \rangle \hookrightarrow T\text{booleanAlgebra}_{8a}\ (Stat1\star) \Rightarrow \quad Stat2: \{h \in hh_\Theta \mid h \restriction \mathbf{arb}(bb \setminus \{zz_\Theta, ee\}) = 1\} \neq \emptyset$

$\langle h \rangle \hookrightarrow Stat2(\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

DEF booleanAlgebra$_5$: [standard isomorphism between this algebra and a family of subsets of hh_Θ] $\quad phi_\Theta =_{\text{Def}} \{[b, \{h \in hh_\Theta \mid h \restriction b = 1\}] : b \in bb\}$

-- images under the standard homomorphism

THM booleanAlgebra$_9$. $X \in bb \to phi_\Theta \restriction X = \{h \in hh_\Theta \mid h \restriction X = 1\}$. PROOF:

Suppose_not$(x_0) \Rightarrow$ AUTO

ELEM $\Rightarrow \quad Stat1: [x_0, \{h \in hh_\Theta \mid h \restriction x_0 = 1\}]^{[1]} = x_0\ \&\ [x_0, \{h \in hh_\Theta \mid h \restriction x_0 = 1\}]^{[2]} = \{h \in hh_\Theta \mid h \restriction x_0 = 1\}$

$\langle \{[b, \{h \in hh_\Theta \mid h \restriction b = 1\}] : b \in bb\}, [x_0, \{h \in hh_\Theta \mid h \restriction x_0 = 1\}] \rangle \hookrightarrow T74\ (Stat1\star) \Rightarrow$

$[x_0, \{h \in hh_\Theta \mid h \restriction x_0 = 1\}] \in \{[b, \{h \in hh_\Theta \mid h \restriction b = 1\}] : b \in bb\} \to$

$\{[b, \{h \in hh_\Theta \mid h \restriction b = 1\}] : b \in bb\} \restriction [x_0, \{h \in hh_\Theta \mid h \restriction x_0 = 1\}]^{[1]} = \{h \in hh_\Theta \mid h \restriction x_0 = 1\}$

Use_def $(phi_\Theta) \Rightarrow \quad phi_\Theta = \{[b, \{h \in hh_\Theta \mid h \restriction b = 1\}] : b \in bb\}$

EQUAL $\langle \mathit{Stat1} \rangle \Rightarrow$ $[\mathsf{x}_0, \{\mathsf{h} \in \mathsf{hh}_\Theta \,|\, \mathsf{h}\restriction\mathsf{x}_0 = 1\}] \in \{[\mathsf{b}, \{\mathsf{h} \in \mathsf{hh}_\Theta \,|\, \mathsf{h}\restriction\mathsf{b} = 1\}] : \mathsf{b} \in \mathsf{bb}\} \rightarrow$
$\mathsf{phi}_\Theta\restriction\mathsf{x}_0 = \{\mathsf{h} \in \mathsf{hh}_\Theta \,|\, \mathsf{h}\restriction\mathsf{x}_0 = 1\}$
Suppose $\Rightarrow$ $\mathit{Stat2}: [\mathsf{x}_0, \{\mathsf{h} \in \mathsf{hh}_\Theta \,|\, \mathsf{h}\restriction\mathsf{x}_0 = 1\}] \notin \{[\mathsf{b}, \{\mathsf{h} \in \mathsf{hh}_\Theta \,|\, \mathsf{h}\restriction\mathsf{b} = 1\}] : \mathsf{b} \in \mathsf{bb}\}$
$\langle \mathsf{x}_0 \rangle \hookrightarrow \mathit{Stat2}(\star) \Rightarrow$ false; Discharge $\Rightarrow$ false
Discharge $\Rightarrow$ QED

-- Boolean homomorphism property , 1

THM booleanAlgebra$_{10}$. $\{\mathsf{X}, \mathsf{Y}\} \subseteq \mathsf{bb} \rightarrow \mathsf{phi}_\Theta\restriction(\mathsf{X} \cdot \mathsf{Y}) = \mathsf{phi}_\Theta\restriction\mathsf{X} \cap \mathsf{phi}_\Theta\restriction\mathsf{Y} \;\&\; \mathsf{phi}_\Theta\restriction(\mathsf{X} \div \mathsf{Y}) = \mathsf{phi}_\Theta\restriction\mathsf{X} \bigtriangleup \mathsf{phi}_\Theta\restriction\mathsf{Y}$. PROOF:
Suppose_not$(\mathsf{x}_0, \mathsf{y}_0) \Rightarrow$ AUTO
$\langle \mathsf{x}_0 \rangle \hookrightarrow T$booleanAlgebra$_9$ $\Rightarrow$ $\mathsf{phi}_\Theta\restriction\mathsf{x}_0 = \{\mathsf{h} \in \mathsf{hh}_\Theta \,|\, \mathsf{h}\restriction\mathsf{x}_0 = 1\}$
$\langle \mathsf{y}_0 \rangle \hookrightarrow T$booleanAlgebra$_9$ $\Rightarrow$ $\mathsf{phi}_\Theta\restriction\mathsf{y}_0 = \{\mathsf{h} \in \mathsf{hh}_\Theta \,|\, \mathsf{h}\restriction\mathsf{y}_0 = 1\}$
Suppose $\Rightarrow$ $\mathsf{phi}_\Theta\restriction(\mathsf{x}_0 \cdot \mathsf{y}_0) \neq \mathsf{phi}_\Theta\restriction\mathsf{x}_0 \cap \mathsf{phi}_\Theta\restriction\mathsf{y}_0$
Assump $\Rightarrow$ $\mathit{Stat1}: \langle \forall \mathsf{x}, \mathsf{y} \,|\, \{\mathsf{x}, \mathsf{y}\} \subseteq \mathsf{bb} \rightarrow \mathsf{x} \cdot \mathsf{y} \in \mathsf{bb} \rangle$
$\langle \mathsf{x}_0, \mathsf{y}_0 \rangle \hookrightarrow \mathit{Stat1} \Rightarrow$ $\mathsf{x}_0 \cdot \mathsf{y}_0 \in \mathsf{bb}$
$\langle \mathsf{x}_0 \cdot \mathsf{y}_0 \rangle \hookrightarrow T$booleanAlgebra$_9$ $\Rightarrow$ $\mathit{Stat2}: \{\mathsf{h} \in \mathsf{hh}_\Theta \,|\, \mathsf{h}\restriction(\mathsf{x}_0 \cdot \mathsf{y}_0) = 1\} \neq \{\mathsf{h} \in \mathsf{hh}_\Theta \,|\, \mathsf{h}\restriction\mathsf{x}_0 = 1\} \cap \{\mathsf{h} \in \mathsf{hh}_\Theta \,|\, \mathsf{h}\restriction\mathsf{y}_0 = 1\}$
$\langle \mathsf{c} \rangle \hookrightarrow \mathit{Stat2} \Rightarrow$ AUTO
Suppose $\Rightarrow$ $\mathit{Stat13}: \mathsf{c} \notin \mathsf{hh}_\Theta$
Suppose $\Rightarrow$ $\mathit{Stat14}: \mathsf{c} \in \{\mathsf{h} \in \mathsf{hh}_\Theta \,|\, \mathsf{h}\restriction\mathsf{x}_0 = 1\}$
$\langle\,\rangle \hookrightarrow \mathit{Stat14}(\mathit{Stat13}\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO
Suppose $\Rightarrow$ $\mathit{Stat15}: \mathsf{c} \in \{\mathsf{h} \in \mathsf{hh}_\Theta \,|\, \mathsf{h}\restriction\mathsf{y}_0 = 1\}$
$\langle\,\rangle \hookrightarrow \mathit{Stat15}(\mathit{Stat13}\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO
$(\mathit{Stat2}\star)$ELEM $\Rightarrow$ $\mathit{Stat16}: \mathsf{c} \in \{\mathsf{h} \in \mathsf{hh}_\Theta \,|\, \mathsf{h}\restriction(\mathsf{x}_0 \cdot \mathsf{y}_0) = 1\}$
$\langle\,\rangle \hookrightarrow \mathit{Stat16}(\mathit{Stat13}\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO
Use_def$(\mathsf{hh}_\Theta) \Rightarrow$ $\mathit{Stat9}: \mathsf{c} \in \{\mathsf{h} \subseteq \mathsf{bb} \times 2 \,|\, \mathsf{BooHom}_\Theta(\mathsf{h})\}$
$\langle\,\rangle \hookrightarrow \mathit{Stat9} \Rightarrow$ AUTO
$\langle \mathsf{c}, \mathsf{bb}, 2 \rangle \hookrightarrow T141\ (\mathit{Stat9}\star) \Rightarrow$ $\mathbf{range}(\mathsf{c}) \subseteq 2$
Use_def$(\mathsf{BooHom}_\Theta) \Rightarrow$ $\mathit{Stat10}: \langle \forall \mathsf{x} \in \mathsf{bb}, \mathsf{y} \in \mathsf{bb} \,|\, \mathsf{c}\restriction(\mathsf{x} \cdot \mathsf{y}) = \mathsf{c}\restriction\mathsf{x} \cap \mathsf{c}\restriction\mathsf{y} \;\&\; \mathsf{c}\restriction(\mathsf{x} \div \mathsf{y}) = \mathsf{c}\restriction\mathsf{x} \bigtriangleup \mathsf{c}\restriction\mathsf{y} \rangle \;\&\; \mathbf{domain}(\mathsf{c}) = \mathsf{bb}$
$\langle \mathsf{x}_0, \mathsf{c} \rangle \hookrightarrow T71\ (\star) \Rightarrow$ $\mathsf{c}\restriction\mathsf{x}_0 \in 2$
$\langle \mathsf{y}_0, \mathsf{c} \rangle \hookrightarrow T71\ (\star) \Rightarrow$ $\mathsf{c}\restriction\mathsf{y}_0 \in 2$
$\langle \mathsf{x}_0, \mathsf{y}_0 \rangle \hookrightarrow \mathit{Stat10}(\star) \Rightarrow$ $\mathsf{c}\restriction(\mathsf{x}_0 \cdot \mathsf{y}_0) = \mathsf{c}\restriction\mathsf{x}_0 \cap \mathsf{c}\restriction\mathsf{y}_0$
TELEM $\Rightarrow$ $2 = \{\emptyset, 1\}$
Suppose $\Rightarrow$ $\mathit{Stat7}: \mathsf{c} \notin \{\mathsf{h} \in \mathsf{hh}_\Theta \,|\, \mathsf{h}\restriction\mathsf{x}_0 = 1\}$
$\langle \mathsf{c} \rangle \hookrightarrow \mathit{Stat7}(\mathit{Stat2}\star) \Rightarrow$ $\mathsf{c}\restriction(\mathsf{x}_0 \cdot \mathsf{y}_0) = \emptyset \;\&\; \emptyset \neq 1$
$(\mathit{Stat2}\star)$ELEM $\Rightarrow$ $\mathit{Stat8}: \mathsf{c} \in \{\mathsf{h} \in \mathsf{hh}_\Theta \,|\, \mathsf{h}\restriction(\mathsf{x}_0 \cdot \mathsf{y}_0) = 1\}$
$\langle\,\rangle \hookrightarrow \mathit{Stat8}(\mathit{Stat2}\star) \Rightarrow$ false; Discharge $\Rightarrow$ $\mathit{Stat6}: \mathsf{c} \in \{\mathsf{h} \in \mathsf{hh}_\Theta \,|\, \mathsf{h}\restriction\mathsf{x}_0 = 1\}$
Suppose $\Rightarrow$ $\mathit{Stat3}: \mathsf{c} \notin \{\mathsf{h} \in \mathsf{hh}_\Theta \,|\, \mathsf{h}\restriction\mathsf{y}_0 = 1\}$
$\langle \mathsf{c} \rangle \hookrightarrow \mathit{Stat3}(\mathit{Stat2}\star) \Rightarrow$ $\mathsf{c}\restriction(\mathsf{x}_0 \cdot \mathsf{y}_0) = \emptyset \;\&\; \emptyset \neq 1$
$(\mathit{Stat2}\star)$ELEM $\Rightarrow$ $\mathit{Stat4}: \mathsf{c} \in \{\mathsf{h} \in \mathsf{hh}_\Theta \,|\, \mathsf{h}\restriction(\mathsf{x}_0 \cdot \mathsf{y}_0) = 1\}$
$\langle\,\rangle \hookrightarrow \mathit{Stat4}(\mathit{Stat2}\star) \Rightarrow$ false; Discharge $\Rightarrow$ $\mathit{Stat11}: \mathsf{c} \in \{\mathsf{h} \in \mathsf{hh}_\Theta \,|\, \mathsf{h}\restriction\mathsf{y}_0 = 1\}$
$(\mathit{Stat2}\star)$ELEM $\Rightarrow$ $\mathit{Stat12}: \mathsf{c} \notin \{\mathsf{h} \in \mathsf{hh}_\Theta \,|\, \mathsf{h}\restriction(\mathsf{x}_0 \cdot \mathsf{y}_0) = 1\}$

