Reasoning about Connectivity without Paths*
Accompanying proof-scenario

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1 Introductory note

This scenario undertakes the study of connectivity, treated here as a property of hypergraphs. HYPERGRAPHS are defined to be just finite sets of edges, each edge being, in its turn, a finite set of cardinality two or more. The elements of the edges of a hypergraph are called its VERTICES, or NODES (deliberately, we are leaving out of our study hypergraphs with isolated nodes).

Ordinary GRAPHS simply are, from this perspective, finite sets of doubletons.

A hypergraph G is said to be CONNECTED when it has no subset P, other than $\emptyset$ and G itself, such that P and $\mathsf{G}\backslash\mathsf{P}$ share no nodes. Trivial examples of connected hypergraphs are: $\emptyset$; every singleton $\{\mathsf{e}\}$ whose element e is finite, non-null, and non-singleton; every doubleton hypergraph $\{\mathsf{e}_0, \mathsf{e}_1\}$ with $\mathsf{e}_0 \cap \mathsf{e}_1 \neq \emptyset$.

We will prove that one can always remove one of the nodes of a connected non-null hypergraph without disrupting its connectivity; but, preliminary to showing that, we must explain how nodes get removed. Removing a node w from an *ordinary* graph G amounts to withdrawing from G all edges to which w belongs; but when G is a hypergraph endowed with at least one non-doubleton edge, node removal becomes a more complicated operation. Speaking in general, REMOVING a node w will amount to withdrawing all doubleton edges to which w belongs, at the same time deleting w from every other edge (the edges to which w does not belong are nohow affected by the removal operation).

Now we are ready to define a CUT VERTEX of an hypergraph G (typically connected) as being a node w such that G does not have the form $\mathsf{G} = \{\{\mathsf{u}, \mathsf{w}\}\}$, and either a disconnectd graph results from the removal of w from G, or some node different from w gets lost in consequence of the removal of w. Our target theorem hence is that *every non-null graph* G *has a node which is not a cut vertex of* G.
Henceforth hypergraphs will be simply called graphs, for brevity.

The proofs carried out in this scenario follow rather closely the guidelines of the conference paper

[CO14] A. Casagrande, E. G. Omodeo, Reasoning about Connectivity without Paths, ICTCS 2014, Perugia.

The propositions in that paper are identified by the names: Lemmas 1, ..., 21, Theorems 1 and 2, and Corollary 1; a reference to each of them is provided near the corresponding Theorem $\mathsf{hgraph}_{\mathrm{XX}}$ within this scenario. The only missing proposition here is Theorem 1, whose proof has been embedded in the proof of Corollary 1 (Theorem hgraph_{24} in this document); on the other hand, a few (relatively marginal) propositions not proved in that paper will enter into play more visibly here.

2 Prerequisites

2.1 Basic laws on the power-set and sum-set global operations

DEF $\mathcal{P}$: [Family of all subsets of a given set] $\mathcal{P}\mathsf{S} \;=_{\text{Def}}\; \{\mathsf{x} : \mathsf{x} \subseteq \mathsf{S}\}$

|| Our next theorem characterizes the powerset formation operation in more usable terms than the very definition of this construct. It also proves that no set can equal is own powerset (else it should belong to itself, against the acyclicity of membership).

THEOREM pow_0: [Characterization of powerset; also: no set equals its own powerset] $(\mathsf{X} \supseteq \mathsf{Y} \leftrightarrow \mathsf{Y} \in \mathcal{P}\mathsf{X})\ \&\ \mathsf{X} \neq \mathcal{P}\mathsf{X}$. PROOF:

Suppose_not$(\mathsf{x}_0, \mathsf{y}_0) \Rightarrow$ AUTO

|| We begin by excluding the possibility that $\mathsf{x}_0 = \mathcal{P}\mathsf{x}_0$:

Suppose $\Rightarrow$ *Stat0*: $\mathsf{x}_0 \notin \{\mathsf{y} : \mathsf{y} \subseteq \mathsf{x}_0\}$
$\langle \mathsf{x}_0 \rangle \hookrightarrow$ *Stat0* $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO
Use_def$(\mathcal{P}\mathsf{x}_0) \Rightarrow$ AUTO

|| Arguing by contradiction, if $\mathsf{x}_0, \mathsf{y}_0$ constituted a counterexample, then either one of the literals $\mathsf{x}_0 \supseteq \mathsf{y}_0$ and $\mathsf{y}_0 \in \{\mathsf{y} : \mathsf{y} \subseteq \mathsf{x}_0\}$ would be true and the other one would be false.

EQUAL $\Rightarrow$ *Stat1*: $\mathsf{x}_0 \supseteq \mathsf{y}_0 \neq \mathsf{y}_0 \in \{\mathsf{y} : \mathsf{y} \subseteq \mathsf{x}_0\}$

|| If it is the second that is true then, via a substitution in the setformer, we would contradict the falsity of the first.

Suppose $\Rightarrow$ *Stat2*: $\mathsf{y}_0 \in \{\mathsf{y} : \mathsf{y} \subseteq \mathsf{x}_0\}$
$\langle \mathsf{y}_1 \rangle \hookrightarrow$ *Stat2*(*Stat1*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat3*: $\mathsf{y}_0 \notin \{\mathsf{y} : \mathsf{y} \subseteq \mathsf{x}_0\}$

|| But then the literals $\mathsf{x}_0 \supseteq \mathsf{y}_0$ and $\mathsf{y}_0 \notin \{\mathsf{y} : \mathsf{y} \subseteq \mathsf{x}_0\}$ should hold together, which gives us a contradiction if we replace the bounded variable y of the setformer by y_0.

$\langle \mathsf{y}_0 \rangle \hookrightarrow$ *Stat3*(*Stat1*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ QED

THEOREM pow_1: [Monotonicity of powerset] $\mathsf{S} \supseteq \mathsf{X} \rightarrow \mathcal{P}\mathsf{X} \cup \{\emptyset, \mathsf{X}\} \subseteq \mathcal{P}\mathsf{S}$. PROOF:

Suppose_not$(\mathsf{s}_0, \mathsf{x}_0) \Rightarrow$ AUTO
Set_monot $\Rightarrow$ $\{\mathsf{x} : \mathsf{x} \subseteq \mathsf{x}_0\} \subseteq \{\mathsf{x} : \mathsf{x} \subseteq \mathsf{s}_0\}$
Use_def$(\mathcal{P}) \Rightarrow$ *Stat1*: $\emptyset \notin \{\mathsf{x} : \mathsf{x} \subseteq \mathsf{s}_0\} \vee \mathsf{x}_0 \notin \{\mathsf{x} : \mathsf{x} \subseteq \mathsf{s}_0\}$
$\langle \emptyset, \mathsf{x}_0 \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ false; Discharge $\Rightarrow$ QED

THEOREM pow_2: [Powerset of null set and of singletons] $\mathcal{P}\emptyset = \{\emptyset\}$ & $\mathcal{P}\{\mathsf{X}\} = \{\emptyset, \{\mathsf{X}\}\}$. PROOF:

Suppose_not(x_0) ⇒ AUTO

Suppose ⇒ $\mathcal{P}\emptyset \neq \{\emptyset\}$

$\langle \emptyset, \emptyset \rangle \hookrightarrow T\mathsf{pow}_1$ ⇒ *Stat0*: $\mathcal{P}\emptyset \not\subseteq \{\emptyset\}$

$\langle y_0 \rangle \hookrightarrow$ *Stat0*(*Stat0*⋆) ⇒ *Stat1*: $y_0 \in \mathcal{P}\emptyset$ & $y_0 \notin \{\emptyset\}$

$\langle \emptyset, y_0 \rangle \hookrightarrow T\mathsf{pow}_0$(*Stat1*⋆) ⇒ false; Discharge ⇒ $\mathcal{P}\{x_0\} \neq \{\emptyset, \{x_0\}\}$

$\langle \{x_0\}, \{x_0\} \rangle \hookrightarrow T\mathsf{pow}_1$ ⇒ *Stat2*: $\mathcal{P}\{x_0\} \not\subseteq \{\emptyset, \{x_0\}\}$

$\langle y_1 \rangle \hookrightarrow$ *Stat2* ⇒ *Stat3*: $y_1 \in \mathcal{P}\{x_0\}$ & $y_1 \notin \{\emptyset, \{x_0\}\}$

$\langle \{x_0\}, y_1 \rangle \hookrightarrow T\mathsf{pow}_0$(*Stat3*⋆) ⇒ false; Discharge ⇒ QED

DEF $\bigcup$: [Family of all members of members of a set] $\bigcup S =_{\mathrm{Def}} \{u : v \in S, u \in v\}$

THEOREM un_0: [Unionset operation yielding null result] $\bigcup X = \emptyset \leftrightarrow X \subseteq \{\emptyset\}$. PROOF:

Suppose_not(x_0) ⇒ AUTO

Use_def($\bigcup x_0$) ⇒ AUTO

Suppose ⇒ *Stat1*: $x_0 \not\subseteq \{\emptyset\}$ & $\{y : x \in x_0, y \in x\} = \emptyset$

$\langle x_1, x_1, \mathbf{arb}(x_1) \rangle \hookrightarrow$ *Stat1*(*Stat1*) ⇒ false; Discharge ⇒ *Stat2*: $\{y : x \in x_0, y \in x\} \neq \emptyset$ & $x_0 \subseteq \{\emptyset\}$

$\langle x_2, y_2 \rangle \hookrightarrow$ *Stat2*(*Stat2*⋆) ⇒ false; Discharge ⇒ QED

THEOREM un_2: [Unionset operation combined with set adjunction] $X \cup \{Y\} = Z \rightarrow \bigcup Z = \bigcup X \cup Y$. PROOF:

Suppose_not(x_0, y_0, z_0) ⇒ AUTO

EQUAL ⇒ *Stat0*: $\bigcup(x_0 \cup \{y_0\}) \neq \bigcup x_0 \cup y_0$

Use_def($\bigcup$) ⇒ *Stat1*: $\{y : x \in x_0 \cup \{y_0\}, y \in x\} \neq \{y' : x' \in x_0, y' \in x'\} \cup y_0$

$\langle c \rangle \hookrightarrow$ *Stat1*(*Stat1*⋆) ⇒ $c \in \{y : x \in x_0 \cup \{y_0\}, y \in x\} \neq c \in \{y' : x' \in x_0, y' \in x'\} \cup y_0$

Suppose ⇒ *Stat2*: $c \in \{y : x \in x_0 \cup \{y_0\}, y \in x\}$ & $c \notin \{y' : x' \in x_0, y' \in x'\}$ & $c \notin y_0$

$\langle x_1, y_1, x_1, y_1 \rangle \hookrightarrow$ *Stat2*(*Stat2*⋆) ⇒ false; Discharge ⇒ AUTO

Suppose ⇒ *Stat3*: $c \notin \{y : x \in x_0 \cup \{y_0\}, y \in x\}$ & $c \in y_0$

$\langle y_0, c \rangle \hookrightarrow$ *Stat3*(*Stat3*⋆) ⇒ false; Discharge ⇒ *Stat4*: $c \in \{y' : x' \in x_0, y' \in x'\}$ & $c \notin \{y : x \in x_0 \cup \{y_0\}, y \in x\}$

$\langle x_2, y_2, x_2, y_2 \rangle \hookrightarrow$ *Stat4*(*Stat4*⋆) ⇒ false; Discharge ⇒ QED

THEOREM un_4: [Unionset operation applied to a singleton] $\{Y\} = Z \rightarrow \bigcup Z = Y$. PROOF:

Suppose_not(y_0, z_0) ⇒ AUTO

$\langle \emptyset \rangle \hookrightarrow T\mathsf{un}_0(\star)$ ⇒ $\bigcup \emptyset = \emptyset$

$\langle \emptyset, y_0, z_0 \rangle \hookrightarrow T\mathsf{un}_2(\star)$ ⇒ false; Discharge ⇒ QED

2.2 Basic laws on the finiteness property, and associated 'Theory's

Traditionally, finiteness is defined through the notion of cardinality of a set: a set is finite if its cardinality precedes the first infinite ordinal. As a shortcut, to begin developing an acceptable formal treatment of finiteness without much preparatory work, we adopt here the following definition (reminiscent of Tarski's 1924 paper "Sur les ensembles fini"): a set F is *finite* if every non-null family of subsets of F owns an inclusion-minimal element. This notion can be specified very succinctly in terms of the powerset operator.

DEF Fin: [Finiteness property] $\mathsf{Finite}(\mathsf{F}) \leftrightarrow_{\mathrm{Def}} \langle \forall g \in \mathcal{P}(\mathcal{P}F) \setminus \{\emptyset\}, \exists m \mid g \cap \mathcal{P}m = \{m\} \rangle$

THEOREM fin_0: [Monotonicity of finiteness] $Y \supseteq X \;\&\; \mathsf{Finite}(Y) \to \mathsf{Finite}(X)$. PROOF:

$\mathsf{Suppose\_not}(y_0, x_0) \Rightarrow$ AUTO

$\langle y_0, x_0 \rangle \hookrightarrow T\mathsf{pow}_1(\star) \Rightarrow \mathcal{P}y_0 \supseteq \mathcal{P}x_0$

$\mathsf{Use\_def}(\mathsf{Finite}) \Rightarrow$ *Stat1* : $\neg\langle \forall g \in \mathcal{P}(\mathcal{P}x_0) \setminus \{\emptyset\}, \exists m \mid g \cap \mathcal{P}m = \{m\} \rangle \;\&\; \langle \forall g' \in \mathcal{P}(\mathcal{P}y_0) \setminus \{\emptyset\}, \exists m \mid g' \cap \mathcal{P}m = \{m\} \rangle$

$\langle \mathcal{P}y_0, \mathcal{P}x_0 \rangle \hookrightarrow T\mathsf{pow}_1(\star) \Rightarrow \mathcal{P}(\mathcal{P}y_0) \supseteq \mathcal{P}(\mathcal{P}x_0)$

$\langle g_0, g_0 \rangle \hookrightarrow$ *Stat1*(*Stat1*$\star$) $\Rightarrow \neg\langle \exists m \mid g_0 \cap \mathcal{P}m = \{m\} \rangle \;\&\; \langle \exists m \mid g_0 \cap \mathcal{P}m = \{m\} \rangle$

Discharge $\Rightarrow$ QED

THEOREM fin_1: [Finiteness of the union of a finite set with a singleton] $\mathsf{Finite}(F) \to \mathsf{Finite}(F \cup \{X\})$. PROOF:

$\mathsf{Suppose\_not}(f_0, x_0) \Rightarrow$ AUTO

Arguing by contradiction, suppose that f_0 and x_0 are such that f_0 is finite but $f_0 \cup \{x_0\}$ is not. A nonnull familiy g_0 of subsets of $f_0 \cup \{x_0\}$ must then exist none of whose elements is minimal. On the other hand $\{y \setminus \{x_0\} : y \in g_0\}$, which is also nonnull but consists entirely of subsets of f_0, must have a minimal element $m_0 = y_0 \setminus \{x_0\}$, with $y_0 \in g_0$.