$\langle\,\rangle\hookrightarrow$ *Stat6* $\Rightarrow$ AUTO

$\langle\,\rangle\hookrightarrow$ *Stat11* $\Rightarrow$ AUTO

$\langle c\rangle\hookrightarrow$ *Stat12(Stat10⋆)* $\Rightarrow$ false; Discharge $\Rightarrow$ $\mathsf{phi}_\Theta\restriction(x_0 \div y_0) \neq \mathsf{phi}_\Theta\restriction x_0 \bigtriangleup \mathsf{phi}_\Theta\restriction y_0$

Assump $\Rightarrow$ *Stat21*: $\langle\forall x, y \mid \{x, y\} \subseteq bb \rightarrow x \div y \in bb\rangle$

$\langle x_0, y_0\rangle\hookrightarrow$ *Stat21* $\Rightarrow$ $x_0 \div y_0 \in bb$

$\langle x_0 \div y_0\rangle\hookrightarrow T\text{booleanAlgebra}_9$ $\Rightarrow$ $\mathsf{phi}_\Theta\restriction(x_0 \div y_0) = \{h \in hh_\Theta \mid h\restriction(x_0 \div y_0) = 1\}$

EQUAL $\Rightarrow$ *Stat22*: $\{h \in hh_\Theta \mid h\restriction(x_0 \div y_0) = 1\} \neq \{h \in hh_\Theta \mid h\restriction x_0 = 1\} \bigtriangleup \{h \in hh_\Theta \mid h\restriction y_0 = 1\}$

$\langle d\rangle\hookrightarrow$ *Stat22(Stat22⋆)* $\Rightarrow$ AUTO

Use_def$(\bigtriangleup)$ $\Rightarrow$ *Stat32*: $d \in \{h \in hh_\Theta \mid h\restriction(x_0 \div y_0) = 1\} \neq d \in \{h \in hh_\Theta \mid h\restriction x_0 = 1\} \setminus \{h \in hh_\Theta \mid h\restriction y_0 = 1\} \cup (\{h \in hh_\Theta \mid h\restriction y_0 = 1\} \setminus \{h \in hh_\Theta \mid h\restriction x_0 = 1\})$

Suppose $\Rightarrow$ *Stat23*: $d \notin hh_\Theta$

Suppose $\Rightarrow$ *Stat24*: $d \in \{h \in hh_\Theta \mid h\restriction x_0 = 1\}$

$\langle\,\rangle\hookrightarrow$ *Stat24(Stat23⋆)* $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO

Suppose $\Rightarrow$ *Stat25*: $d \in \{h \in hh_\Theta \mid h\restriction y_0 = 1\}$

$\langle\,\rangle\hookrightarrow$ *Stat25(Stat23⋆)* $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO

(*Stat32⋆*)ELEM $\Rightarrow$ *Stat26*: $d \in \{h \in hh_\Theta \mid h\restriction(x_0 \div y_0) = 1\}$

$\langle\,\rangle\hookrightarrow$ *Stat26(Stat23⋆)* $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO

Use_def(hh_Θ) $\Rightarrow$ *Stat29*: $d \in \{h \subseteq bb \times 2 \mid \mathsf{BooHom}_\Theta(h)\}$

$\langle\,\rangle\hookrightarrow$ *Stat29* $\Rightarrow$ AUTO

$\langle d, bb, 2\rangle\hookrightarrow T141$ (*Stat29⋆*) $\Rightarrow$ $\mathbf{range}(d) \subseteq 2$

Use_def(BooHom_Θ) $\Rightarrow$ *Stat30*: $\langle\forall x \in bb, y \in bb \mid d\restriction(x \cdot y) = d\restriction x \cap d\restriction y \;\&\; d\restriction(x \div y) = d\restriction x \bigtriangleup d\restriction y\rangle \;\&\; \mathbf{domain}(d) = bb$

$\langle x_0, y_0\rangle\hookrightarrow$ *Stat30(⋆)* $\Rightarrow$ $d\restriction(x_0 \div y_0) = d\restriction x_0 \bigtriangleup d\restriction y_0$

Use_def$(\bigtriangleup)$ $\Rightarrow$ *Stat31*: $d\restriction(x_0 \div y_0) = d\restriction x_0 \setminus d\restriction y_0 \cup (d\restriction y_0 \setminus d\restriction x_0)$