$\mathsf{Use\_def}(\mathsf{Finite}) \Rightarrow$ *Stat0* : $\neg\langle \forall g \in \mathcal{P}(\mathcal{P}(f_0 \cup \{x_0\})) \setminus \{\emptyset\}, \exists m \mid g \cap \mathcal{P}m = \{m\} \rangle \;\&$ *Stat1* : $\langle \forall h \in \mathcal{P}(\mathcal{P}f_0) \setminus \{\emptyset\}, \exists m \mid h \cap \mathcal{P}m = \{m\} \rangle$

$\langle g_0 \rangle \hookrightarrow$ *Stat0*(*Stat0*) $\Rightarrow$ *Stat2* : $\neg\langle \exists m \mid g_0 \cap \mathcal{P}m = \{m\} \rangle \;\&\; g_0 \in \mathcal{P}(\mathcal{P}(f_0 \cup \{x_0\})) \;\&\; g_0 \neq \emptyset$

$\mathsf{Loc\_def} \Rightarrow$ *Stat3* : $h_0 = \{y \setminus \{x_0\} : y \in g_0\}$

$\mathsf{Suppose} \Rightarrow h_0 \notin \mathcal{P}(\mathcal{P}f_0) \setminus \{\emptyset\}$

$\mathsf{Suppose} \Rightarrow$ *Stat4* : $\{y \setminus \{x_0\} : y \in g_0\} = \emptyset$

$\langle \mathbf{arb}(g_0) \rangle \hookrightarrow$ *Stat4*(*Stat2*, *Stat2*) $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO

$\mathsf{Use\_def}(\mathcal{P}) \Rightarrow$ *Stat5* : $h_0 \notin \{h : h \subseteq \{k : k \subseteq f_0\}\}$

$\langle h_0 \rangle \hookrightarrow$ *Stat5*(*Stat5*$\star$) $\Rightarrow$ *Stat6* : $h_0 \not\subseteq \{k : k \subseteq f_0\}$

$\langle k_0 \rangle \hookrightarrow$ *Stat6*(*Stat3*$\star$) $\Rightarrow$ *Stat7* : $k_0 \in \{y \setminus \{x_0\} : y \in g_0\} \;\&\; k_0 \notin \{k : k \subseteq f_0\}$

$\langle y_1, k_0 \rangle \hookrightarrow$ *Stat7*(*Stat7*$\star$) $\Rightarrow y_1 \in g_0 \;\&\; y_1 \not\subseteq f_0 \cup \{x_0\}$

Use_def($\mathcal{P}$) $\Rightarrow$ *Stat8*: $g_0 \in \{h : h \subseteq \{k : k \subseteq f_0 \cup \{x_0\}\}\}$
$\langle h_1 \rangle \hookrightarrow$ *Stat8*(*Stat7*$\star$) $\Rightarrow$ *Stat9*: $y_1 \in \{k : k \subseteq f_0 \cup \{x_0\}\}$
$\langle k_1 \rangle \hookrightarrow$ *Stat9*(*Stat7*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO
$\langle h_0, m_0 \rangle \hookrightarrow$ *Stat1*(*Stat3*$\star$) $\Rightarrow$ *Stat10*: $m_0 \in \{y \setminus \{x_0\} : y \in g_0\}$ & $h_0 \cap \mathcal{P}m_0 = \{m_0\}$
$\langle y_0 \rangle \hookrightarrow$ *Stat10*(*Stat10*$\star$) $\Rightarrow$ *Stat11*: $m_0 = y_0 \setminus \{x_0\}$ & $y_0 \in g_0$

> We will reach the desired contradiction by showing that either m_0 or $y_0 = m_0 \cup \{x_0\}$ is minimal in g_0. We check first that m_0 itself must be minimal when $m_0 \in g_0$.

Suppose $\Rightarrow$ $m_0 \in g_0$
$\langle m_0 \rangle \hookrightarrow$ *Stat2*(*Stat10*$\star$) $\Rightarrow$ *Stat12*: $g_0 \cap \mathcal{P}m_0 \not\subseteq \{m_0\}$
Use_def($\mathcal{P}m_0$) $\Rightarrow$ AUTO
$\langle z_0 \rangle \hookrightarrow$ *Stat12*(*Stat3*$\star$) $\Rightarrow$ *Stat13*: $z_0 \in \{h : h \subseteq m_0\}$ & $z_0 \notin \{y \setminus \{x_0\} : y \in g_0\}$ & $z_0 \in g_0$
$\langle h_2, z_0 \rangle \hookrightarrow$ *Stat13*(*Stat11*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO

> Suppose next that $m_0 \notin g_0$; we will reach a contradiction by showing that y_0 is minimal in g_0.

Suppose $\Rightarrow$ $y_0 \notin \mathcal{P}y_0$
Use_def($\mathcal{P}$) $\Rightarrow$ *Stat13a*: $y_0 \notin \{s : s \subseteq y_0\}$
$\langle y_0 \rangle \hookrightarrow$ *Stat13a*(*Stat13a*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO
$\langle y_0 \rangle \hookrightarrow$ *Stat2*(*Stat11*$\star$) $\Rightarrow$ *Stat14*: $g_0 \cap \mathcal{P}y_0 \not\subseteq \{y_0\}$
Use_def($\mathcal{P}y_0$) $\Rightarrow$ AUTO
$\langle z_1 \rangle \hookrightarrow$ *Stat14*(*Stat11*$\star$) $\Rightarrow$ *Stat15*: $z_1 \in \{h : h \subseteq y_0\}$ & $z_1 \in g_0$ & $z_1 \setminus \{x_0\} \neq y_0 \setminus \{x_0\}$
EQUAL $\langle$*Stat10*$\rangle$ $\Rightarrow$ $h_0 \cap \mathcal{P}(y_0 \setminus \{x_0\}) = \{y_0 \setminus \{x_0\}\}$
Suppose $\Rightarrow$ $z_1 \setminus \{x_0\} \notin \mathcal{P}(y_0 \setminus \{x_0\})$
Use_def($\mathcal{P}$) $\Rightarrow$ *Stat16*: $z_1 \setminus \{x_0\} \notin \{h : h \subseteq y_0 \setminus \{x_0\}\}$
$\langle z_1 \setminus \{x_0\} \rangle \hookrightarrow$ *Stat16*(*Stat16*$\star$) $\Rightarrow$ $z_1 \setminus \{x_0\} \not\subseteq y_0 \setminus \{x_0\}$
$\langle h_3 \rangle \hookrightarrow$ *Stat15*(*Stat16*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO
Suppose $\Rightarrow$ *Stat17*: $z_1 \setminus \{x_0\} \notin \{y \setminus \{x_0\} : y \in g_0\}$
$\langle z_1 \rangle \hookrightarrow$ *Stat17*(*Stat15*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ $z_1 \setminus \{x_0\} \in h_0$
(*Stat15*$\star$)Discharge $\Rightarrow$ QED

THEOREM fin_2: [Singletons are finite] Finite($\{X\}$) & Finite($\emptyset$). PROOF:
Suppose_not(x_0) $\Rightarrow$ AUTO
$\langle \{x_0\}, \emptyset \rangle \hookrightarrow T\text{fin}_0$ $\Rightarrow$ $\neg$Finite($\{x_0\}$)
Use_def(Finite) $\Rightarrow$ *Stat1*: $\neg\langle \forall g \in \mathcal{P}(\mathcal{P}\{x_0\}) \setminus \{\emptyset\}, \exists m \mid g \cap \mathcal{P}m = \{m\} \rangle$
$\langle g_0 \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ *Stat2*: $\neg\langle \exists m \mid g_0 \cap \mathcal{P}m = \{m\} \rangle$ & $g_0 \in \mathcal{P}(\mathcal{P}\{x_0\}) \setminus \{\emptyset\}$
Use_def($\mathcal{P}$) $\Rightarrow$ *Stat3*: $g_0 \in \{y : y \subseteq \mathcal{P}\{x_0\}\}$
$\langle x_0 \rangle \hookrightarrow T\text{pow}_2$ $\Rightarrow$ AUTO

$\langle y_0 \rangle \hookrightarrow Stat3(Stat2\star) \Rightarrow \quad Stat4: \; g_0 \neq \emptyset \;\&\; g_0 \subseteq \{\emptyset, \{x_0\}\}$
Suppose $\Rightarrow \quad \emptyset \in g_0$
$\langle \emptyset \rangle \hookrightarrow Stat2(Stat3\star) \Rightarrow$ false; Discharge $\Rightarrow \quad g_0 = \{\{x_0\}\}$
$\langle \{x_0\} \rangle \hookrightarrow Stat2(Stat3\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

THEORY finiteInduction$(s_0, P(S))$
Finite(s_0) & $P(s_0)$
END finiteInduction

ENTER_THEORY finiteInduction

THEOREM finiteInduction$_0$. $\langle \exists m \mid \{s \subseteq s_0 \mid P(s)\} \cap \mathcal{P}m = \{m\} \rangle$. PROOF:
Suppose_not( ) $\Rightarrow$ AUTO
Assump $\Rightarrow$ Finite(s_0) & $P(s_0)$
Use_def(Finite) $\Rightarrow \quad Stat1: \; \langle \forall g \in \mathcal{P}(\mathcal{P}s_0) \setminus \{\emptyset\}, \exists m \mid g \cap \mathcal{P}m = \{m\} \rangle$
$\langle \{s \subseteq s_0 \mid P(s)\} \rangle \hookrightarrow Stat1 \Rightarrow \quad \{s \subseteq s_0 \mid P(s)\} \notin \mathcal{P}(\mathcal{P}s_0) \setminus \{\emptyset\}$
Suppose $\Rightarrow \quad Stat2: \; s_0 \notin \{s \subseteq s_0 \mid P(s)\}$
$\langle s_0 \rangle \hookrightarrow Stat2 \Rightarrow$ false; Discharge $\Rightarrow \quad \{s \subseteq s_0 \mid P(s)\} \notin \mathcal{P}(\mathcal{P}s_0)$
Use_def$(\mathcal{P}) \Rightarrow \quad Stat3: \; \{s \subseteq s_0 \mid P(s)\} \notin \{y : \; y \subseteq \{z : \; z \subseteq s_0\}\}$
$\langle \{s \subseteq s_0 \mid P(s)\} \rangle \hookrightarrow Stat3 \Rightarrow \quad Stat4: \; \{s \subseteq s_0 \mid P(s)\} \not\subseteq \{z : \; z \subseteq s_0\}$
$\langle s_1 \rangle \hookrightarrow Stat4 \Rightarrow \quad Stat5: \; s_1 \in \{s : \; s \subseteq s_0 \mid P(s)\} \;\&\; s_1 \notin \{z : \; z \subseteq s_0\}$
$\langle s, s_1 \rangle \hookrightarrow Stat5(Stat5\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

APPLY $\langle v1_\Theta : \; fin_\Theta \rangle$ Skolem$\Rightarrow$ THEOREM finiteInduction$_1$. $\{s \subseteq s_0 \mid P(s)\} \cap \mathcal{P}fin_\Theta = \{fin_\Theta\}$.

THEOREM finiteInduction$_2$: [Minimal finite set satisfying P] $S \subseteq fin_\Theta \rightarrow$ Finite(S) & $(P(S) \leftrightarrow S = fin_\Theta)$. PROOF:
Suppose_not$(s_1) \Rightarrow$ AUTO
$\langle \rangle \hookrightarrow T$finiteInduction$_1 \Rightarrow \quad \{s \subseteq s_0 \mid P(s)\} \cap \mathcal{P}fin_\Theta = \{fin_\Theta\} \;\&\; Stat1: \; fin_\Theta \in \{s \subseteq s_0 \mid P(s)\}$
$\langle \rangle \hookrightarrow Stat1 \Rightarrow \quad fin_\Theta \subseteq s_0 \;\&\; P(fin_\Theta)$
Assump $\Rightarrow$ Finite(s_0)
$\langle s_0, fin_\Theta \rangle \hookrightarrow T fin_0 \Rightarrow$ Finite(fin_Θ)
$\langle fin_\Theta, s_1 \rangle \hookrightarrow T fin_0 \Rightarrow \quad P(s_1) \neq s_1 = fin_\Theta$
Suppose $\Rightarrow \quad s_1 = fin_\Theta$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow \quad s_1 \notin \{s \subseteq s_0 \mid P(s)\} \cap \mathcal{P}fin_\Theta \;\&\; P(s_1)$
Suppose $\Rightarrow \quad s_1 \notin \mathcal{P}fin_\Theta$
Use_def$(\mathcal{P}) \Rightarrow \quad Stat2: \; s_1 \notin \{y : \; y \subseteq fin_\Theta\}$
$\langle s_1 \rangle \hookrightarrow Stat2 \Rightarrow$ false; Discharge $\Rightarrow \quad Stat3: \; s_1 \notin \{s \subseteq s_0 \mid P(s)\}$
$\langle s_1 \rangle \hookrightarrow Stat3 \Rightarrow$ false; Discharge $\Rightarrow$ QED

ENTER_THEORY Set_theory

DISPLAY finiteInduction

THEORY finiteInduction$(s_0, P(S))$
$\quad$ $\mathsf{Finite}(s_0)$ & $P(s_0)$
$\Rightarrow$ $(\mathsf{fin}_\ominus)$
$\quad$ $\langle \forall S \mid S \subseteq \mathsf{fin}_\ominus \rightarrow \mathsf{Finite}(S)$ & $(P(S) \leftrightarrow S = \mathsf{fin}_\ominus)\rangle$
END finiteInduction

THEOREM fin_3: [Finiteness of the union of two finites sets] $\mathsf{Finite}(X)$ & $\mathsf{Finite}(Y) \rightarrow \mathsf{Finite}(X \cup Y)$. PROOF:
$\quad$ Suppose_not$(x_0, y_1) \Rightarrow$ AUTO

> Arguing by contradiction, suppose that x_0 and y_1 are finite sets whose union is not finite. The finite induction enables us to take a minimal subset y_0 of y_1 for which $x_0 \cup y_0$ is not finite.