$\langle x_0, d\rangle\hookrightarrow T71$ (⋆) $\Rightarrow$ $d\restriction x_0 \in 2$

$\langle y_0, d\rangle\hookrightarrow T71$ (⋆) $\Rightarrow$ $d\restriction y_0 \in 2$

TELEM $\Rightarrow$ $2 = \{\emptyset, 1\}$

Suppose $\Rightarrow$ *Stat33*: $d \notin \{h \in hh_\Theta \mid h\restriction(x_0 \div y_0) = 1\}$

Suppose $\Rightarrow$ *Stat34*: $d \in \{h \in hh_\Theta \mid h\restriction x_0 = 1\}$

(*Stat32⋆*)ELEM $\Rightarrow$ *Stat35*: $d \notin \{h \in hh_\Theta \mid h\restriction y_0 = 1\}$

$\langle\,\rangle\hookrightarrow$ *Stat34(Stat34⋆)* $\Rightarrow$ AUTO

$\langle d\rangle\hookrightarrow$ *Stat35(Stat34⋆)* $\Rightarrow$ $d\restriction y_0 \neq 1$

$\langle d\rangle\hookrightarrow$ *Stat33(Stat31⋆)* $\Rightarrow$ $d\restriction(x_0 \div y_0) \neq 1$

(*Stat31⋆*)Discharge $\Rightarrow$ *Stat36*: $d \notin \{h \in hh_\Theta \mid h\restriction x_0 = 1\}$

Suppose $\Rightarrow$ *Stat44*: $d \in \{h \in hh_\Theta \mid h\restriction y_0 = 1\}$

$\langle\,\rangle\hookrightarrow$ *Stat44(Stat44⋆)* $\Rightarrow$ AUTO

$\langle d\rangle\hookrightarrow$ *Stat36(Stat44⋆)* $\Rightarrow$ $d\restriction x_0 \neq 1$

$\langle d\rangle\hookrightarrow$ *Stat33(Stat44⋆)* $\Rightarrow$ $d\restriction(x_0 \div y_0) \neq 1$

(*Stat31⋆*)Discharge $\Rightarrow$ *Stat45*: $d \notin \{h \in hh_\Theta \mid h\restriction y_0 = 1\}$

(*Stat32⋆*)ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat46*: $d \in \{h \in hh_\Theta \mid h\restriction(x_0 \div y_0) = 1\}$

$\langle\,\rangle\hookrightarrow$ *Stat46(Stat31⋆)* $\Rightarrow$ $d\restriction x_0 \setminus d\restriction y_0 \cup (d\restriction y_0 \setminus d\restriction x_0) = 1$

Suppose $\Rightarrow$ *Stat47*: $d \notin \{h \in hh_\Theta \mid h\restriction x_0 = 1\} \;\&\; d \notin \{h \in hh_\Theta \mid h\restriction y_0 = 1\}$

$\langle \mathsf{d}, \mathsf{d} \rangle \hookrightarrow Stat47(Stat32\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO
$(Stat32\star)$ELEM $\Rightarrow$ $Stat48: \mathsf{d} \in \{\mathsf{h} \in \mathsf{hh}_\Theta \mid \mathsf{h} \restriction \mathsf{x}_0 = 1\}$
$(Stat32\star)$ELEM $\Rightarrow$ $Stat49: \mathsf{d} \in \{\mathsf{h} \in \mathsf{hh}_\Theta \mid \mathsf{h} \restriction \mathsf{y}_0 = 1\}$
TELEM $\Rightarrow$ $\emptyset \neq 1$
$\langle \, \rangle \hookrightarrow Stat48(Stat48\star) \Rightarrow$ $\mathsf{d} \restriction \mathsf{x}_0 = 1$
$\langle \, \rangle \hookrightarrow Stat49(Stat46\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- Boolean homomorphism property , 2

THM booleanAlgebra$_{11}$. $\mathsf{phi}_\Theta \restriction \mathsf{ee} = \bigcup \mathbf{range}(\mathsf{phi}_\Theta)$ & $\mathsf{phi}_\Theta \restriction \mathsf{zz}_\Theta = \emptyset$ & $\mathsf{phi}_\Theta \restriction \mathsf{ee} \neq \mathsf{phi}_\Theta \restriction \mathsf{zz}_\Theta$. PROOF:
Suppose_not() $\Rightarrow$ AUTO
Assump $\Rightarrow$ $Stat0: \mathsf{ee} \in \mathsf{bb}$
Suppose $\Rightarrow$ $\mathsf{phi}_\Theta \restriction \mathsf{zz}_\Theta \neq \emptyset$
Loc_def $\Rightarrow$ $\mathsf{h}_2 = \mathbf{arb}(\mathsf{phi}_\Theta \restriction \mathsf{zz}_\Theta)$
$(Stat0)$ELEM $\Rightarrow$ $\mathsf{h}_2 \in \mathsf{phi}_\Theta \restriction \mathsf{zz}_\Theta$
TbooleanAlgebra$_0$ $(Stat0\star) \Rightarrow$ $Stat0a: \langle \forall \mathsf{x} \mid (\mathsf{x} \in \mathsf{bb} \rightarrow \mathsf{x} \div \mathsf{x} = \mathsf{zz}_\Theta \,\&\, \mathsf{x} \div \mathsf{zz}_\Theta = \mathsf{x} \,\&\, \mathsf{zz}_\Theta \div \mathsf{x} = \mathsf{x}) \,\&\, \mathsf{zz}_\Theta \in \mathsf{bb} \rangle$
$\langle \mathsf{ee} \rangle \hookrightarrow Stat0a(Stat0a\star) \Rightarrow$ $\mathsf{zz}_\Theta \in \mathsf{bb}$
$\langle \mathsf{zz}_\Theta \rangle \hookrightarrow T$booleanAlgebra$_9$ $(Stat0) \Rightarrow$ $Stat1a: \mathsf{h}_2 \in \{\mathsf{h} \in \mathsf{hh}_\Theta \mid \mathsf{h} \restriction \mathsf{zz}_\Theta = 1\}$
$\langle \, \rangle \hookrightarrow Stat1a(Stat1a\star) \Rightarrow$ $\mathsf{h}_2 \in \mathsf{hh}_\Theta \,\&\, \mathsf{h}_2 \restriction \mathsf{zz}_\Theta = 1$
TELEM $\Rightarrow$ $\emptyset \neq 1$
$\langle \mathsf{h}_2 \rangle \hookrightarrow T$booleanAlgebra$_{6c}$ $(Stat1a\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO
TbooleanAlgebra$_{8c}$ $\Rightarrow$ $\mathsf{hh}_\Theta \neq \emptyset$
Suppose $\Rightarrow$ $\mathsf{phi}_\Theta \restriction \mathsf{ee} \neq \mathsf{hh}_\Theta$
$\langle \mathsf{ee} \rangle \hookrightarrow T$booleanAlgebra$_9$ $(\star) \Rightarrow$ $\{\mathsf{h} \in \mathsf{hh}_\Theta \mid \mathsf{h} \restriction \mathsf{ee} = 1\} \neq \mathsf{hh}_\Theta$
Set_monot $\Rightarrow$ $\{\mathsf{h} : \mathsf{h} \in \mathsf{hh}_\Theta \mid \mathsf{h} \restriction \mathsf{ee} = 1\} \subseteq \{\mathsf{h} : \mathsf{h} \in \mathsf{hh}_\Theta\}$
ELEM $\Rightarrow$ $Stat1: \mathsf{hh}_\Theta \not\subseteq \{\mathsf{h} \in \mathsf{hh}_\Theta \mid \mathsf{h} \restriction \mathsf{ee} = 1\}$
$\langle \mathsf{h}_0 \rangle \hookrightarrow Stat1(Stat1\star) \Rightarrow$ $Stat2: \mathsf{h}_0 \notin \{\mathsf{h} \in \mathsf{hh}_\Theta \mid \mathsf{h} \restriction \mathsf{ee} = 1\} \,\&\, \mathsf{h}_0 \in \mathsf{hh}_\Theta$
$\langle \mathsf{h}_0 \rangle \hookrightarrow Stat2(Stat2\star) \Rightarrow$ $\mathsf{h}_0 \restriction \mathsf{ee} \neq 1$
$\langle \mathsf{h}_0 \rangle \hookrightarrow T$booleanAlgebra$_{6c}$ $(Stat2\star) \Rightarrow$ false; Discharge $\Rightarrow$ $\mathsf{hh}_\Theta \neq \bigcup \mathbf{range}(\mathsf{phi}_\Theta)$
Use_def$(\mathsf{phi}_\Theta) \Rightarrow$ $\mathbf{range}(\mathsf{phi}_\Theta) = \{\{\mathsf{h} \in \mathsf{hh}_\Theta \mid \mathsf{h} \restriction \mathsf{b} = 1\} : \mathsf{b} \in \mathsf{bb}\}$
EQUAL $\Rightarrow$ $Stat6: \mathsf{hh}_\Theta \neq \bigcup \{\{\mathsf{h} \in \mathsf{hh}_\Theta \mid \mathsf{h} \restriction \mathsf{b} = 1\} : \mathsf{b} \in \mathsf{bb}\}$
Use_def$(\bigcup \{\{\mathsf{h} \in \mathsf{hh}_\Theta \mid \mathsf{h} \restriction \mathsf{b} = 1\} : \mathsf{b} \in \mathsf{bb}\}) \Rightarrow$ AUTO
$\langle \mathsf{h}_1 \rangle \hookrightarrow Stat6(Stat6\star) \Rightarrow$ $\mathsf{h}_1 \in \mathsf{hh}_\Theta \neq \mathsf{h}_1 \in \{\mathsf{u} : \mathsf{v} \in \{\{\mathsf{h} \in \mathsf{hh}_\Theta \mid \mathsf{h} \restriction \mathsf{b} = 1\} : \mathsf{b} \in \mathsf{bb}\}, \mathsf{u} \in \mathsf{v}\}$
Suppose $\Rightarrow$ $Stat7: \mathsf{h}_1 \in \{\mathsf{u} : \mathsf{v} \in \{\{\mathsf{h} \in \mathsf{hh}_\Theta \mid \mathsf{h} \restriction \mathsf{b} = 1\} : \mathsf{b} \in \mathsf{bb}\}, \mathsf{u} \in \mathsf{v}\} \,\&\, \mathsf{h}_1 \notin \mathsf{hh}_\Theta$
$\langle \mathsf{v}_1, \mathsf{u}_1 \rangle \hookrightarrow Stat7(Stat7\star) \Rightarrow$ $Stat8: \mathsf{v}_1 \in \{\{\mathsf{h} \in \mathsf{hh}_\Theta \mid \mathsf{h} \restriction \mathsf{b} = 1\} : \mathsf{b} \in \mathsf{bb}\} \,\&\, \mathsf{h}_1 \in \mathsf{v}_1$
$\langle \mathsf{b}_1 \rangle \hookrightarrow Stat8(Stat8\star) \Rightarrow$ $Stat9: \mathsf{h}_1 \in \{\mathsf{h} \in \mathsf{hh}_\Theta \mid \mathsf{h} \restriction \mathsf{b}_1 = 1\}$
$\langle \, \rangle \hookrightarrow Stat9(Stat7\star) \Rightarrow$ false; Discharge $\Rightarrow$ $Stat10: \mathsf{h}_1 \notin \{\mathsf{u} : \mathsf{v} \in \{\{\mathsf{h} \in \mathsf{hh}_\Theta \mid \mathsf{h} \restriction \mathsf{b} = 1\} : \mathsf{b} \in \mathsf{bb}\}, \mathsf{u} \in \mathsf{v}\} \,\&\, \mathsf{h}_1 \in \mathsf{hh}_\Theta$
$\langle \{\mathsf{h} \in \mathsf{hh}_\Theta \mid \mathsf{h} \restriction \mathsf{ee} = 1\}, \mathsf{h}_1 \rangle \hookrightarrow Stat10(Stat10\star) \Rightarrow$ $\{\mathsf{h} \in \mathsf{hh}_\Theta \mid \mathsf{h} \restriction \mathsf{ee} = 1\} \notin \{\{\mathsf{h} \in \mathsf{hh}_\Theta \mid \mathsf{h} \restriction \mathsf{b} = 1\} : \mathsf{b} \in \mathsf{bb}\} \vee$
$\mathsf{h}_1 \notin \{\mathsf{h} \in \mathsf{hh}_\Theta \mid \mathsf{h} \restriction \mathsf{ee} = 1\}$
Suppose $\Rightarrow$ $Stat11: \{\mathsf{h} \in \mathsf{hh}_\Theta \mid \mathsf{h} \restriction \mathsf{ee} = 1\} \notin \{\{\mathsf{h} \in \mathsf{hh}_\Theta \mid \mathsf{h} \restriction \mathsf{b} = 1\} : \mathsf{b} \in \mathsf{bb}\}$
$\langle \mathsf{ee} \rangle \hookrightarrow Stat11(Stat0, Stat0\star) \Rightarrow$ false; Discharge $\Rightarrow$ $Stat12: \mathsf{h}_1 \notin \{\mathsf{h} \in \mathsf{hh}_\Theta \mid \mathsf{h} \restriction \mathsf{ee} = 1\}$