APPLY $\langle \mathsf{fin}_\ominus : y_0\rangle$ finiteInduction$(s_0 \mapsto y_1, P(S) \mapsto \neg\mathsf{Finite}(x_0 \cup S)) \Rightarrow$
$\quad$ *Stat1* : $\langle \forall s \mid s \subseteq y_0 \rightarrow \mathsf{Finite}(s)$ & $(\neg\mathsf{Finite}(x_0 \cup s) \leftrightarrow s = y_0)\rangle$
$\langle y_0\rangle \hookrightarrow$ *Stat1*(*Stat1*$\star$) $\Rightarrow$ $\mathsf{Finite}(y_0)$ & $\neg\mathsf{Finite}(x_0 \cup y_0)$
Loc_def $\Rightarrow$ $a_0 = \mathbf{arb}(y_0)$

> Since y_0 cannot be empty, the union $x_0 \cup y_0$ can be decomposed as $x_0 \cup (y_0 \setminus \{\mathbf{arb}(y_0)\}) \cup \{\mathbf{arb}(y_0)\}$, where $x_0 \cup (y_0 \setminus \{\mathbf{arb}(y_0)\})$ is finite by inductive hypothesis. But then $x_0 \cup y_0$ must also be finite by Theorem fin_1

Suppose $\Rightarrow$ $x_0 \cup y_0 = x_0$
EQUAL $\Rightarrow$ false; $\quad$ Discharge $\Rightarrow$ *Stat2* : $y_0 \setminus \{a_0\} \neq y_0$
$\langle y_0 \setminus \{a_0\}\rangle \hookrightarrow$ *Stat1*(*Stat1*$\star$) $\Rightarrow$ $\mathsf{Finite}(x_0 \cup (y_0 \setminus \{a_0\}))$
$\langle x_0 \cup (y_0 \setminus \{a_0\}), a_0\rangle \hookrightarrow$ *T*fin_1(*Stat2*$\star$) $\Rightarrow$ $\mathsf{Finite}(x_0 \cup (y_0 \setminus \{a_0\}) \cup \{a_0\})$
EQUAL $\langle$*Stat1*$\rangle$ $\Rightarrow$ false; $\quad$ Discharge $\Rightarrow$ QED

THEOREM fin_4: [Every nonnull finite set has an inclusion-maximal element] $\mathsf{Finite}(S)$ & $S \neq \emptyset \rightarrow \langle \exists m \in S \mid \{x \in S \setminus \{m\} \mid m \subseteq x\} = \emptyset\rangle$. PROOF:
$\quad$ Suppose_not$(s_1) \Rightarrow$ AUTO

> To start an inductive argument by contradiction, suppose s_2 is an inclusion-minimal counterexample.

APPLY $\langle \mathsf{fin}_\ominus : s_2\rangle$ finiteInduction$(s_0 \mapsto s_1, P(S) \mapsto (S \neq \emptyset$ & $\neg\langle \exists m \in S \mid \{x \in S \setminus \{m\} \mid m \subseteq x\} = \emptyset\rangle)) \Rightarrow$
$\quad$ *Stat1* : $\langle \forall s \mid s \subseteq s_2 \rightarrow \mathsf{Finite}(s)$ & $(s \neq \emptyset$ & $\neg\langle \exists m \in S \mid \{x \in S \setminus \{m\} \mid m \subseteq x\} = \emptyset\rangle \leftrightarrow s = s_2)\rangle$

$\langle s_2\rangle \hookrightarrow$ *Stat1(Stat1⋆)* $\Rightarrow$ *Stat2* : $s_2 \neq \emptyset$ & *Stat3* : $\neg\langle \exists m \in s_2 \mid \{x \in s_2 \setminus \{m\} \mid m \subseteq x\} = \emptyset\rangle$

‖ Draw from this set s_2, where no element is maximal, an element x_2. Since x_2 is not maximal, there is an $x_3 \in s_2$ exceeding it.

$\langle x_2, x_2\rangle \hookrightarrow$ *Stat2(Stat2⋆)* $\Rightarrow$ *Stat4* : $\{x \in s_2 \setminus \{x_2\} \mid x_2 \subseteq x\} \neq \emptyset$ & $x_2 \in s_2$
$\langle x_3\rangle \hookrightarrow$ *Stat4(Stat4⋆)* $\Rightarrow$ $x_3 \in s_2 \setminus \{x_2\}$ & $x_2 \subseteq x_3$

‖ Due to the minimality assumption concerning s_2, there is a maximal element m_0 in $s_2 \setminus \{x_2\}$.

$\langle s_2 \setminus \{x_2\}\rangle \hookrightarrow$ *Stat1(Stat4⋆)* $\Rightarrow$ *Stat5* : $\langle \exists m \in s_2 \setminus \{x_2\} \mid \{x \in s_2 \setminus \{x_2\} \setminus \{m\} \mid m \subseteq x\} = \emptyset\rangle$
$\langle m_0\rangle \hookrightarrow$ *Stat5(Stat5⋆)* $\Rightarrow$ *Stat6* : $\{x \in s_2 \setminus \{x_2\} \setminus \{m_0\} \mid m_0 \subseteq x\} = \emptyset$ & $m_0 \in s_2 \setminus \{x_2\}$

‖ This m_0 is not maximal in s_2; therefore, unlike x_3, it must be included in x_2.

$\langle m_0\rangle \hookrightarrow$ *Stat3(Stat6⋆)* $\Rightarrow$ *Stat7* : $\{x \in s_2 \setminus \{m_0\} \mid m_0 \subseteq x\} \neq \emptyset$ & $\{x \in s_2 \setminus \{x_2\} \setminus \{m_0\} \mid m_0 \subseteq x\} = \emptyset$
$\langle x_4, x_4\rangle \hookrightarrow$ *Stat7(Stat7⋆)* $\Rightarrow$ $m_0 \subseteq x_2$

‖ But then x_3 exceeds m_0, which contradicts the maximality of m_0 in $s_2 \setminus \{x_2\}$.

$\langle x_3\rangle \hookrightarrow$ *Stat6(Stat4⋆)* $\Rightarrow$ false; Discharge $\Rightarrow$ QED

THEORY finiteImage$(s_0, f(X))$
Finite(s_0)
END finiteImage

ENTER_THEORY finiteImage

THEOREM finiteImage. Finite$(\{f(x) : x \in s_0\})$. PROOF:
Suppose_not( ) $\Rightarrow$ AUTO

‖ We can prove the claim by means of finite induction. Arguing by contradiction, let us assume that s_0 has, via the global function $f(X)$, infinite image; then take an s_1 which is finite and minimal (w. r. t. inclusion) and has, much like s_0, infinite image $\{f(x) : x \in s_1\}$. As one sees easily, $s_1 \neq \emptyset$; hence, if we remove an element a from s_1, we find that $\{f(x) : x \in s_1 \setminus \{a\}\}$ is finite in consequence of the supposed minimality of s_1. Since the union of two finite sets is finite, we get the finiteness of $\{f(x) : x \in s_1 \setminus \{a\}\} \cup \{f\}(a)$, which hence must differ from $\{f(x) : x \in s_1\}$.

Assump $\Rightarrow$ $\mathsf{Finite}(s_0)$

APPLY $\langle \mathsf{fin}_\ominus : s_1 \rangle$ finiteInduction$\big(s_0 \mapsto s_0, P(S) \mapsto \neg\mathsf{Finite}(\{f(x) : x \in S\})\big) \Rightarrow$

$Stat1 : \langle \forall s \mid s \subseteq s_1 \rightarrow \mathsf{Finite}(s) \ \& \ \big(\neg\mathsf{Finite}(\{f(x) : x \in s\}) \leftrightarrow s = s_1\big)\rangle$

$\langle s_1 \rangle \hookrightarrow Stat1 \Rightarrow$ $\neg\mathsf{Finite}(\{f(x) : x \in s_1\})$

Loc_def $\Rightarrow$ $Stat0 : a = \mathbf{arb}(s_1)$

$\langle f(a) \rangle \hookrightarrow T\mathsf{fin}_2 \Rightarrow$ $\mathsf{Finite}(\{f\}(a)) \ \& \ \mathsf{Finite}(\emptyset)$

Suppose $\Rightarrow$ $s_1 = \emptyset$

ELEM $\Rightarrow$ $\{f(x) : x \in \emptyset\} = \emptyset$

EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO

$(Stat0)$ELEM $\Rightarrow$ $Stat2 : s_1 \setminus \{a\} \subseteq s_1 \ \& \ s_1 \setminus \{a\} \neq s_1$

Suppose $\Rightarrow$ $\{f(x) : x \in s_1\} = \{f(x) : x \in s_1 \setminus \{a\}\} \cup \{f\}(a)$

$\langle s_1 \setminus \{a\} \rangle \hookrightarrow Stat1(Stat2\star) \Rightarrow$ $\mathsf{Finite}(\{f(x) : x \in s_1 \setminus \{a\}\})$

$\langle \{f(x) : x \in s_1 \setminus \{a\}\}, f(a) \rangle \hookrightarrow T\mathsf{fin}_1(Stat1\star) \Rightarrow$

$\mathsf{Finite}(\{f(x) : x \in s_1 \setminus \{a\}\} \cup \{f\}(a))$

EQUAL $\langle Stat1 \rangle \Rightarrow$ false; Discharge $\Rightarrow$ AUTO

‖ On the other hand, $\{f(x) : x \in s_1 \setminus \{a\}\} \cup \{f\}(a)$ and $\{f(x) : x \in s_1\}$ must be equal: in fact $a \in s_1$, and therefore $f(a) \in \{f(x) : x \in s_1\}$; moreover, by monotonicity, $\{f(x) : x \in s_1 \setminus \{a\}\} \subseteq \{f(x) : x \in s_1\}$ and ...

Set_monot $\Rightarrow$ $\{f(x) : x \in s_1 \setminus \{a\}\} \subseteq \{f(x) : x \in s_1\}$

Suppose $\Rightarrow$ $Stat3 : f(a) \notin \{f(x) : x \in s_1\}$

$\langle a \rangle \hookrightarrow Stat3(Stat2, Stat2\star) \Rightarrow$ false; Discharge $\Rightarrow$ $Stat4 : \{f(x) : x \in s_1\} \not\subseteq \{f(x) : x \in s_1 \setminus \{a\}\} \cup \{f\}(a)$

‖ ... one easily sees that $\{f(x) : x \in s_1\} \subseteq \{f(x) : x \in s_1 \setminus \{a\}\} \cup \{f\}(a)$, ...

$\langle b \rangle \hookrightarrow Stat4(Stat4\star) \Rightarrow$ $Stat5 : b \in \{f(x) : x \in s_1\} \ \& \ b \notin \{f(x) : x \in s_1 \setminus \{a\}\} \cup \{f\}(a)$

$\langle x_0 \rangle \hookrightarrow Stat5(Stat5\star) \Rightarrow$ $f(x_0) \notin \{f(x) : x \in s_1 \setminus \{a\}\} \ \& \ x_0 \in s_1 \ \& \ f(x_0) \neq f(a)$

Suppose $\Rightarrow$ $x_0 = a$

EQUAL $\langle Stat5 \rangle \Rightarrow$ false; Discharge $\Rightarrow$ $Stat6 : f(x_0) \notin \{f(x) : x \in s_1 \setminus \{a\}\} \ \& \ x_0 \neq a \ \& \ x_0 \in s_1$

$\langle x_0 \rangle \hookrightarrow Stat6(Stat6\star) \Rightarrow$ false; -

‖ which leads us to the sought contradiction.

Discharge $\Rightarrow$ QED

ENTER_THEORY Set_theory

DISPLAY finiteImage

THEORY $\mathsf{finiteImage}(s_0, f(X))$
$\quad \mathsf{Finite}(s_0)$
$\Rightarrow$
$\quad \mathsf{Finite}(\{f(x) : x \in s_0\})$
END finiteImage

THEOREM fin_5: [Powersets of finite sets are finite] $\mathsf{Finite}(F) \rightarrow \mathsf{Finite}(\mathcal{P}F)$. PROOF:
$\mathsf{Suppose\_not}(f_1) \Rightarrow$ AUTO

> We can prove the claim by means of finite induction. Arguing by contradiction, let us assume that f_1 is finite but has an infinite powerset; then take an f_0 which is finite and minimal (w. r. t. inclusion) and has, much like f_1, infinite powerset. As one sees easily, $f_0 \neq \emptyset$; hence, if we remove an element a from f_0, we find that $\mathcal{P}(f_0 \setminus \{a\})$ is finite in consequence of the supposed minimality of f_0.

APPLY $\langle \mathsf{fin}_\Theta : f_0 \rangle$ $\mathsf{finiteInduction}(s_0 \mapsto f_1, P(S) \mapsto \neg\mathsf{Finite}(\mathcal{P}S)) \Rightarrow$
$\quad$ *Stat1* : $\langle \forall s \mid s \subseteq f_0 \rightarrow \mathsf{Finite}(s) \;\&\; (\neg\mathsf{Finite}(\mathcal{P}s) \leftrightarrow s = f_0) \rangle$
$\langle f_0 \rangle \hookrightarrow Stat1 \Rightarrow \quad \neg\mathsf{Finite}(\mathcal{P}f_0)$
Suppose $\Rightarrow \quad f_0 = \emptyset$
$\langle \emptyset \rangle \hookrightarrow T\mathsf{fin}_2(Stat1\star) \Rightarrow \quad \mathsf{Finite}(\{\emptyset\})$
Suppose $\Rightarrow \quad \mathcal{P}\emptyset = \{\emptyset\}$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO
$\mathsf{Use\_def}(\mathcal{P}\emptyset) \Rightarrow$ AUTO
$\langle \emptyset, \emptyset \rangle \hookrightarrow T\mathsf{pow}_1(Stat1\star) \Rightarrow \quad Stat2 : \{x : x \subseteq \emptyset\} \not\subseteq \{\emptyset\}$
$\langle x_0 \rangle \hookrightarrow Stat2(Stat2\star) \Rightarrow \quad Stat3 : x_0 \in \{x : x \subseteq \emptyset\} \;\&\; x_0 \neq \emptyset$
$\langle x_1 \rangle \hookrightarrow Stat3(Stat3\star) \Rightarrow$ false; Discharge $\Rightarrow \quad Stat4 : f_0 \neq \emptyset$
$\langle a \rangle \hookrightarrow Stat4(Stat4\star) \Rightarrow \quad a \in f_0$
$\langle f_0 \setminus \{a\} \rangle \hookrightarrow Stat1(Stat4\star) \Rightarrow \quad \mathsf{Finite}(\mathcal{P}(f_0 \setminus \{a\}))$

$\langle f_0, f_0 \setminus \{a\} \rangle \hookrightarrow T\mathsf{pow}_1(Stat4\star) \Rightarrow \quad \mathcal{P}f_0 = \mathcal{P}(f_0 \setminus \{a\}) \cup (\mathcal{P}f_0 \setminus \mathcal{P}(f_0 \setminus \{a\}))$
EQUAL $\langle Stat1 \rangle \Rightarrow \quad \neg\mathsf{Finite}\Big(\mathcal{P}(f_0 \setminus \{a\}) \cup (\mathcal{P}f_0 \setminus \mathcal{P}(f_0 \setminus \{a\}))\Big)$
$\langle \mathcal{P}(f_0 \setminus \{a\}), \mathcal{P}f_0 \setminus \mathcal{P}(f_0 \setminus \{a\}) \rangle \hookrightarrow T\mathsf{fin}_3(Stat1\star) \Rightarrow \quad \neg\mathsf{Finite}(\mathcal{P}f_0 \setminus \mathcal{P}(f_0 \setminus \{a\}))$
Suppose $\Rightarrow \quad Stat5 : \mathcal{P}f_0 \setminus \mathcal{P}(f_0 \setminus \{a\}) \neq \{x \cup \{a\} : x \in \mathcal{P}(f_0 \setminus \{a\})\}$
$\langle b \rangle \hookrightarrow Stat5 \Rightarrow$ AUTO
$\mathsf{Use\_def}(\mathcal{P}f_0) \Rightarrow$ AUTO
$\mathsf{Use\_def}(\mathcal{P}(f_0 \setminus \{a\})) \Rightarrow$ AUTO
Suppose $\Rightarrow \quad Stat6 : b \in \{x \cup \{a\} : x \in \mathcal{P}(f_0 \setminus \{a\})\}$

$\langle x_2 \rangle \hookrightarrow Stat6(Stat5\star) \Rightarrow$ $Stat7: x_2 \in \{y : y \subseteq f_0 \setminus \{a\}\} \ \& \ b = x_2 \cup \{a\}$
$\langle y_2 \rangle \hookrightarrow Stat7(Stat7\star) \Rightarrow$ $x_2 \subseteq f_0 \setminus \{a\}$
$(Stat5\star)$ELEM $\Rightarrow$ $Stat8: b \in \{y : y \subseteq f_0 \setminus \{a\}\} \vee b \notin \{y : y \subseteq f_0\}$
$\langle y_1, b \rangle \hookrightarrow Stat8(Stat4\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO
$(Stat5\star)$ELEM $\Rightarrow$ $Stat9: b \in \{y : y \subseteq f_0\} \ \& \ b \notin \{y : y \subseteq f_0 \setminus \{a\}\} \ \& \ b \notin \{x \cup \{a\} : x \in \mathcal{P}(f_0 \setminus \{a\})\}$
$\langle y_0, y_0, y_0 \setminus \{a\} \rangle \hookrightarrow Stat9(Stat5\star) \Rightarrow$ $Stat10:$
$y_0 \setminus \{a\} \notin \{y : y \subseteq f_0 \setminus \{a\}\} \ \& \ y_0 \subseteq f_0 \ \& \ a \in y_0$
$\langle y_0 \setminus \{a\} \rangle \hookrightarrow Stat10(Stat10\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO
APPLY $\langle\,\rangle$ finiteImage$(s_0 \mapsto \mathcal{P}(f_0 \setminus \{a\}), f(X) \mapsto x \cup \{a\}) \Rightarrow$ $\mathsf{Finite}(\{x \cup \{a\} : x \in \mathcal{P}(f_0 \setminus \{a\})\})$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

THEOREM fin_6: [Sets whose sum-set is finite are finite] $\mathsf{Finite}(\bigcup F) \rightarrow \mathsf{Finite}(F)$. PROOF:
Suppose_not$(f_0) \Rightarrow$ AUTO
Suppose $\Rightarrow$ $f_0 \subseteq \mathcal{P}(\bigcup f_0)$
$\langle \mathcal{P}(\bigcup f_0), f_0 \rangle \hookrightarrow T\mathsf{fin}_0(\star) \Rightarrow$ $\neg \mathsf{Finite}(\mathcal{P}(\bigcup f_0))$
$\langle \bigcup f_0 \rangle \hookrightarrow T\mathsf{fin}_5(\star) \Rightarrow$ false; Discharge $\Rightarrow$ $Stat1: f_0 \nsubseteq \mathcal{P}(\bigcup f_0)$
Use_def$(\mathcal{P}(\bigcup f_0)) \Rightarrow$ AUTO
$\langle x_0 \rangle \hookrightarrow Stat1(Stat1\star) \Rightarrow$ $Stat2: x_0 \notin \{y : y \subseteq \bigcup f_0\} \ \& \ x_0 \in f_0$
$\langle x_0 \rangle \hookrightarrow Stat2(Stat2\star) \Rightarrow$ $x_0 \nsubseteq \bigcup f_0 \ \& \ x_0 \in f_0$
$\langle f_0, x_0, f_0 \rangle \hookrightarrow T\mathsf{un}_2(\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

3 Connected hypergraphs have non-cut vertices

3.1 Sum-set will conveniently represent the operation supplying the set of nodes underlying a hypergraph

DEF $\mathsf{roughCard}_2$: [Having at least two elements] $\mathsf{CardAtLeast2}(E) \quad \leftrightarrow_{\mathrm{Def}} \quad E \nsubseteq \{\mathbf{arb}(E)\}$

DEF hgraph_0: ['nodes' is an alias of sum-set] $\mathsf{nodes}(G) \quad =_{\mathrm{Def}} \quad \bigcup G$

DEF hgraph_1: [An edge to which a given vertex belongs] $\mathsf{edgeOf}(V, G) \quad =_{\mathrm{Def}} \quad \mathbf{arb}(\{e \in G \mid V \in e\})$

|| The following proposition corresponds to Lemma 9 of [CO14].

THEOREM hgraph_a: [Sumset monotonicity] $P \subseteq G \rightarrow \mathsf{nodes}(P) \subseteq \mathsf{nodes}(G)$. PROOF:
Suppose_not$(p_0, g_0) \Rightarrow$ AUTO
Use_def$(\bigcup) \Rightarrow$ $\bigcup p_0 = \{v : e \in p_0, v \in e\} \ \& \ \bigcup g_0 = \{v : e \in g_0, v \in e\}$
Set_monot $\Rightarrow$ $\{v : e \in p_0, v \in e\} \subseteq \{v : e \in g_0, v \in e\}$

Use_def(nodes) $\Rightarrow$ false; Discharge $\Rightarrow$ QED

‖ The following proposition corresponds to Lemma 10 of [CO14].

THEOREM hgraph_b: [Sumset distributes over dyadic union] $\mathsf{nodes}(P \cup Q) = \mathsf{nodes}(P) \cup \mathsf{nodes}(Q)$. PROOF:

Suppose_not$(p_0, q_0) \Rightarrow$ AUTO

$\langle p_0, p_0 \cup q_0 \rangle \hookrightarrow T\mathsf{hgraph}_a \Rightarrow$ AUTO

$\langle q_0, p_0 \cup q_0 \rangle \hookrightarrow T\mathsf{hgraph}_a \Rightarrow$ *Stat1* : $\mathsf{nodes}(p_0 \cup q_0) \nsubseteq \mathsf{nodes}(p_0) \cup \mathsf{nodes}(q_0)$

$\langle v_0 \rangle \hookrightarrow Stat1(Stat1\star) \Rightarrow$ $v_0 \in \mathsf{nodes}(p_0 \cup q_0)\ \&\ v_0 \notin \mathsf{nodes}(p_0) \cup \mathsf{nodes}(q_0)$

Use_def(nodes) $\Rightarrow$ $v_0 \in \bigcup(p_0 \cup q_0)\ \&\ v_0 \notin \bigcup p_0 \cup \bigcup q_0$

Use_def$(\bigcup) \Rightarrow$ *Stat2* : $v_0 \in \{v : e \in p_0 \cup q_0, v \in e\}\ \&\ v_0 \notin \{v : e \in p_0, v \in e\}\ \&\ v_0 \notin \{v : e \in q_0, v \in e\}$

$\langle e_1, v_1, e_1, v_0, e_1, v_0 \rangle \hookrightarrow Stat2(Stat2\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

‖ The following proposition corresponds to Lemma 11 of [CO14].

THEOREM hgraph_c: [Sumset disjointness law for two sets of which one includes the other] $P \subseteq G \rightarrow$
$\left(\mathsf{nodes}(P) \cap \mathsf{nodes}(G \backslash P) = \emptyset \leftrightarrow \langle \forall e \in G \mid \emptyset \in \{\mathsf{nodes}(P) \cap e, \mathsf{nodes}(G \backslash P) \cap e\} \rangle\right)$. PROOF:

Suppose_not$(p_0, g_0) \Rightarrow$ AUTO

Use_def(nodes) $\Rightarrow$ *Stat1* : $p_0 \subseteq g_0\ \&\ \left(\bigcup p_0 \cap \bigcup(g_0 \backslash p_0) = \emptyset \neq \langle \forall e \in g_0 \mid \emptyset \in \{\bigcup p_0 \cap e, \bigcup(g_0 \backslash p_0) \cap e\} \rangle\right)$

‖ Arguing by contradiction, suppose that p_0, g_0 form a counterexample. Recall that 'nodes' simply is an alias for the sum-set operator. Thus either (1) $\bigcup p_0 \cap \bigcup(g_0 \backslash p_0) = \emptyset$ and $\neg\langle \forall e \in g_0 \mid \emptyset \in \{\bigcup p_0 \cap e, \bigcup(g_0 \backslash p_0) \cap e\} \rangle$ hold together; or (2) $\bigcup p_0 \cap \bigcup(g_0 \backslash p_0) \neq \emptyset$ and $\langle \forall e \in g_0 \mid \emptyset \in \{\bigcup p_0 \cap e, \bigcup(g_0 \backslash p_0) \cap e\} \rangle$ hold together. We will exclude both possibilities, arguing as follows.

Suppose $\Rightarrow$ *Stat3* : $\neg\langle \forall e \in g_0 \mid \emptyset \in \{\bigcup p_0 \cap e, \bigcup(g_0 \backslash p_0) \cap e\} \rangle$

‖ Possibility (1) leads to a contradiction: in fact, if $e_0 \in g_0$ is such that $\bigcup p_0 \cap e_0$ and $\bigcup(g_0 \backslash p_0) \cap e_0$ are non-null, then either $e_0 \in p_0$ or $e_0 \in g_0 \backslash p_0$ must hold, but in the former case $e_0 \subseteq \bigcup p_0$ and hence $\bigcup p_0$ and $\bigcup(g_0 \backslash p_0)$ would intersect; likewise, in the latter case $e_0 \subseteq \bigcup(g_0 \backslash p_0)$ and hence $\bigcup p_0$ and $\bigcup(g_0 \backslash p_0)$ intersect again.

$\langle e_0 \rangle \hookrightarrow Stat3(Stat1\star) \Rightarrow$ *Stat4* : $\emptyset \neq \bigcup p_0 \cap e_0\ \&\ \emptyset \neq \bigcup(g_0 \backslash p_0) \cap e_0\ \&\ \bigcup p_0 \cap \bigcup(g_0 \backslash p_0) = \emptyset\ \&\ e_0 \in g_0$

Suppose $\Rightarrow$ $e_0 \in p_0$

$\langle p_0, e_0, p_0 \rangle \hookrightarrow T\mathsf{un}_2(Stat4\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO

$\langle g_0 \backslash p_0, e_0, g_0 \backslash p_0 \rangle \hookrightarrow T\mathsf{un}_2(Stat4\star) \Rightarrow$ false; Discharge $\Rightarrow$ *Stat5* : $\bigcup p_0 \cap \bigcup(g_0 \backslash p_0) \neq \emptyset\ \&$ *Stat6* : $\langle \forall e \in g_0 \mid \emptyset \in \{\bigcup p_0 \cap e, \bigcup(g_0 \backslash p_0) \cap e\} \rangle$

‖ Possibility (2) also leads to a contradiction: in fact, if $e_1 \in p_0$ and $v_1 \in e_1$ are such that $v_1 \in \bigcup(g_0 \backslash p_0)$ then either $\bigcup p_0 \cap e_1$ is null or $\bigcup(g_0 \backslash p_0) \cap e_1$ must be null; but then it is untenable that $v_1 \in \bigcup(g_0 \backslash p_0)$ holds.