$\langle h_1 \rangle \hookrightarrow Stat12(Stat10, Stat10\star) \Rightarrow \quad h_1 \restriction ee \neq 1$
$\langle h_1 \rangle \hookrightarrow T\mathsf{booleanAlgebra}_{6c}\ (Stat10\star) \Rightarrow \quad$ false; Discharge $\Rightarrow$ QED

-- one more Boolean homomorphism property

THM booleanAlgebra$_{12}$. $X \in bb \to phi_\Theta \restriction cmp_\Theta(X) = phi_\Theta \restriction ee \backslash phi_\Theta \restriction X$. PROOF:

Suppose_not$(x_0) \Rightarrow$ AUTO
$T\mathsf{booleanAlgebra}_{11} \Rightarrow \quad phi_\Theta \restriction ee = \bigcup \mathbf{range}(phi_\Theta)$
Use_def$(cmp_\Theta) \Rightarrow \quad phi_\Theta \restriction (ee \div x_0) \neq \bigcup \mathbf{range}(phi_\Theta) \backslash phi_\Theta \restriction x_0$
Assump $\Rightarrow \quad ee \in bb$
$\langle ee, x_0 \rangle \hookrightarrow T\mathsf{booleanAlgebra}_{10} \Rightarrow \quad Stat1:\ phi_\Theta \restriction ee \bigtriangleup phi_\Theta \restriction x_0 \neq \bigcup \mathbf{range}(phi_\Theta) \backslash phi_\Theta \restriction x_0$
Use_def$(phi_\Theta) \Rightarrow \quad \mathbf{domain}(phi_\Theta) = bb$
$\langle x_0, phi_\Theta \rangle \hookrightarrow T71 \Rightarrow \quad phi_\Theta \restriction x_0 \in \mathbf{range}(phi_\Theta)$
$\langle phi_\Theta \restriction x_0, \mathbf{range}(phi_\Theta) \rangle \hookrightarrow T108 \Rightarrow$ AUTO
Use_def$(\bigtriangleup) \Rightarrow$ false; Discharge $\Rightarrow$ QED

-- injectivity of the standard homomorphism , 0

THM booleanAlgebra$_{13a}$. $\{X, Y\} \subseteq bb \ \&\ X \cdot Y \neq X \to phi_\Theta \restriction X \neq phi_\Theta \restriction Y$. PROOF:

Suppose_not$(x_0, x_1) \Rightarrow$ AUTO
Assump $\Rightarrow \quad Stat5a:\ \langle \forall x, y \mid \{x, y\} \subseteq bb \to x \div y \in bb \rangle \ \&\ Stat6:\ \langle \forall x, y \mid \{x, y\} \subseteq bb \to x \cdot y \in bb \rangle \ \&$
$\quad Stat7:\ \langle \forall x, y, z \mid \{x, y, z\} \subseteq bb \to x \div (y \div z) = (x \div y) \div z \rangle \ \&\ Stat8:\ \langle \forall x, y \mid \{x, y\} \subseteq bb \to x \div (y \div x) = y \rangle \ \&\ Stat14:\ \langle \forall x \mid x \in bb \to x \cdot x = x \rangle$
Suppose $\Rightarrow \quad x_0 = x_1$
$\langle x_0 \rangle \hookrightarrow Stat14(\star) \Rightarrow \quad x_0 \cdot x_0 = x_0$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO
ELEM $\Rightarrow \quad Stat5:\ x_0 \neq x_1 \ \&\ phi_\Theta \restriction x_0 = phi_\Theta \restriction x_1 \ \&\ x_0, x_1 \in bb$
$T\mathsf{booleanAlgebra}_0 \Rightarrow \quad Stat9:\ \langle \forall x \mid (x \in bb \to x \div x = zz_\Theta \ \&\ x \div zz_\Theta = x \ \&\ zz_\Theta \div x = x) \ \&\ zz_\Theta \in bb \rangle \ \&\ Stat9a:\ \langle \forall x, y \mid \{x, y\} \subseteq bb \to x \cdot y = y \cdot x \rangle$
Assump $\Rightarrow \quad Stat16:\ \langle \forall x, y, z \mid \{x, y, z\} \subseteq bb \to (x \div y) \cdot z = z \cdot y \div z \cdot x \rangle \ \&\ Stat16a:\ \langle \forall x, y, z \mid \{x, y, z\} \subseteq bb \to x \cdot (y \cdot z) = (x \cdot y) \cdot z \rangle$
$\langle x_0, x_1 \rangle \hookrightarrow Stat9a(Stat5, Stat5\star) \Rightarrow \quad Stat19a:\ x_0 \cdot x_1 = x_1 \cdot x_0$
TELEM $\Rightarrow \quad Stat55a:\ 1 = \{\emptyset\} \ \&\ 2 = \{\emptyset, 1\}$
ELEM $\Rightarrow \quad x_0 \cdot x_1 \neq x_0$

> The contradiction will be obtained by finding a homomorphism h_0 in hh_Θ which sends x_0 and x_1 to different images; this homomorphism will witness that $phi_\Theta \restriction x_0 \neq phi_\Theta \restriction x_1$.

Loc_def $\Rightarrow \quad Stat12:\ e_0 = x_0 \div x_0 \cdot x_1$
Suppose $\Rightarrow \quad e_0 = zz_\Theta$
EQUAL $\langle Stat9 \rangle \Rightarrow \quad zz_\Theta \div x_0 = x_0 \div x_0 \cdot x_1 \div x_0$
$\langle x_0 \rangle \hookrightarrow Stat9(Stat5\star) \Rightarrow \quad x_0 = x_0 \div x_0 \cdot x_1 \div x_0$
$\langle x_0, x_1, x_0, x_0 \cdot x_1, x_0, x_0, x_0 \cdot x_1 \rangle \hookrightarrow Stat6(Stat5\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO
$\langle x_0, x_0 \cdot x_1, x_0, x_1 \rangle \hookrightarrow Stat5a(Stat5, Stat12\star) \Rightarrow \quad Stat13:\ x_0 \cdot x_1, e_0 \in bb$

$\langle e_0 \rangle \hookrightarrow T\text{booleanAlgebra}_2$ (*Stat16*⋆) ⇒ $e_0 \notin \{zz_\Theta, ee\} \to cmp_\Theta(e_0) \in bb \setminus \{zz_\Theta, ee\}$
$\langle e_0 \rangle \hookrightarrow T\text{booleanAlgebra}_{8a}$ (*Stat12*⋆) ⇒ *Stat20* : $\{h \in hh_\Theta \mid h \restriction e_0 = 1\} \neq \emptyset$
$\langle h_0 \rangle \hookrightarrow$ *Stat20*(*Stat20*⋆) ⇒ *Stat20a* : $h_0 \in hh_\Theta$ & $h_0 \restriction e_0 = 1$
Suppose ⇒ $e_0 \cdot x_1 \neq zz_\Theta$
$\langle x_0, x_0 \cdot x_1, x_1 \rangle \hookrightarrow$ *Stat16*(*Stat5*, *Stat13*⋆) ⇒ $(x_0 \div x_0 \cdot x_1) \cdot x_1 = x_1 \cdot (x_0 \cdot x_1) \div x_1 \cdot x_0$
$\langle x_1, x_1, x_0 \rangle \hookrightarrow$ *Stat16a*(*Stat5*, *Stat5*⋆) ⇒ $x_1 \cdot (x_1 \cdot x_0) = (x_1 \cdot x_1) \cdot x_0$
$\langle x_1 \rangle \hookrightarrow$ *Stat14*(*Stat5*, *Stat5*⋆) ⇒ $x_1 \cdot x_1 = x_1$
$\langle x_1 \cdot x_0 \rangle \hookrightarrow$ *Stat9*(*Stat13*, *Stat19a*⋆) ⇒ $x_1 \cdot x_0 \div x_1 \cdot x_0 = zz_\Theta$
EQUAL $\langle$*Stat19a*$\rangle$ ⇒ false; Discharge ⇒ $e_0 \cdot x_1 = zz_\Theta$
Use_def (hh_Θ) ⇒ *Stat51* : $h_0 \in \{h \subseteq bb \times 2 \mid \text{BooHom}_\Theta(h)\}$
Use_def $(\text{BooHom}_\Theta(h_0))$ ⇒ AUTO
$\langle\ \rangle \hookrightarrow$ *Stat51*(*Stat51*⋆) ⇒ *Stat52* : $\langle \forall x \in bb, y \in bb \mid h_0 \restriction (x \cdot y) = h_0 \restriction x \cap h_0 \restriction y \;\&\; h_0 \restriction (x \div y) = h_0 \restriction x \bigtriangleup h_0 \restriction y \rangle \;\&\; h_0 \subseteq bb \times 2 \;\&\; \mathbf{domain}(h_0) = bb$
$\langle h_0, bb, 2 \rangle \hookrightarrow T141$ (*Stat52*⋆) ⇒ *Stat52a* : $\mathbf{range}(h_0) \subseteq 2$
$\langle x_1, h_0 \rangle \hookrightarrow T71$ (*Stat5*, *Stat52*, *Stat52a*, *Stat55a*, *Stat20a*⋆) ⇒ *Stat54* : $h_0 \restriction e_0 \cap h_0 \restriction x_1 = h_0 \restriction x_1$
$\langle zz_\Theta \rangle \hookrightarrow$ *Stat9*(*Stat52*⋆) ⇒ $zz_\Theta \div zz_\Theta = zz_\Theta$ & $zz_\Theta \in bb$
EQUAL $\langle$*Stat20a*$\rangle$ ⇒ *Stat53* : $h_0 \restriction (e_0 \cdot x_1) = h_0 \restriction (zz_\Theta \div zz_\Theta)$
$\langle e_0, x_1 \rangle \hookrightarrow$ *Stat52*(*Stat13*, *Stat5*, *Stat53*⋆) ⇒ $h_0 \restriction (zz_\Theta \div zz_\Theta) = h_0 \restriction e_0 \cap h_0 \restriction x_1$
$\langle h_0 \restriction zz_\Theta, \emptyset \rangle \hookrightarrow T1000$ (*Stat53*⋆) ⇒ $h_0 \restriction zz_\Theta \bigtriangleup h_0 \restriction zz_\Theta = \emptyset$
$\langle zz_\Theta, zz_\Theta \rangle \hookrightarrow$ *Stat52*(*Stat52*⋆) ⇒ *Stat55* : $h_0 \restriction (e_0 \cdot x_1) = \emptyset$
$\langle e_0, x_1 \rangle \hookrightarrow$ *Stat52*(*Stat13*, *Stat5*, *Stat54*, *Stat55*⋆) ⇒ $h_0 \restriction x_1 = \emptyset$
Suppose ⇒ *Stat56* : $h_0 \in phi_\Theta \restriction x_1$
$\langle x_1 \rangle \hookrightarrow T\text{booleanAlgebra}_9$ (*Stat5*, *Stat56*⋆) ⇒ *Stat57* : $h_0 \in \{h \in hh_\Theta \mid h \restriction x_1 = 1\}$
$\langle\ \rangle \hookrightarrow$ *Stat57*(*Stat55*⋆) ⇒ $\emptyset = 1$
(*Stat55a*⋆)Discharge ⇒ AUTO
EQUAL $\langle$*Stat5*$\rangle$ ⇒ *Stat56a* : $h_0 \notin phi_\Theta \restriction x_0$
$\langle x_0 \rangle \hookrightarrow T\text{booleanAlgebra}_9$ (*Stat5*, *Stat56a*⋆) ⇒ *Stat58* : $h_0 \notin \{h \in hh_\Theta \mid h \restriction x_0 = 1\}$
$\langle h_0 \rangle \hookrightarrow$ *Stat58*(*Stat20a*, *Stat20a*⋆) ⇒ *Stat59* : $h_0 \restriction x_0 \neq 1$
EQUAL $\langle$*Stat12*, *Stat20a*$\rangle$ ⇒ *Stat59a* : $h_0 \restriction (x_0 \div x_0 \cdot x_1) = 1$
$\langle h_0, x_0, x_1 \rangle \hookrightarrow T\text{booleanAlgebra}_{6d}$ (*Stat20a*, *Stat5*, *Stat59a*, *Stat59*⋆) ⇒ $h_0 \restriction x_1 \neq \emptyset$
(*Stat55*⋆)Discharge ⇒ QED

-- injectivity of the standard homomorphism

THM booleanAlgebra$_{13}$. $1\text{–}1(phi_\Theta)$ & $\mathbf{domain}(phi_\Theta) = bb$. PROOF:
Suppose_not() ⇒ AUTO

> Arguing by contradiction, let us assume that the claim is false. Then, since by its very definition phi_Θ is a single-valued map and has the domain indicated in the claim, there must be distinct pairs p, q in phi_Θ whose second components coincide.