$\mathsf{Use\_def}(\bigcup \mathsf{p}_0) \Rightarrow$ AUTO
$\langle \mathsf{v}_1 \rangle \hookrightarrow Stat5(Stat5\star) \Rightarrow \quad Stat7:\ \mathsf{v}_1 \in \{\mathsf{u}:\ \mathsf{e} \in \mathsf{p}_0, \mathsf{u} \in \mathsf{e}\}\ \&\ \mathsf{v}_1 \in \bigcup \mathsf{p}_0 \cap \bigcup (\mathsf{g}_0 \setminus \mathsf{p}_0)$
$\langle \mathsf{e}_1, \mathsf{u}_1 \rangle \hookrightarrow Stat7(Stat7, Stat1\star) \Rightarrow \quad \mathsf{e}_1 \in \mathsf{g}_0\ \&\ \mathsf{v}_1 \in \mathsf{e}_1$

∥ Both possibilities have led us to contradiction; hence the claim of this theorem holds.

$\langle \mathsf{e}_1 \rangle \hookrightarrow Stat6(Stat7\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

THEOREM $\mathsf{hgraph}_\mathsf{d}$: [The nodes of a singleton graph are the elements of its edge] $\mathsf{nodes}(\{\mathsf{E}\}) = \mathsf{E}\ \&\ \mathsf{nodes}(\emptyset) = \emptyset$. PROOF:
$\mathsf{Suppose\_not}(\mathsf{e}_0) \Rightarrow$ AUTO
$\langle \mathsf{e}_0, \{\mathsf{e}_0\} \rangle \hookrightarrow T\mathsf{un}_4 \Rightarrow \quad \bigcup \{\mathsf{e}_0\} = \mathsf{e}_0$
$\langle \emptyset \rangle \hookrightarrow T\mathsf{un}_0(\star) \Rightarrow \quad \bigcup \emptyset = \emptyset$
$\mathsf{Use\_def}(\mathsf{nodes}) \Rightarrow$ false; Discharge $\Rightarrow$ QED

∥ The following proposition corresponds to Lemma 8 of [CO14].

THEOREM $\mathsf{hgraph}_\mathsf{e}$: [Every vertex is brought in by an edge] $\mathsf{V} \in \mathsf{nodes}(\mathsf{G}) \rightarrow \mathsf{V} \in \mathsf{edgeOf}(\mathsf{V},\mathsf{G})\ \&\ \mathsf{edgeOf}(\mathsf{V},\mathsf{G}) \in \mathsf{G}\ \&\ \mathsf{edgeOf}(\mathsf{V},\mathsf{G}) \subseteq \mathsf{nodes}(\mathsf{G})$. PROOF:
$\mathsf{Suppose\_not}(\mathsf{v}_0, \mathsf{g}_0) \Rightarrow$ AUTO
$\mathsf{Use\_def}(\mathsf{nodes}(\mathsf{g}_0)) \Rightarrow$ AUTO
Suppose $\Rightarrow \quad \mathsf{edgeOf}(\mathsf{v}_0, \mathsf{g}_0) \notin \{\mathsf{e} \in \mathsf{g}_0 \mid \mathsf{v}_0 \in \mathsf{e}\}$
$\mathsf{Use\_def}(\mathsf{edgeOf}) \Rightarrow \quad Stat1:\ \{\mathsf{e} \in \mathsf{g}_0 \mid \mathsf{v}_0 \in \mathsf{e}\} = \emptyset$
$\mathsf{Use\_def}(\bigcup) \Rightarrow \quad Stat2:\ \mathsf{v}_0 \in \{\mathsf{v}:\ \mathsf{e} \in \mathsf{g}_0, \mathsf{v} \in \mathsf{e}\}$
$\langle \mathsf{e}_1, \mathsf{v}_1 \rangle \hookrightarrow Stat2(Stat2\star) \Rightarrow \quad \mathsf{e}_1 \in \mathsf{g}_0\ \&\ \mathsf{v}_0 \in \mathsf{e}_1$
$\langle \mathsf{e}_1 \rangle \hookrightarrow Stat1(Stat1\star) \Rightarrow$ false; Discharge $\Rightarrow \quad Stat3:\ \mathsf{edgeOf}(\mathsf{v}_0, \mathsf{g}_0) \in \{\mathsf{e} \in \mathsf{g}_0 \mid \mathsf{v}_0 \in \mathsf{e}\}$
$\langle\ \rangle \hookrightarrow Stat3(\star) \Rightarrow \quad \mathsf{edgeOf}(\mathsf{v}_0, \mathsf{g}_0) \in \mathsf{g}_0\ \&\ \mathsf{v}_0 \in \mathsf{edgeOf}(\mathsf{v}_0, \mathsf{g}_0)\ \&\ \mathsf{edgeOf}(\mathsf{v}_0, \mathsf{g}_0) \not\subseteq \bigcup \mathsf{g}_0$
$\langle \mathsf{g}_0, \mathsf{edgeOf}(\mathsf{v}_0, \mathsf{g}_0), \mathsf{g}_0 \rangle \hookrightarrow T\mathsf{un}_2(Stat3\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

3.2 Hypergraphs viewed as sets of edges

DEF hgraph_2: [Hypergraphs seen as sets of edges] $\quad \mathsf{HGraph}(\mathsf{G}) \quad \leftrightarrow_{\mathrm{Def}} \quad \mathsf{Finite}(\mathsf{nodes}(\mathsf{G}))\ \&\ \langle \forall \mathsf{e} \in \mathsf{G} \mid \mathsf{CardAtLeast2}(\mathsf{e}) \rangle$

THEOREM hgraph_0: [An ur-graph] $\mathsf{HGraph}(\{\{\emptyset, \{\emptyset\}\}\})$. PROOF:
$\mathsf{Suppose\_not}(\,) \Rightarrow$ AUTO
$\mathsf{Use\_def}(\mathsf{HGraph}) \Rightarrow \quad \neg \mathsf{Finite}(\mathsf{nodes}(\{\{\emptyset, \{\emptyset\}\}\})) \vee \neg \langle \forall \mathsf{e} \in \{\{\emptyset, \{\emptyset\}\}\} \mid \mathsf{CardAtLeast2}(\mathsf{e}) \rangle$
$\langle \{\emptyset, \{\emptyset\}\} \rangle \hookrightarrow T\mathsf{hgraph}_\mathsf{d}(\star) \Rightarrow \quad \mathsf{nodes}(\{\{\emptyset, \{\emptyset\}\}\}) = \{\emptyset, \{\emptyset\}\}$
$\langle \{\emptyset\} \rangle \hookrightarrow T\mathsf{fin}_2(\star) \Rightarrow \quad \mathsf{Finite}(\{\{\emptyset\}\})\ \&\ \{\emptyset, \{\emptyset\}\} = \{\{\emptyset\}\} \cup \{\emptyset\}$
$\langle \{\{\emptyset\}\}, \emptyset \rangle \hookrightarrow T\mathsf{fin}_1(\star) \Rightarrow \quad \mathsf{Finite}(\{\{\emptyset\}\} \cup \{\emptyset\})$
$\mathsf{EQUAL} \Rightarrow \quad Stat1:\ \neg \langle \forall \mathsf{e} \in \{\{\emptyset, \{\emptyset\}\}\} \mid \mathsf{CardAtLeast2}(\mathsf{e}) \rangle$

$\langle e_1 \rangle \hookrightarrow Stat1(Stat1\star) \Rightarrow \quad Stat2: \ e_1 = \{\emptyset, \{\emptyset\}\} \ \& \ \neg \mathsf{CardAtLeast2}(e_1)$
$\mathsf{Use\_def}(\mathsf{CardAtLeast2}) \Rightarrow \quad \neg e_1 \not\subseteq \{\mathbf{arb}(e_1)\}$
$(Stat2)\mathsf{Discharge} \Rightarrow \quad$ QED

THEOREM hgraph_1: [Every graph has finitely many edges; and it is null iff its set of nodes is null] $\mathsf{HGraph}(G) \rightarrow \mathsf{Finite}(G) \ \& \ \big(G = \emptyset \leftrightarrow \mathsf{nodes}(G) = \emptyset\big)$. PROOF:
$\mathsf{Suppose\_not}(g_0) \Rightarrow \quad$ AUTO

> A hypothetical counterexample g_0 cannot be null, else it would be finite and its sum-set would be null; but then, unless finite, g_0 would equal $\{\emptyset\}$, which conflicts with the fact that every edge must be at least doubleton.

$\langle \emptyset \rangle \hookrightarrow T\mathsf{fin}_2(\star) \Rightarrow \quad \mathsf{Finite}(\emptyset)$
$\langle g_0 \rangle \hookrightarrow T\mathsf{un}_0(\star) \Rightarrow \quad \bigcup g_0 = \emptyset \leftrightarrow g_0 \subseteq \{\emptyset\}$
$\mathsf{Use\_def}(\mathsf{nodes}) \Rightarrow \quad Stat1: \ \neg \mathsf{Finite}(g_0) \vee \emptyset \in g_0$
$\mathsf{Use\_def}(\mathsf{HGraph}) \Rightarrow \quad Stat2: \ \langle \forall e \in g_0 \mid \mathsf{CardAtLeast2}(e) \rangle \ \& \ \mathsf{Finite}\big(\mathsf{nodes}(g_0)\big)$
$\mathsf{Use\_def}(\mathsf{CardAtLeast2}) \Rightarrow \quad \neg \mathsf{CardAtLeast2}(\emptyset)$
$\langle \emptyset \rangle \hookrightarrow Stat2(Stat2\star) \Rightarrow \quad \emptyset \notin g_0$
$\mathsf{Use\_def}(\mathsf{nodes}) \Rightarrow \quad \mathsf{Finite}(\bigcup g_0)$
$\langle g_0 \rangle \hookrightarrow T\mathsf{fin}_6(Stat1\star) \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

THEOREM hgraph_2: [Every graph edge consists of two or more nodes] $\mathsf{HGraph}(G) \ \& \ E \in G \rightarrow E \subseteq \mathsf{nodes}(G) \ \& \ E \not\subseteq \{V\} \ \& \ \mathsf{CardAtLeast2}(E)$. PROOF:
$\mathsf{Suppose\_not}(g_0, e_0, v_0) \Rightarrow \quad$ AUTO
$\langle g_0, e_0, g_0 \rangle \hookrightarrow T\mathsf{un}_2(\star) \Rightarrow \quad e_0 \subseteq \bigcup g_0$
$\mathsf{Use\_def}\big(\mathsf{nodes}(g_0)\big) \Rightarrow \quad$ AUTO
$\mathsf{Use\_def}(\mathsf{HGraph}) \Rightarrow \quad Stat1: \ \langle \forall e \in g_0 \mid \mathsf{CardAtLeast2}(e) \rangle \ \& \ e_0 \in g_0 \ \& \ \neg\big(e_0 \not\subseteq \{v_0\} \ \& \ \mathsf{CardAtLeast2}(e_0)\big)$
$\langle e_0 \rangle \hookrightarrow Stat1(Stat1\star) \Rightarrow \quad \mathsf{CardAtLeast2}(e_0)$
$\mathsf{Use\_def}(\mathsf{CardAtLeast2}) \Rightarrow \quad \mathsf{CardAtLeast2}(e_0) \leftrightarrow e_0 \not\subseteq \{\mathbf{arb}(e_0)\}$
$(Stat1)\mathsf{Discharge} \Rightarrow \quad$ QED

THEOREM hgraph_3: [Graphs induce subgraphs] $\mathsf{HGraph}(G) \ \& \ P \subseteq G \rightarrow \mathsf{HGraph}(P) \ \& \ \mathsf{nodes}(P) \subseteq \mathsf{nodes}(G)$. PROOF:
$\mathsf{Suppose\_not}(g_0, p_0) \Rightarrow \quad$ AUTO
$\mathsf{Set\_monot} \Rightarrow \quad \langle \forall e \in g_0 \mid \mathsf{CardAtLeast2}(e) \rangle \rightarrow \langle \forall e \in p_0 \mid \mathsf{CardAtLeast2}(e) \rangle$
$\mathsf{Use\_def}(\mathsf{HGraph}) \Rightarrow \quad Stat1: \ \mathsf{nodes}(p_0) \not\subseteq \mathsf{nodes}(g_0) \ \& \ g_0 = g_0 \cup p_0$
$\mathsf{EQUAL} \ \langle Stat1 \rangle \Rightarrow \quad \mathsf{nodes}(p_0) \not\subseteq \mathsf{nodes}(g_0 \cup p_0)$
$\langle g_0, p_0 \rangle \hookrightarrow T\mathsf{hgraph}_b(Stat1\star) \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

3.3 Edges incident to a vertex or to a set of vertices

DEF hgraph$_3$: [Edges incident to a given node] $\mathsf{contains}(G, V) =_{\text{Def}} \{e \in G \mid V \in e\}$

DEF hgraph$_4$: [Edges touching a given set] $\mathsf{cov}(G, P) =_{\text{Def}} \{e \in G \mid e \cap \mathsf{nodes}(P) \neq \emptyset\}$

‖ The following proposition roughly corresponds to Lemma 12 of [CO14].