Use_def(phi$_\Theta$) $\Rightarrow$ $\mathsf{Svm}(\mathsf{phi}_\Theta)$ & $\mathbf{domain}(\mathsf{phi}_\Theta) = \mathsf{bb}$
Use_def(1–1) $\Rightarrow$ *Stat1* : $\neg\langle\forall p \in \mathsf{phi}_\Theta, q \in \mathsf{phi}_\Theta \mid p^{[2]} = q^{[2]} \rightarrow p = q\rangle$
$\langle p, q\rangle \hookrightarrow$ *Stat1*(*Stat1*$\star$) $\Rightarrow$ *Stat2* : $p, q \in \mathsf{phi}_\Theta$ & $p^{[2]} = q^{[2]}$ & $p \neq q$

It follows from the definition of phi_Θ that if x_0, x_1 are the respective first components of p, q, then the corresponding images $\{h \in \mathsf{hh}_\Theta \mid h\restriction x_0 = 1\} = \mathsf{phi}_\Theta\restriction x_0$ and $\{h \in \mathsf{hh}_\Theta \mid h\restriction x_1 = 1\} = \mathsf{phi}_\Theta\restriction x_1$ coincide.

Use_def(phi$_\Theta$) $\Rightarrow$ *Stat3* : $p, q \in \{[b, \{h \in \mathsf{hh}_\Theta \mid h\restriction b = 1\}] : b \in \mathsf{bb}\}$
$\langle x_0, x_1\rangle \hookrightarrow$ *Stat3*(*Stat3*$\star$) $\Rightarrow$ *Stat4* : $x_0, x_1 \in \mathsf{bb}$ & $p = [x_0, \{h \in \mathsf{hh}_\Theta \mid h\restriction x_0 = 1\}]$ & $q = [x_1, \{h \in \mathsf{hh}_\Theta \mid h\restriction x_1 = 1\}]$
(*Stat4*$\star$)ELEM $\Rightarrow$ $[x_0, \{h \in \mathsf{hh}_\Theta \mid h\restriction x_0 = 1\}]^{[1]} = x_0$ & $[x_0, \{h \in \mathsf{hh}_\Theta \mid h\restriction x_0 = 1\}]^{[2]} = \{h \in \mathsf{hh}_\Theta \mid h\restriction x_0 = 1\}$ &
$[x_1, \{h \in \mathsf{hh}_\Theta \mid h\restriction x_1 = 1\}]^{[1]} = x_1$ & $[x_1, \{h \in \mathsf{hh}_\Theta \mid h\restriction x_1 = 1\}]^{[2]} = \{h \in \mathsf{hh}_\Theta \mid h\restriction x_1 = 1\}$
EQUAL $\langle$*Stat2*$\rangle$ $\Rightarrow$ $\{h \in \mathsf{hh}_\Theta \mid h\restriction x_0 = 1\} = \{h \in \mathsf{hh}_\Theta \mid h\restriction x_1 = 1\}$
Suppose $\Rightarrow$ $x_0 = x_1$
EQUAL $\langle$*Stat2*$\rangle$ $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO
$\langle x_0\rangle \hookrightarrow T$booleanAlgebra$_9$ $\Rightarrow$ AUTO
$\langle x_1\rangle \hookrightarrow T$booleanAlgebra$_9$ $\Rightarrow$ AUTO
(*Stat4*$\star$)ELEM $\Rightarrow$ *Stat5* : $x_0 \neq x_1$ & $\mathsf{phi}_\Theta\restriction x_0 = \mathsf{phi}_\Theta\restriction x_1$ & $x_0, x_1 \in \mathsf{bb}$

Since $x_0 \neq x_1$, either $x_0 \neq x_0 \cdot x_1$ or $x_1 \neq x_0 \cdot x_1$ We will first get a contradiction by assuming that the first is the case; then, by proceeding analogously, we will get a contradiction in the other case as well. The contradiction is obtained through the preceding theorem.

TbooleanAlgebra$_0$ $\Rightarrow$ *Stat10* : $\langle\forall u, v \mid \{u, v\} \subseteq \mathsf{bb}$ & $u \cdot v = u$ & $v \cdot u = v \rightarrow u = v\rangle$
$\langle x_0, x_1\rangle \hookrightarrow$ *Stat10* $\Rightarrow$ AUTO
$\langle x_0, x_1\rangle \hookrightarrow T$booleanAlgebra$_3$ (*Stat5*$\star$) $\Rightarrow$ *Stat11* : $x_0 \cdot x_1 \neq \mathsf{ee}$ & $x_0 \cdot x_1 \neq x_0 \vee x_1 \cdot x_0 \neq x_1$
Suppose $\Rightarrow$ $x_0 \cdot x_1 \neq x_0$
$\langle x_0, x_1\rangle \hookrightarrow T$booleanAlgebra$_{13a}$ (*Stat5*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO
(*Stat11*$\star$)ELEM $\Rightarrow$ $x_1 \cdot x_0 \neq x_1$
$\langle x_1, x_0\rangle \hookrightarrow T$booleanAlgebra$_{13a}$ (*Stat5*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ QED

-- standard Boolean isomorphism
THM booleanAlgebra$_{14}$. $\mathsf{BooHom}_\Theta(\mathsf{phi}_\Theta)$. PROOF:
Suppose_not() $\Rightarrow$ AUTO

We already know from earlier theorems that phi_Θ is a one-one function and that it satisfies $\mathsf{phi}_\Theta\restriction\mathsf{ee} = \bigcup\mathbf{range}(\mathsf{phi}_\Theta)$

Use_def(1–1(phi$_\Theta$)) $\Rightarrow$ AUTO
TbooleanAlgebra$_{13}$ $\Rightarrow$ $\mathsf{Svm}(\mathsf{phi}_\Theta)$ & $\mathbf{domain}(\mathsf{phi}_\Theta) = \mathsf{bb}$

TbooleanAlgebra$_{11}$ $\Rightarrow$ $\mathsf{phi}_\Theta \upharpoonright \mathsf{ee} = \bigcup \mathbf{range}(\mathsf{phi}_\Theta)$ & $\mathsf{phi}_\Theta \upharpoonright \mathsf{ee} = \bigcup \mathbf{range}(\mathsf{phi}_\Theta)$ & $\mathsf{phi}_\Theta \upharpoonright \mathsf{ee} \neq \mathsf{phi}_\Theta \upharpoonright \mathsf{zz}_\Theta$
Use_def $(\mathsf{BooHom}_\Theta(\mathsf{phi}_\Theta))$ $\Rightarrow$ AUTO

We already know from earlier theorems that phi_Θ is a one-one function and that it satisfies $\mathsf{phi}_\Theta \upharpoonright \mathsf{ee} = \bigcup \mathbf{range}(\mathsf{phi}_\Theta)$; hence it may only fail to be a Boolean homomorphism due to one of the foollowing reasons:

ELEM $\Rightarrow$ *Stat1* : $\neg \langle \forall x \in \mathsf{bb}, y \in \mathsf{bb} \mid \mathsf{phi}_\Theta \upharpoonright (x \cdot y) = \mathsf{phi}_\Theta \upharpoonright x \cap \mathsf{phi}_\Theta \upharpoonright y \ \& \ \mathsf{phi}_\Theta \upharpoonright (x \div y) = \mathsf{phi}_\Theta \upharpoonright x \triangle \mathsf{phi}_\Theta \upharpoonright y \rangle$
$\langle x_0, y_0 \rangle \hookrightarrow$ *Stat1*(*Stat1*$\star$) $\Rightarrow$ $x_0, y_0 \in \mathsf{bb}$ &
$\neg(\mathsf{phi}_\Theta \upharpoonright (x_0 \cdot y_0) = \mathsf{phi}_\Theta \upharpoonright x_0 \cap \mathsf{phi}_\Theta \upharpoonright y_0 \ \& \ \mathsf{phi}_\Theta \upharpoonright (x_0 \div y_0) = \mathsf{phi}_\Theta \upharpoonright x_0 \triangle \mathsf{phi}_\Theta \upharpoonright y_0)$
$\langle x_0, y_0 \rangle \hookrightarrow T$booleanAlgebra$_{10}$ $\Rightarrow$ false; Discharge $\Rightarrow$ QED

ENTER_THEORY Set_theory

DISPLAY booleanAlgebra

THEORY booleanAlgebra(bb, $\cdot$, $\div$, ee)
$\mathsf{ee} \in \mathsf{bb}$
$\mathsf{ee} \neq \mathsf{ee} \div \mathsf{ee}$
$\langle \forall x, y \mid \{x, y\} \subseteq \mathsf{bb} \rightarrow x \cdot y \in \mathsf{bb} \rangle$
$\langle \forall x, y \mid \{x, y\} \subseteq \mathsf{bb} \rightarrow x \div y \in \mathsf{bb} \rangle$
$\langle \forall x, y, z \mid \{x, y, z\} \subseteq \mathsf{bb} \rightarrow x \cdot (y \cdot z) = (x \cdot y) \cdot z \rangle$
$\langle \forall x, y, z \mid \{x, y, z\} \subseteq \mathsf{bb} \rightarrow x \div (y \div z) = (x \div y) \div z \rangle$
$\langle \forall x, y, z \mid \{x, y, z\} \subseteq \mathsf{bb} \rightarrow (x \div y) \cdot z = z \cdot y \div z \cdot x \rangle$
$\langle \forall x, y \mid \{x, y\} \subseteq \mathsf{bb} \rightarrow x \div x = y \div y \rangle$
$\langle \forall x, y \mid \{x, y\} \subseteq \mathsf{bb} \rightarrow x \div (y \div x) = y \rangle$
$\langle \forall x \mid x \in \mathsf{bb} \rightarrow x \cdot x = x \rangle$
$\langle \forall x \mid x \in \mathsf{bb} \rightarrow \mathsf{ee} \cdot x = x \rangle$

$\Rightarrow (\mathsf{zz}_\Theta, \mathsf{cmp}_\Theta, \mathsf{Ideal}_\Theta, \mathsf{BooHom}_\Theta, \mathsf{hh}_\Theta, \mathsf{phi}_\Theta)$

$\mathsf{zz}_\Theta = \mathbf{arb}(\mathsf{bb}) \div \mathbf{arb}(\mathsf{bb})$

$\langle \forall \mathsf{x} \mid (\mathsf{x} \in \mathsf{bb} \rightarrow \mathsf{x} \div \mathsf{x} = \mathsf{zz}_\Theta \;\&\; \mathsf{x} \div \mathsf{zz}_\Theta = \mathsf{x} \;\&\; \mathsf{zz}_\Theta \div \mathsf{x} = \mathsf{x}) \;\&\; \mathsf{zz}_\Theta \in \mathsf{bb} \rangle$

$\langle \forall \mathsf{x}, \mathsf{y} \mid \{\mathsf{x}, \mathsf{y}\} \subseteq \mathsf{bb} \rightarrow \mathsf{x} \div \mathsf{y} = \mathsf{y} \div \mathsf{x} \rangle$

$\langle \forall \mathsf{x}, \mathsf{y} \mid \{\mathsf{x}, \mathsf{y}\} \subseteq \mathsf{bb} \rightarrow \mathsf{x} \cdot \mathsf{y} = \mathsf{y} \cdot \mathsf{x} \rangle$

$\langle \forall \mathsf{x}, \mathsf{y} \mid \mathsf{x}, \mathsf{y} \in \mathsf{bb} \rightarrow \mathsf{x} \div \mathsf{y} = \mathsf{y} \div \mathsf{x} \rangle$

$\langle \forall \mathsf{x}, \mathsf{y} \mid \mathsf{x}, \mathsf{y} \in \mathsf{bb} \rightarrow \mathsf{x} \cdot \mathsf{y} = \mathsf{y} \cdot \mathsf{x} \rangle$

$\langle \forall \mathsf{u}, \mathsf{v} \mid \{\mathsf{u}, \mathsf{v}\} \subseteq \mathsf{bb} \;\&\; \mathsf{u} \cdot \mathsf{v} = \mathsf{u} \;\&\; \mathsf{v} \cdot \mathsf{u} = \mathsf{v} \rightarrow \mathsf{u} = \mathsf{v} \rangle$

$\langle \forall \mathsf{x} \mid \mathsf{x} \in \mathsf{bb} \rightarrow \mathsf{cmp}_\Theta(\mathsf{x}) = \mathsf{ee} \div \mathsf{x} \rangle$

$\langle \forall \mathsf{x} \mid \big(\mathsf{x} \in \mathsf{bb} \rightarrow \mathsf{cmp}_\Theta(\mathsf{x}) \in \mathsf{bb} \;\&\; \mathsf{cmp}_\Theta(\mathsf{cmp}_\Theta(\mathsf{x})) = \mathsf{x}\big) \;\&\; \mathsf{cmp}_\Theta(\mathsf{ee}) = \mathsf{zz}_\Theta \;\&\; \mathsf{cmp}_\Theta(\mathsf{zz}_\Theta) = \mathsf{ee} \rangle$

$\langle \forall \mathsf{x}, \mathsf{y} \mid \{\mathsf{x}, \mathsf{y}\} \subseteq \mathsf{bb} \rightarrow \mathsf{cmp}_\Theta(\mathsf{x}) \div \mathsf{x} = \mathsf{ee} \;\&\; \mathsf{y} \cdot \mathsf{x} \div \mathsf{y} \cdot \mathsf{cmp}_\Theta(\mathsf{x}) = \mathsf{y} \;\&\; \mathsf{y} \cdot \mathsf{x} \cdot (\mathsf{y} \cdot \mathsf{cmp}_\Theta(\mathsf{x})) = \mathsf{zz}_\Theta \rangle$

$\langle \forall \mathsf{x}, \mathsf{y} \mid \{\mathsf{x}, \mathsf{y}\} \subseteq \mathsf{bb} \rightarrow \;\rightarrow \mathsf{cmp}_\Theta(\mathsf{x} \div \mathsf{y}) = \mathsf{x} \cdot \mathsf{y} \div \mathsf{cmp}_\Theta(\mathsf{x}) \cdot \mathsf{cmp}_\Theta(\mathsf{y}) \rangle$

$\langle \forall \mathsf{x} \mid \mathsf{x} \in \mathsf{bb} \rightarrow \mathsf{cmp}_\Theta(\mathsf{x}) \neq \mathsf{x} \;\&\; (\mathsf{x} \notin \{\mathsf{zz}_\Theta, \mathsf{ee}\} \rightarrow \mathsf{cmp}_\Theta(\mathsf{x}) \in \mathsf{bb} \setminus \{\mathsf{zz}_\Theta, \mathsf{ee}\}) \rangle$

$\langle \forall \mathsf{u}, \mathsf{v} \mid \{\mathsf{u}, \mathsf{v}\} \subseteq \mathsf{bb} \;\&\; \mathsf{u} \cdot \mathsf{v} = \mathsf{ee} \rightarrow \mathsf{u} = \mathsf{ee} \;\&\; \mathsf{v} = \mathsf{ee} \rangle$

$\langle \forall \mathsf{u}, \mathsf{v}, \mathsf{x}, \mathsf{y} \mid \{\mathsf{u}, \mathsf{v}, \mathsf{x}, \mathsf{y}\} \subseteq \mathsf{bb} \rightarrow \mathsf{u} \cdot \mathsf{cmp}_\Theta(\mathsf{x}) \div \mathsf{v} \cdot \mathsf{cmp}_\Theta(\mathsf{y}) = (\mathsf{u} \cdot \mathsf{cmp}_\Theta(\mathsf{x}) \div \mathsf{v} \cdot \mathsf{cmp}_\Theta(\mathsf{y})) \cdot \mathsf{cmp}_\Theta(\mathsf{x} \cdot \mathsf{y}) \rangle$

$\langle \forall \mathsf{i} \mid \mathsf{Ideal}_\Theta(\mathsf{i}) \leftrightarrow \{\mathsf{x} \div \mathsf{y} : \mathsf{x} \in \mathsf{i}, \mathsf{y} \in \mathsf{i}\} \subseteq \mathsf{i} \;\&\; \{\mathsf{x} \cdot \mathsf{y} : \mathsf{x} \in \mathsf{bb}, \mathsf{y} \in \mathsf{i}\} \subseteq \mathsf{i} \;\&\; \mathsf{i} \subseteq \mathsf{bb} \setminus \{\mathsf{ee}\} \;\&\; \mathsf{i} \not\subseteq \{\mathsf{zz}_\Theta\} \rangle$

$\langle \forall \mathsf{i}, \mathsf{x}, \mathsf{y} \mid \mathsf{Ideal}_\Theta(\mathsf{i}) \;\&\; \{\mathsf{x}, \mathsf{y}\} \subseteq \mathsf{i} \rightarrow \mathsf{x} \div \mathsf{y} \in \mathsf{i} \rangle$

$\langle \forall \mathsf{i} \mid \mathsf{Ideal}_\Theta(\mathsf{i}) \rightarrow \mathsf{zz}_\Theta \in \mathsf{i} \;\&\; (\mathsf{x} \in \mathsf{i} \;\&\; \mathsf{y} \in \mathsf{bb} \rightarrow \mathsf{x} \cdot \mathsf{y}, \mathsf{y} \cdot \mathsf{x} \in \mathsf{i} \;\&\; \mathsf{cmp}_\Theta(\mathsf{x}) \notin \mathsf{i}) \;\&\; \mathsf{ee} \notin \mathsf{i} \rangle$