THEOREM hgraph$_4$. $\mathsf{HGraph}(P) \rightarrow \mathsf{cov}(G, P) = \{a : e \in P, v \in e, a \in \mathsf{contains}(G, v)\}$. PROOF:

Suppose_not$(p_0, g_0) \Rightarrow$ AUTO

Use_def(cov) $\Rightarrow$ *Stat1*: $\{e \in g_0 \mid e \cap \mathsf{nodes}(p_0) \neq \emptyset\} \neq \{a : e \in p_0, v \in e, a \in \mathsf{contains}(g_0, v)\}$

$\langle e_0 \rangle \hookrightarrow$ *Stat1*(*Stat1*$\star$) $\Rightarrow$ *Stat2*: $e_0 \in \{e \in g_0 \mid e \cap \mathsf{nodes}(p_0) \neq \emptyset\} \neq e_0 \in \{a : e \in p_0, v \in e, a \in \mathsf{contains}(g_0, v)\}$

Suppose $\Rightarrow$ *Stat3*: $e_0 \in \{a : e \in p_0, v \in e, a \in \mathsf{contains}(g_0, v)\}$

$\langle e_1, v_1, a_1 \rangle \hookrightarrow$ *Stat3*(*Stat2*$\star$) $\Rightarrow$ *Stat4*:

$e_0 \notin \{e \in g_0 \mid e \cap \mathsf{nodes}(p_0) \neq \emptyset\} \;\&\; e_1 \in p_0 \;\&\; v_1 \in e_1 \;\&\; e_0 \in \mathsf{contains}(g_0, v_1)$

$\langle p_0, e_1, v_1 \rangle \hookrightarrow$ *T*hgraph$_2(\star) \Rightarrow$ $v_1 \in \mathsf{nodes}(p_0)$

Use_def$(\mathsf{contains}(g_0, v_1)) \Rightarrow$ AUTO

$\langle e_0 \rangle \hookrightarrow$ *Stat4*(*Stat4*$\star$) $\Rightarrow$ *Stat5*: $e_0 \in \{e \in g_0 \mid v_1 \in e\} \;\&\; (e_0 \in g_0 \rightarrow v_1 \notin e_0)$

$\langle\, \rangle \hookrightarrow$ *Stat5*(*Stat5*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat6*: $e_0 \in \{e \in g_0 \mid e \cap \mathsf{nodes}(p_0) \neq \emptyset\} \;\&\; e_0 \notin \{a : e \in p_0, v \in e, a \in \mathsf{contains}(g_0, v)\}$

$\langle\, \rangle \hookrightarrow$ *Stat6*(*Stat6*$\star$) $\Rightarrow$ *Stat7*: $e_0 \cap \mathsf{nodes}(p_0) \neq \emptyset \;\&\; e_0 \notin \{a : e \in p_0, v \in e, a \in \mathsf{contains}(g_0, v)\} \;\&\; e_0 \in g_0$

$\langle v_2, p_0 \rangle \hookrightarrow$ *T*hgraph$_e \Rightarrow$ AUTO

Use_def$(\mathsf{contains}(g_0, v_2)) \Rightarrow$ AUTO

$\langle v_2, \mathsf{edgeOf}(v_2, p_0), v_2, e_0 \rangle \hookrightarrow$ *Stat7*(*Stat7*$\star$) $\Rightarrow$ *Stat8*: $e_0 \notin \{e \in g_0 \mid v_2 \in e\} \;\&\; v_2 \in e_0$

$\langle e_0 \rangle \hookrightarrow$ *Stat8*(*Stat8*, *Stat7*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ QED

THEOREM hgraph$_5$. $\mathsf{HGraph}(G) \rightarrow \mathsf{cov}(G, Q) \supseteq Q \cap G$. PROOF:

Suppose_not$(g_0, q_0) \Rightarrow$ AUTO

Use_def(cov) $\Rightarrow$ *Stat1*: $\{e \in g_0 \mid e \cap \mathsf{nodes}(q_0) \neq \emptyset\} \not\supseteq q_0 \cap g_0 \;\&\; \mathsf{HGraph}(g_0)$

$\langle e_0 \rangle \hookrightarrow$ *Stat1*(*Stat1*$\star$) $\Rightarrow$ *Stat2*: $e_0 \notin \{e \in g_0 \mid e \cap \mathsf{nodes}(q_0) \neq \emptyset\} \;\&\; e_0 \in q_0 \cap g_0$

$\langle g_0, e_0 \rangle \hookrightarrow$ Thgraph$_2$(*Stat1*$\star$) $\Rightarrow$ $e_0 \subseteq \mathsf{nodes}(g_0) \;\&\; e_0 \neq \emptyset$

$\langle e_0 \rangle \hookrightarrow$ *Stat2*(*Stat2*$\star$) $\Rightarrow$ *Stat3*: $e_0 \cap \mathsf{nodes}(q_0) = \emptyset \;\&\; e_0 \in q_0$

$\langle \{e_0\}, q_0 \rangle \hookrightarrow$ *T*hgraph$_a$(*Stat3*$\star$) $\Rightarrow$ $e_0 \cap \mathsf{nodes}(\{e_0\}) = \emptyset$

$\langle e_0 \rangle \hookrightarrow$ *T*hgraph$_d$(*Stat2*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ QED

‖ The following proposition corresponds to Lemma 13 of [CO14].

THEOREM hgraph$_6$. $\mathsf{cov}(G, P) \subseteq P \leftrightarrow \mathsf{nodes}(G \backslash P) \cap \mathsf{nodes}(P) = \emptyset$. PROOF:

Suppose_not$(g_0, p_0) \Rightarrow$ AUTO

Use_def$(\mathsf{cov}(g_0, p_0)) \Rightarrow$ AUTO

Use_def(nodes) $\Rightarrow$ *Stat0*: $\{e \in g_0 \mid e \cap \bigcup p_0 \neq \emptyset\} \subseteq p_0 \neq \bigcup(g_0 \backslash p_0) \cap \bigcup p_0 = \emptyset$

Suppose ⇒ *Stat1* : $\bigcup(g_0 \backslash p_0) \cap \bigcup p_0 \neq \emptyset \;\&\; \{e \in g_0 \mid e \cap \bigcup p_0 \neq \emptyset\} \subseteq p_0$
Use_def$(\bigcup(g_0\backslash p_0))$ ⇒ AUTO
$\langle v_0 \rangle \hookrightarrow$ *Stat1(Stat1⋆)* ⇒ *Stat2* : $v_0 \in \{u : v \in g_0\backslash p_0, u \in v\} \;\&\; v_0 \in \bigcup p_0$
$\langle e_0, v_1 \rangle \hookrightarrow$ *Stat2(Stat1⋆)* ⇒ *Stat3* : $e_0 \notin \{e \in g_0 \mid e \cap \bigcup p_0 \neq \emptyset\} \;\&\; e_0 \in g_0 \;\&\; v_0 \in e_0$
$\langle e_0 \rangle \hookrightarrow$ *Stat3(Stat2⋆)* ⇒ false; Discharge ⇒ *Stat4* : $\{e \in g_0 \mid e \cap \bigcup p_0 \neq \emptyset\} \not\subseteq p_0 \;\&\; \bigcup(g_0\backslash p_0) \cap \bigcup p_0 = \emptyset$
$\langle e_1 \rangle \hookrightarrow$ *Stat4(Stat4⋆)* ⇒ *Stat5* : $e_1 \in \{e \in g_0 \mid e \cap \bigcup p_0 \neq \emptyset\} \;\&\; e_1 \notin p_0$
$\langle\,\rangle \hookrightarrow$ *Stat5(Stat5⋆)* ⇒ $e_1 \in g_0 \;\&\; e_1 \cap \bigcup p_0 \neq \emptyset$
$\langle g_0\backslash p_0, e_1, g_0\backslash p_0 \rangle \hookrightarrow Tun_2$*(Stat4⋆)* ⇒ false; Discharge ⇒ QED

‖ The following proposition corresponds to Lemma 14 of [CO14], which it slightly generalizes.

THEOREM hgraph$_7$. $\mathsf{nodes}(G) \supseteq \mathsf{nodes}(F) \;\&\; \mathsf{HGraph}(F) \;\&\; F \supseteq P \;\&\; P \neq \emptyset \to \mathsf{cov}(G, P) \neq \emptyset$. PROOF:
Suppose_not(g_0, f_0, p_0) ⇒ *Stat0* : $p_0 \neq \emptyset \;\&\; f_0 \supseteq p_0 \;\&\; \mathsf{HGraph}(f_0) \;\&\; \mathsf{nodes}(g_0) \supseteq \mathsf{nodes}(f_0) \;\&\; \mathsf{cov}(g_0, p_0) = \emptyset$
$\langle f_0, p_0 \rangle \hookrightarrow$ *Thgraph$_3$(⋆)* ⇒ $\mathsf{HGraph}(p_0) \;\&\; \mathsf{nodes}(p_0) \subseteq \mathsf{nodes}(f_0)$
$\langle e_0 \rangle \hookrightarrow$ *Stat0(⋆)* ⇒ $e_0 \in p_0$
$\langle p_0, e_0 \rangle \hookrightarrow$ *Thgraph$_2$(⋆)* ⇒ *Stat1* : $e_0 \neq \emptyset \;\&\; e_0 \subseteq \mathsf{nodes}(p_0) \;\&\; e_0 \subseteq \mathsf{nodes}(f_0)$
$\langle v_0 \rangle \hookrightarrow$ *Stat1(⋆)* ⇒ $v_0 \in \mathsf{nodes}(p_0) \;\&\; v_0 \in \mathsf{nodes}(g_0)$
$\langle v_0, g_0 \rangle \hookrightarrow$ *Thgraph$_e$(Stat1⋆)* ⇒ $v_0 \in \mathsf{edgeOf}(v_0, g_0) \;\&\; \mathsf{edgeOf}(v_0, g_0) \in g_0$
Use_def(cov) ⇒ *Stat2* : $\{e \in g_0 \mid e \cap \mathsf{nodes}(p_0) \neq \emptyset\} = \emptyset$
$\langle \mathsf{edgeOf}(v_0, g_0) \rangle \hookrightarrow$ *Stat2(Stat1⋆)* ⇒ false; Discharge ⇒ QED

3.4 Vertex removal

DEF hgraph$_5$: [Filtering out of a graph the edges incident to a fixed vertex] $\mathsf{filter}(G, V) =_{\mathrm{Def}} \{e \backslash \{V\} : e \in G \mid \mathsf{CardAtLeast2}(e \backslash \{V\})\}$

‖ The following proposition roughly corresponds to Lemma 15 of [CO14], which it slightly enriches.

THEOREM hgraph$_8$: [Filtering transforms graphs into graphs and eliminates at least the operand vertex]
$\mathsf{contains}(G, V) \subseteq G \;\&\; \mathsf{nodes}(\mathsf{filter}(G, V)) \subseteq \mathsf{nodes}(G) \backslash \{V\} \;\&\; \big(\mathsf{HGraph}(G) \to \mathsf{HGraph}(\mathsf{filter}(G, V))\big)$. PROOF:
Suppose_not(g_0, v_0) ⇒ AUTO
Suppose ⇒ $\mathsf{contains}(g_0, v_0) \not\subseteq g_0$
Set_monot ⇒ $\{e \in g_0 \mid v_0 \in e\} \subseteq \{e : e \in g_0\}$
Use_def$(\mathsf{contains}(g_0, v_0))$ ⇒ AUTO
Discharge ⇒ AUTO
Suppose ⇒ $\mathsf{nodes}(\mathsf{filter}(g_0, v_0)) \not\subseteq \mathsf{nodes}(g_0) \backslash \{v_0\}$
Use_def(nodes) ⇒ $\bigcup \mathsf{filter}(g_0, v_0) \not\subseteq \bigcup g_0 \backslash \{v_0\}$
Use_def(filter) ⇒ *Stat1* : $\bigcup \{e \backslash \{v_0\} : e \in g_0 \mid \mathsf{CardAtLeast2}(e \backslash \{v_0\})\} \not\subseteq \bigcup g_0 \backslash \{v_0\}$
Use_def$(\bigcup \{e \backslash \{v_0\} : e \in g_0 \mid \mathsf{CardAtLeast2}(e \backslash \{v_0\})\})$ ⇒ AUTO

$\langle v_1 \rangle \hookrightarrow Stat1(Stat1\star) \Rightarrow$ $Stat2: \; v_1 \in \{u: \; v \in \{e \setminus \{v_0\}: \; e \in g_0 \mid \mathsf{CardAtLeast2}(e \setminus \{v_0\})\}, u \in v\} \; \& \; v_1 \notin \bigcup g_0 \setminus \{v_0\}$
SIMPLF $\langle Stat2\star \rangle \Rightarrow$ $Stat3: \; v_1 \in \{u: \; e \in g_0, u \in e \setminus \{v_0\} \mid \mathsf{CardAtLeast2}(e \setminus \{v_0\})\}$
Use_def($\bigcup g_0$) $\Rightarrow$ AUTO
$\langle e_1, u_1 \rangle \hookrightarrow Stat3(Stat2\star) \Rightarrow$ $Stat4: \; v_1 \notin \{u: \; e \in g_0, u \in e\} \; \& \; e_1 \in g_0 \; \& \; v_1 \in e_1$
$\langle e_1, v_1 \rangle \hookrightarrow Stat4(Stat4\star) \Rightarrow$ false; Discharge $\Rightarrow$ $Stat5: \; \mathsf{nodes}(\mathsf{filter}(g_0, v_0)) \subseteq \mathsf{nodes}(g_0) \; \& \; \mathsf{HGraph}(g_0) \; \& \; \neg \mathsf{HGraph}(\mathsf{filter}(g_0, v_0))$
$\langle g_0 \rangle \hookrightarrow T\mathsf{hgraph}_1 \Rightarrow$ AUTO
Use_def(HGraph) $\Rightarrow$ $\mathsf{Finite}(\mathsf{nodes}(g_0)) \; \& \; \langle \forall e \in g_0 \mid \mathsf{CardAtLeast2}(e) \rangle \; \& \; \neg \Big(\mathsf{Finite}\big(\mathsf{nodes}(\mathsf{filter}(g_0, v_0))\big) \; \& \; \langle \forall e \in \mathsf{filter}(g_0, v_0) \mid \mathsf{CardAtLeast2}(e) \rangle \Big)$
Suppose $\Rightarrow$ $\neg \mathsf{Finite}(\mathsf{filter}(g_0, v_0))$
Use_def(filter) $\Rightarrow$ $\neg \mathsf{Finite}(\{e \setminus \{v_0\}: \; e \in g_0 \mid \mathsf{CardAtLeast2}(e \setminus \{v_0\})\})$
Set_monot $\Rightarrow$ $\{e \setminus \{v_0\}: \; e \in g_0 \mid \mathsf{CardAtLeast2}(e \setminus \{v_0\})\} \subseteq \{e \setminus \{v_0\}: \; e \in g_0\}$
APPLY $\langle \rangle$ finiteImage($s_0 \mapsto g_0, f(X) \mapsto X \setminus \{v_0\}$) $\Rightarrow$ $\mathsf{Finite}(\{e \setminus \{v_0\}: \; e \in g_0\})$
($Stat5$)ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ $\neg \langle \forall e \in \mathsf{filter}(g_0, v_0) \mid \mathsf{CardAtLeast2}(e) \rangle$
Use_def(filter) $\Rightarrow$ $Stat6: \; \neg \langle \forall a \in \{e \setminus \{v_0\}: \; e \in g_0 \mid \mathsf{CardAtLeast2}(e \setminus \{v_0\})\} \mid \mathsf{CardAtLeast2}(a) \rangle$
$\langle e_0 \rangle \hookrightarrow Stat6(Stat6\star) \Rightarrow$ $Stat7: \; e_0 \in \{e \setminus \{v_0\}: \; e \in g_0 \mid \mathsf{CardAtLeast2}(e \setminus \{v_0\})\} \; \& \; \neg \mathsf{CardAtLeast2}(e_0)$
$\langle e \rangle \hookrightarrow Stat7 \Rightarrow$ AUTO
EQUAL $\langle Stat7 \rangle \Rightarrow$ false; Discharge $\Rightarrow$ QED

‖ The following proposition corresponds to Lemma 16 of [CO14].

THEOREM hgraph_9: [Covering splitting rule] $\mathsf{HGraph}(G) \; \& \; \mathsf{nodes}(\mathsf{filter}(G, V)) = \mathsf{nodes}(G) \setminus \{V\} \; \& \; P \subseteq \mathsf{filter}(G, V) \rightarrow \mathsf{cov}(G, P) \cup \mathsf{cov}(G, \mathsf{filter}(G, V) \setminus P) = G$. PROOF:
Suppose_not(g_0, v_0, p_0) $\Rightarrow$ AUTO
Use_def(cov) $\Rightarrow$ $Stat1: \; \{e \in g_0 \mid e \cap \mathsf{nodes}(p_0) \neq \emptyset\} \cup \{e \in g_0 \mid e \cap \mathsf{nodes}(\mathsf{filter}(g_0, v_0) \setminus p_0) \neq \emptyset\} \neq g_0$
Set_monot $\Rightarrow$ $\{e \in g_0 \mid e \cap \mathsf{nodes}(p_0) \neq \emptyset\} \subseteq \{e \in g_0 \mid \mathsf{true}\}$
Set_monot $\Rightarrow$ $\{e \in g_0 \mid e \cap \mathsf{nodes}(\mathsf{filter}(g_0, v_0) \setminus p_0) \neq \emptyset\} \subseteq \{e \in g_0 \mid \mathsf{true}\}$
TELEM $\Rightarrow$ $\{e \in g_0 \mid \mathsf{true}\} = g_0$
$\langle e_0 \rangle \hookrightarrow Stat1(Stat1\star) \Rightarrow$ $Stat2: \; e_0 \notin \{e \in g_0 \mid e \cap \mathsf{nodes}(\mathsf{filter}(g_0, v_0) \setminus p_0) \neq \emptyset\} \; \& \; e_0 \notin \{e \in g_0 \mid e \cap \mathsf{nodes}(p_0) \neq \emptyset\} \; \& \; e_0 \in g_0$
$\langle e_0, e_0 \rangle \hookrightarrow Stat2(Stat2\star) \Rightarrow$ $e_0 \cap \Big(\mathsf{nodes}(\mathsf{filter}(g_0, v_0) \setminus p_0) \cup \mathsf{nodes}(p_0)\Big) = \emptyset$
EQUAL $\Rightarrow$ $Stat3: \; \mathsf{filter}(g_0, v_0) \setminus p_0 \cup p_0 = \mathsf{filter}(g_0, v_0)$
EQUAL $\langle Stat3 \rangle \Rightarrow$ $\mathsf{nodes}(\mathsf{filter}(g_0, v_0) \setminus p_0 \cup p_0) = \mathsf{nodes}(\mathsf{filter}(g_0, v_0))$
$\langle p_0, \mathsf{filter}(g_0, v_0) \setminus p_0 \rangle \hookrightarrow T\mathsf{hgraph}_b(Stat2\star) \Rightarrow$ $Stat4: \; e_0 \cap \mathsf{nodes}(\mathsf{filter}(g_0, v_0)) = \emptyset$
$\langle g_0, e_0, v_0 \rangle \hookrightarrow T\mathsf{hgraph}_2(\star) \Rightarrow$ $e_0 \subseteq \mathsf{nodes}(g_0) \; \& \; e_0 \not\subseteq \{v_0\} \; \& \; \mathsf{nodes}(\mathsf{filter}(g_0, v_0)) \cup \{v_0\} = \mathsf{nodes}(g_0)$
$\langle g_0, v_0 \rangle \hookrightarrow T\mathsf{hgraph}_8(Stat4\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

‖ The following proposition corresponds to Lemma 6 of [CO14].

THEOREM hgraph_{10}: [Basic promoter of the equivalence between connectivity and path connectivity]
$\mathsf{HGraph}(G) \; \& \; P \subseteq \mathsf{filter}(G, V) \; \& \; \mathsf{nodes}(P) \cap \mathsf{nodes}(\mathsf{filter}(G, V) \setminus P) = \emptyset \rightarrow \langle \forall e \in G \mid \emptyset \in \{\mathsf{nodes}(P) \cap e, \mathsf{nodes}(\mathsf{filter}(G, V) \setminus P) \cap e\} \rangle$. PROOF:
Suppose_not(g_0, p_0, v_0) $\Rightarrow$ AUTO

ELEM $\Rightarrow$ *Stat0* : $\neg\langle\forall e \in g_0 \mid \emptyset \in \{\mathsf{nodes}(p_0) \cap e, \mathsf{nodes}(\mathsf{filter}(g_0, v_0)\backslash p_0) \cap e\}\rangle$
Use_def(filter(g_0, v_0)) $\Rightarrow$ AUTO
$\langle e_0\rangle\hookrightarrow$*Stat0*($\star$) $\Rightarrow$ *Stat1* : $\emptyset \neq \mathsf{nodes}(\mathsf{filter}(g_0, v_0)\backslash p_0) \cap e_0$ & $\emptyset \neq \mathsf{nodes}(p_0) \cap e_0$ & $e_0 \in g_0$ & $p_0 \subseteq \mathsf{filter}(g_0, v_0)$ & $\mathsf{nodes}(p_0) \cap \mathsf{nodes}(\mathsf{filter}(g_0, v_0)\backslash p_0) = \emptyset$ & $\mathsf{HGraph}(g_0)$
Use_def(HGraph) $\Rightarrow$ *Stat2* : $\langle\forall e \in g_0 \mid \mathsf{CardAtLeast2}(e)\rangle$
$\langle p_0, \mathsf{filter}(g_0, v_0)\rangle\hookrightarrow$*Thgraph*$_c$(*Stat0*$\star$) $\Rightarrow$ *Stat3* : $\langle\forall e \in \mathsf{filter}(g_0, v_0) \mid \emptyset \in \{\mathsf{nodes}(p_0) \cap e, \mathsf{nodes}(\mathsf{filter}(g_0, v_0)\backslash p_0) \cap e\}\rangle$
$\langle g_0, v_0\rangle\hookrightarrow$*Thgraph*$_8$ $\Rightarrow$ AUTO
Suppose $\Rightarrow$ $e_0 \backslash \{v_0\} = e_0$
$\langle e_0\rangle\hookrightarrow$*Stat2*(*Stat1*$\star$) $\Rightarrow$ $\mathsf{CardAtLeast2}(e_0)$
EQUAL $\Rightarrow$ $\mathsf{CardAtLeast2}(e_0 \backslash \{v_0\})$
Suppose $\Rightarrow$ *Stat4* : $e_0 \notin \{e\backslash \{v_0\} : e \in g_0 \mid \mathsf{CardAtLeast2}(e\backslash \{v_0\})\}$
$\langle e_0\rangle\hookrightarrow$*Stat4*(*Stat1*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ $e_0 \in \mathsf{filter}(g_0, v_0)$
$\langle e_0\rangle\hookrightarrow$*Stat3*(*Stat1*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO
$\langle v_1, v_2\rangle\hookrightarrow$*Stat1*(*Stat1*, *Stat1*$\star$) $\Rightarrow$ *Stat5* : $v_1 \in \mathsf{nodes}(\mathsf{filter}(g_0, v_0)\backslash p_0) \cap e_0$ & $v_2 \in \mathsf{nodes}(p_0) \cap e_0$
$\langle p_0, \mathsf{filter}(g_0, v_0)\rangle\hookrightarrow$*Thgraph*$_a$(*Stat1*$\star$) $\Rightarrow$ $v_0 \notin \mathsf{nodes}(p_0)$
$\langle \mathsf{filter}(g_0, v_0)\backslash p_0, \mathsf{filter}(g_0, v_0)\rangle\hookrightarrow$*Thgraph*$_a$(*Stat3*$\star$) $\Rightarrow$ $v_0 \neq v_1$ & $v_0 \neq v_2$
Loc_def $\Rightarrow$ $e_1 = e_0 \backslash \{v_0\}$
Suppose $\Rightarrow$ $e_1 \notin \mathsf{filter}(g_0, v_0)$
Use_def(CardAtLeast2) $\Rightarrow$ $\mathsf{CardAtLeast2}(e_0 \backslash \{v_0\})$
Use_def(filter) $\Rightarrow$ *Stat6* : $e_1 \notin \{e\backslash \{v_0\} : e \in g_0 \mid \mathsf{CardAtLeast2}(e\backslash \{v_0\})\}$
$\langle e_0\rangle\hookrightarrow$*Stat6*(*Stat1*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO
$\langle e_1\rangle\hookrightarrow$*Stat3*(*Stat5*$\star$) $\Rightarrow$ *Stat7* : $\mathsf{nodes}(p_0) \cap \mathsf{nodes}(\mathsf{filter}(g_0, v_0)\backslash p_0) \neq \emptyset$
(*Stat1*, *Stat7*$\star$)Discharge $\Rightarrow$ QED

‖ The following proposition corresponds to Lemma 17 of [CO14].

THEOREM hgraph$_{11}$: [Variant of the basic promoter of the equivalence between connectivity and path connectivity]
$\mathsf{HGraph}(G)$ & $P \subseteq \mathsf{filter}(G, V)$ & $\mathsf{nodes}(P) \cap \mathsf{nodes}(\mathsf{filter}(G, V)\backslash P) = \emptyset \rightarrow \mathsf{cov}(G, P) \cap \mathsf{cov}(G, \mathsf{filter}(G, V)\backslash P) = \emptyset$. PROOF:
Suppose_not(g_0, p_0, v_0) $\Rightarrow$ AUTO
Use_def(cov) $\Rightarrow$ *Stat1* : $\{e \in g_0 \mid e \cap \mathsf{nodes}(p_0) \neq \emptyset\} \cap \{e \in g_0 \mid e \cap \mathsf{nodes}(\mathsf{filter}(g_0, v_0)\backslash p_0) \neq \emptyset\} \neq \emptyset$
$\langle g_0, p_0, v_0\rangle\hookrightarrow$*Thgraph*$_{10}$ $\Rightarrow$ AUTO
$\langle e_0\rangle\hookrightarrow$*Stat1*($\star$) $\Rightarrow$ *Stat2* :
$e_0 \in \{e : e \in g_0 \mid e \cap \mathsf{nodes}(p_0) \neq \emptyset\}$ & $e_0 \in \{e : e \in g_0 \mid e \cap \mathsf{nodes}(\mathsf{filter}(g_0, v_0)\backslash p_0) \neq \emptyset\}$ & $\langle\forall e \in g_0 \mid \emptyset \in \{\mathsf{nodes}(p_0) \cap e, \mathsf{nodes}(\mathsf{filter}(g_0, v_0)\backslash p_0) \cap e\}\rangle$
$\langle e_1, e_2, e_0\rangle\hookrightarrow$*Stat2*(*Stat2*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ QED

3.5 Connectivity

DEF hgraph$_6$: [Connectivity] $\mathsf{Conn}(G)$ $\leftrightarrow_{\mathrm{Def}}$ $\mathsf{HGraph}(G)$ & $\{p \subseteq G \mid \mathsf{nodes}(p) \cap \mathsf{nodes}(G\backslash p) = \emptyset\} \subseteq \{\emptyset, G\}$

DEF hgraph$_7$: [Nodes of a graph that get 'accidentally' lost in consequence of filtering] $\mathsf{lost}(G,V) \;=_{\text{Def}}\; \mathsf{nodes}(G)\backslash\big(\mathsf{nodes}(\mathsf{filter}(G,V)) \cup \{V\}\big)$

‖ The following proposition corresponds to Lemma 7 of [CO14].