$\langle \forall \mathsf{i} \mid \mathsf{Ideal}_\Theta(\mathsf{i}) \rightarrow \langle \exists \mathsf{m} \mid \mathsf{i} \subseteq \mathsf{m} \;\&\; \langle \forall \mathsf{j} \mid \mathsf{Ideal}_\Theta(\mathsf{j}) \;\&\; \mathsf{m} \subseteq \mathsf{j} \leftrightarrow \mathsf{j} = \mathsf{m} \rangle \rangle \rangle$

$\langle \forall \mathsf{b} \mid \mathsf{b} \subseteq \mathsf{bb} \setminus \{\mathsf{zz}_\Theta\} \;\&\; \{\mathsf{x} \cdot \mathsf{y} : \mathsf{x} \in \mathsf{b}, \mathsf{y} \in \mathsf{b}\} \subseteq \mathsf{b} \;\&\; \mathsf{b} \not\subseteq \{\mathsf{ee}\} \rightarrow \mathsf{Ideal}_\Theta(\{\mathsf{a} \cdot \mathsf{cmp}_\Theta(\mathsf{x}) : \mathsf{a} \in \mathsf{bb}, \mathsf{x} \in \mathsf{b}\}) \rangle$

$\langle \forall \mathsf{x} \mid \mathsf{x} \in \mathsf{bb} \setminus \{\mathsf{zz}_\Theta, \mathsf{ee}\} \rightarrow \mathsf{Ideal}_\Theta(\{\mathsf{a} \cdot \mathsf{x} : \mathsf{a} \in \mathsf{bb}\}) \;\&\; \mathsf{x} \in \{\mathsf{a} \cdot \mathsf{x} : \mathsf{a} \in \mathsf{bb}\} \rangle$

$\langle \forall \mathsf{h} \mid \mathsf{BooHom}_\Theta(\mathsf{h}) \leftrightarrow \mathsf{Svm}(\mathsf{h}) \;\&\; \mathbf{domain}(\mathsf{h}) = \mathsf{bb} \;\&\; \mathsf{h}\!\upharpoonright\!\mathsf{ee} = \bigcup \mathbf{range}(\mathsf{h}) \;\&\; \mathsf{h}\!\upharpoonright\!\mathsf{ee} \neq \mathsf{h}\!\upharpoonright\!\mathsf{zz}_\Theta \;\&\; \langle \forall \mathsf{x} \in \mathsf{bb}, \mathsf{y} \in \mathsf{bb} \mid \mathsf{h}\!\upharpoonright\!(\mathsf{x} \cdot \mathsf{y}) = \mathsf{h}\!\upharpoonright\!\mathsf{x} \cap \mathsf{h}\!\upharpoonright\!\mathsf{y} \;\&\; \mathsf{h}\!\upharpoonright\!(\mathsf{x} \div \mathsf{y}) = \mathsf{h}\!\upharpoonright\!\mathsf{x} \bigtriangleup \mathsf{h}\!\upharpoonright\!\mathsf{y} \rangle \rangle$

$\mathsf{hh}_\Theta = \{\mathsf{h} \subseteq \mathsf{bb} \times 2 \mid \mathsf{BooHom}_\Theta(\mathsf{h})\}$

$\langle \forall \mathsf{h} \mid \mathsf{h} \in \mathsf{hh}_\Theta \rightarrow \mathsf{h}\!\upharpoonright\!\mathsf{zz}_\Theta = \emptyset \;\&\; \mathsf{h}\!\upharpoonright\!\mathsf{ee} = 1 \rangle$

$\langle \forall \mathsf{h} \mid \mathsf{h} \in \mathsf{hh}_\Theta \;\&\; \{\mathsf{x}, \mathsf{y}\} \subseteq \mathsf{bb} \;\&\; \mathsf{h}\!\upharpoonright\!(\mathsf{x} \div \mathsf{x} \cdot \mathsf{y}) = 1 \;\&\; \mathsf{h}\!\upharpoonright\!\mathsf{y} = \emptyset \rightarrow \mathsf{h}\!\upharpoonright\!\mathsf{x} = 1 \rangle$

$\langle \forall \mathsf{x}, \mathsf{m} \mid \mathsf{x} \notin \mathsf{m} \;\&\; \mathsf{x} \in \mathsf{bb} \;\&\; \langle \forall \mathsf{j} \mid \mathsf{Ideal}_\Theta(\mathsf{j}) \;\&\; \mathsf{m} \subseteq \mathsf{j} \leftrightarrow \mathsf{j} = \mathsf{m} \rangle \rightarrow \mathsf{cmp}_\Theta(\mathsf{x}) \in \mathsf{m} \rangle$

$\langle \forall \mathsf{m} \mid \langle \forall \mathsf{j} \mid \mathsf{Ideal}_\Theta(\mathsf{j}) \;\&\; \mathsf{m} \subseteq \mathsf{j} \leftrightarrow \mathsf{j} = \mathsf{m} \rangle \rightarrow \{[\mathsf{x}, \textbf{if } \mathsf{x} \in \mathsf{m} \textbf{ then } \emptyset \textbf{ else } 1 \textbf{ fi}] : \mathsf{x} \in \mathsf{bb}\} \in \mathsf{hh}_\Theta \rangle$

$\mathsf{bb} \subseteq \{\mathsf{zz}_\Theta, \mathsf{ee}\} \rightarrow \{[\mathsf{zz}_\Theta, \emptyset], [\mathsf{ee}, 1]\} \in \mathsf{hh}_\Theta$

$\langle \forall \mathsf{x} \mid \mathsf{x} \in \mathsf{bb} \setminus \{\mathsf{zz}_\Theta\} \rightarrow \{\mathsf{h} \in \mathsf{hh}_\Theta \mid \mathsf{h}\!\upharpoonright\!\mathsf{x} = 1\} \neq \emptyset \rangle$

$\mathsf{phi}_\Theta = \{[\mathsf{b}, \{\mathsf{h} \in \mathsf{hh}_\Theta \mid \mathsf{h}\!\upharpoonright\!\mathsf{b} = 1\}] : \mathsf{b} \in \mathsf{bb}\}$

$\langle \forall \mathsf{x} \mid \mathsf{x} \in \mathsf{bb} \rightarrow \mathsf{phi}_\Theta\!\upharpoonright\!\mathsf{x} = \{\mathsf{h} \in \mathsf{hh}_\Theta \mid \mathsf{h}\!\upharpoonright\!\mathsf{x} = 1\} \rangle$

$\langle \forall \mathsf{x}, \mathsf{y} \mid \mathsf{x}, \mathsf{y} \in \mathsf{bb} \rightarrow \mathsf{phi}_\Theta\!\upharpoonright\!(\mathsf{x} \cdot \mathsf{y}) = \mathsf{phi}_\Theta\!\upharpoonright\!\mathsf{x} \cap \mathsf{phi}_\Theta\!\upharpoonright\!\mathsf{y} \;\&\; \mathsf{phi}_\Theta\!\upharpoonright\!(\mathsf{x} \div \mathsf{y}) = \mathsf{phi}_\Theta\!\upharpoonright\!\mathsf{x} \bigtriangleup \mathsf{phi}_\Theta\!\upharpoonright\!\mathsf{y} \rangle$

$\mathsf{phi}_\Theta\!\upharpoonright\!\mathsf{ee} = \bigcup \mathbf{range}(\mathsf{phi}_\Theta) \;\&\; \mathsf{phi}_\Theta\!\upharpoonright\!\mathsf{zz}_\Theta = \emptyset \;\&\; \mathsf{phi}_\Theta\!\upharpoonright\!\mathsf{ee} \neq \mathsf{phi}_\Theta\!\upharpoonright\!\mathsf{zz}_\Theta$

$\langle \forall \mathsf{x} \mid \mathsf{x} \in \mathsf{bb} \rightarrow \mathsf{phi}_\Theta\!\upharpoonright\!\mathsf{cmp}_\Theta(\mathsf{x}) = \mathsf{phi}_\Theta\!\upharpoonright\!\mathsf{ee} \setminus \mathsf{phi}_\Theta\!\upharpoonright\!\mathsf{x} \rangle$

$1\text{–}1(\mathsf{phi}_\Theta) \;\&\; \mathbf{domain}(\mathsf{phi}_\Theta) = \mathsf{bb}$

$\mathsf{BooHom}_\Theta(\mathsf{phi}_\Theta)$

END booleanAlgebra