THEOREM hgraph$_{12}$: [Covering disjointness rule] $\mathsf{Conn}(G)\ \&\ P \subseteq \mathsf{filter}(G,V)\ \&\ P \notin \{\emptyset, \mathsf{filter}(G,V)\}\ \&\ V \in \mathsf{nodes}(G)\ \&\ \mathsf{nodes}(P) \cap \mathsf{nodes}(\mathsf{filter}(G,V)\backslash P) = \emptyset\ \&\ \mathsf{lost}(G,V) = \emptyset \rightarrow$
$\mathsf{nodes}(\mathsf{cov}(G,P)) \cap \mathsf{nodes}\big(\mathsf{cov}(G, \mathsf{filter}(G,V)\backslash P)\big) = \{V\}$. PROOF:

Suppose_not$(g_0, p_0, v_0) \Rightarrow$ AUTO
Use_def(Conn) $\Rightarrow\ \{p \subseteq g_0 \mid \mathsf{nodes}(p) \cap \mathsf{nodes}(g_0\backslash p) = \emptyset\} \subseteq \{\emptyset, g_0\}\ \&\ \mathsf{HGraph}(g_0)$
$\langle g_0, v_0\rangle \hookrightarrow Thgraph_8 \Rightarrow$ *Stat0*: AUTO
Use_def(lost) $\Rightarrow$ *Stat1*: $\mathsf{nodes}(\mathsf{filter}(g_0,v_0)) = \mathsf{nodes}(g_0)\backslash\{v_0\}\ \&\ v_0 \in \mathsf{nodes}(g_0)$
$\langle g_0, v_0, p_0\rangle \hookrightarrow Thgraph_9(\star) \Rightarrow\ \mathsf{cov}(g_0,p_0) \cup \mathsf{cov}(g_0, \mathsf{filter}(g_0,v_0)\backslash p_0) = g_0$
$\langle g_0, p_0, v_0\rangle \hookrightarrow Thgraph_{11}(\star) \Rightarrow$ *Stat2*: $\mathsf{cov}(g_0, \mathsf{filter}(g_0,v_0)\backslash p_0) = g_0\backslash \mathsf{cov}(g_0,p_0)$
Suppose $\Rightarrow\ \mathsf{nodes}\big(\mathsf{cov}(g_0, \mathsf{filter}(g_0,v_0)\backslash p_0)\big) \cap \mathsf{nodes}(\mathsf{cov}(g_0,p_0)) = \emptyset$
EQUAL $\langle$*Stat2*$\rangle \Rightarrow\ \mathsf{nodes}(g_0\backslash\mathsf{cov}(g_0,p_0)) \cap \mathsf{nodes}(\mathsf{cov}(g_0,p_0)) = \emptyset$
Suppose $\Rightarrow$ *Stat3*: $\mathsf{cov}(g_0,p_0) \notin \{p \subseteq g_0 \mid \mathsf{nodes}(p) \cap \mathsf{nodes}(g_0\backslash p) = \emptyset\}$
$\langle g_0, p_0\rangle \hookrightarrow Thgraph_6 \Rightarrow$ AUTO
$\langle p_0\rangle \hookrightarrow Stat3(\star) \Rightarrow$ false; Discharge $\Rightarrow$ *Stat4*: $p_0 \neq \emptyset\ \&\ \mathsf{cov}(g_0,p_0) \in \{\emptyset, g_0\}$
Use_def$(\mathsf{filter}(g_0,v_0)) \Rightarrow$ AUTO
Use_def$(\mathsf{cov}(g_0,p_0)) \Rightarrow$ AUTO
$\langle g_0, \mathsf{filter}(g_0,v_0), p_0\rangle \hookrightarrow Thgraph_7(\star) \Rightarrow$ *Stat6*: $\mathsf{filter}(g_0,v_0) \neq p_0\ \&\ \{e \in g_0 \mid e \cap \mathsf{nodes}(p_0) \neq \emptyset\} = g_0$
$\langle e'\rangle \hookrightarrow Stat6(\star) \Rightarrow\ e' \in \mathsf{filter}(g_0,v_0)\ \&\ e' \notin p_0$
$\langle \mathsf{filter}(g_0,v_0), \mathsf{filter}(g_0,v_0)\backslash p_0\rangle \hookrightarrow Thgraph_3(\star) \Rightarrow\ \mathsf{HGraph}(\mathsf{filter}(g_0,v_0)\backslash p_0)$
$\langle \mathsf{filter}(g_0,v_0)\backslash p_0, e'\rangle \hookrightarrow Thgraph_2(\star) \Rightarrow\ e' \neq \emptyset\ \&\ e' \cap \mathsf{nodes}(p_0) = \emptyset$
Use_def(filter) $\Rightarrow$ *Stat7*: $e' \in \{e\backslash\{v_0\} : e \in g_0 \mid \mathsf{CardAtLeast2}(e\backslash\{v_0\})\}$
$\langle eq\rangle \hookrightarrow Stat7(Stat6\star) \Rightarrow$ *Stat8*: $eq \in \{e \in g_0 \mid e \cap \mathsf{nodes}(p_0) \neq \emptyset\}\ \&\ e' = eq\backslash\{v_0\}$
$\langle\ \rangle \hookrightarrow Stat8(Stat6\star) \Rightarrow\ v_0 \in \mathsf{nodes}(p_0)$
$\langle p_0, \mathsf{filter}(g_0,v_0)\rangle \hookrightarrow Thgraph_a(\star) \Rightarrow\ \mathsf{nodes}(p_0) \subseteq \mathsf{nodes}(\mathsf{filter}(g_0,v_0))$
(*Stat0*$\star$)Discharge $\Rightarrow$ AUTO
Use_def(cov) $\Rightarrow$ *Stat9*: $\mathsf{nodes}\big(\{e \in g_0 \mid e \cap \mathsf{nodes}(\mathsf{filter}(g_0,v_0)\backslash p_0) \neq \emptyset\}\big) \cap \mathsf{nodes}(\{e \in g_0 \mid e \cap \mathsf{nodes}(p_0) \neq \emptyset\}) \not\subseteq \{v_0\}$
$\langle v_1\rangle \hookrightarrow Stat9(Stat9\star) \Rightarrow$ *Stat10*: $v_1 \in \mathsf{nodes}\big(\{e \in g_0 \mid e \cap \mathsf{nodes}(\mathsf{filter}(g_0,v_0)\backslash p_0) \neq \emptyset\}\big)\ \&\ v_1 \in \mathsf{nodes}(\{e \in g_0 \mid e \cap \mathsf{nodes}(p_0) \neq \emptyset\})\ \&\ v_1 \neq v_0$
Loc_def $\Rightarrow\ e_1 = \mathsf{edgeOf}(v_1, \{e \in g_0 \mid e \cap \mathsf{nodes}(p_0) \neq \emptyset\})\ \&\ e_2 = \mathsf{edgeOf}\big(v_1, \{e \in g_0 \mid e \cap \mathsf{nodes}(\mathsf{filter}(g_0,v_0)\backslash p_0) \neq \emptyset\}\big)$
$\langle v_1, \{e \in g_0 \mid e \cap \mathsf{nodes}(p_0) \neq \emptyset\}\rangle \hookrightarrow Thgraph_e(Stat10\star) \Rightarrow$ *Stat11*: $e_1 \in \{e \in g_0 \mid e \cap \mathsf{nodes}(p_0) \neq \emptyset\}\ \&\ v_1 \in e_1\ \&\ e_1 \subseteq \mathsf{nodes}(\{e \in g_0 \mid e \cap \mathsf{nodes}(p_0) \neq \emptyset\})$
$\langle v_1, \{e \in g_0 \mid e \cap \mathsf{nodes}(\mathsf{filter}(g_0,v_0)\backslash p_0) \neq \emptyset\}\rangle \hookrightarrow Thgraph_e(Stat10\star) \Rightarrow$ *Stat12*: $e_2 \in \{e \in g_0 \mid e \cap \mathsf{nodes}(\mathsf{filter}(g_0,v_0)\backslash p_0) \neq \emptyset\}\ \&$
$v_1 \in e_2\ \&\ e_2 \subseteq \mathsf{nodes}\big(\{e \in g_0 \mid e \cap \mathsf{nodes}(\mathsf{filter}(g_0,v_0)\backslash p_0) \neq \emptyset\}\big)$
Suppose $\Rightarrow\ \neg(e_1 \subseteq \mathsf{nodes}(g_0)\ \&\ e_2 \subseteq \mathsf{nodes}(g_0))$
Set_monot $\Rightarrow\ \{e \in g_0 \mid e \cap \mathsf{nodes}(p_0) \neq \emptyset\} \subseteq \{e \in g_0 \mid \mathsf{true}\}$

Set_monot $\Rightarrow$ $\{e \in g_0 \mid e \cap \mathrm{nodes}(\mathrm{filter}(g_0, v_0)\backslash p_0) \neq \emptyset\} \subseteq \{e \in g_0 \mid \mathrm{true}\}$
TELEM $\Rightarrow$ $\{e \in g_0 \mid \mathrm{true}\} = g_0$
$\langle \{e \in g_0 \mid e \cap \mathrm{nodes}(p_0) \neq \emptyset\}, g_0 \rangle \hookrightarrow Thgraph_a(Stat12\star) \Rightarrow$ $\mathrm{nodes}(\{e \in g_0 \mid e \cap \mathrm{nodes}(p_0) \neq \emptyset\}) \subseteq \mathrm{nodes}(g_0)$
$\langle \{e \in g_0 \mid e \cap \mathrm{nodes}(\mathrm{filter}(g_0, v_0)\backslash p_0) \neq \emptyset\}, g_0 \rangle \hookrightarrow Thgraph_a(Stat12\star) \Rightarrow$ $\mathrm{nodes}\Big(\{e \in g_0 \mid e \cap \mathrm{nodes}(\mathrm{filter}(g_0, v_0)\backslash p_0) \neq \emptyset\}\Big) \subseteq \mathrm{nodes}(g_0)$
$(Stat11\star)$Discharge $\Rightarrow$ $Stat13:\ e_1 \subseteq \mathrm{nodes}(g_0)\ \&\ e_2 \subseteq \mathrm{nodes}(g_0)\ \&\ v_1 \in e_1 \cap e_2 \cap \mathrm{nodes}(g_0)\ \&\ \mathrm{filter}(g_0, v_0)\backslash p_0 \cup p_0 = \mathrm{filter}(g_0, v_0)$
$\langle\ \rangle \hookrightarrow Stat11(Stat11\star) \Rightarrow$ $e_1 \in g_0\ \&\ e_1 \cap \mathrm{nodes}(p_0) \neq \emptyset$
$\langle\ \rangle \hookrightarrow Stat12(Stat12\star) \Rightarrow$ $e_2 \in g_0\ \&\ e_2 \cap \mathrm{nodes}(\mathrm{filter}(g_0, v_0)\backslash p_0) \neq \emptyset$
$(Stat1, Stat1\star)$ELEM $\Rightarrow$ $\mathrm{nodes}(g_0) = \mathrm{nodes}(\mathrm{filter}(g_0, v_0)) \cup \{v_0\}$
$\langle \mathrm{filter}(g_0, v_0)\backslash p_0, p_0 \rangle \hookrightarrow Thgraph_b(Stat12\star) \Rightarrow$ $\mathrm{nodes}(\mathrm{filter}(g_0, v_0)\backslash p_0 \cup p_0) = \mathrm{nodes}(\mathrm{filter}(g_0, v_0)\backslash p_0) \cup \mathrm{nodes}(p_0)$
EQUAL $\Rightarrow$ $\mathrm{nodes}(g_0) = \mathrm{nodes}(\mathrm{filter}(g_0, v_0)\backslash p_0) \cup \mathrm{nodes}(p_0) \cup \{v_0\}$
$\langle g_0, p_0, v_0 \rangle \hookrightarrow Thgraph_{10}(\star) \Rightarrow$ $Stat14:\ \langle \forall e \in g_0 \mid \emptyset \in \{\mathrm{nodes}(p_0) \cap e, \mathrm{nodes}(\mathrm{filter}(g_0, v_0)\backslash p_0) \cap e\} \rangle$
$\langle e_1 \rangle \hookrightarrow Stat14 \Rightarrow$ Auto
$\langle e_2 \rangle \hookrightarrow Stat14 \Rightarrow$ Auto
$(Stat13\star)$ELEM $\Rightarrow$ $v_1 \in e_1 \cap e_2 \cap \{v_0\}$
$(Stat9\star)$Discharge $\Rightarrow$ Qed

|| The following proposition corresponds to Lemma 18 of [CO14].

Theorem hgraph$_{13}$. $\mathrm{nodes}(Q) \cap \mathrm{nodes}(G\backslash Q) = \emptyset\ \&\ P \subseteq G\ \&\ \mathrm{Conn}(P) \rightarrow P \subseteq G\backslash Q \vee P \subseteq Q$. Proof:
Suppose_not$(q_0, g_0, p_0) \Rightarrow$ $Stat1:\ p_0 \nsubseteq g_0\backslash q_0\ \&\ p_0 \nsubseteq q_0\ \&\ \mathrm{nodes}(q_0) \cap \mathrm{nodes}(g_0\backslash q_0) = \emptyset\ \&\ p_0 \subseteq g_0\ \&\ \mathrm{Conn}(p_0)$
Use_def$(\mathrm{Conn}(p_0)) \Rightarrow$ Auto
$\langle e_0, e_1 \rangle \hookrightarrow Stat1(Stat1\star) \Rightarrow$ $Stat2:\ p_0\backslash q_0 \notin \{p \subseteq p_0 \mid \mathrm{nodes}(p) \cap \mathrm{nodes}(p_0\backslash p) = \emptyset\}\ \&\ p_0\backslash(p_0\backslash q_0) \subseteq q_0\ \&\ p_0\backslash q_0 \subseteq g_0\backslash q_0$
$\langle p_0\backslash q_0 \rangle \hookrightarrow Stat2(Stat2\star) \Rightarrow$ $\mathrm{nodes}(p_0\backslash q_0) \cap \mathrm{nodes}(p_0\backslash(p_0\backslash q_0)) \neq \emptyset$
$\langle p_0\backslash(p_0\backslash q_0), q_0 \rangle \hookrightarrow Thgraph_a \Rightarrow$ Auto
$\langle p_0\backslash q_0, g_0\backslash q_0 \rangle \hookrightarrow Thgraph_a(Stat2\star) \Rightarrow$ $Stat3:\ \mathrm{nodes}(q_0) \cap \mathrm{nodes}(g_0\backslash q_0) \neq \emptyset$
$(Stat3, Stat1\star)$Discharge $\Rightarrow$ Qed

|| The following proposition corresponds to Lemma 19 of [CO14].

Theorem hgraph$_{14}$. $\mathrm{Conn}(P \cup Q)\ \&\ \mathrm{nodes}(P) \cap \mathrm{nodes}(Q) = \{V\} \rightarrow \mathrm{Conn}(P)$. Proof:
Suppose_not$(p_0, q_0, v_0) \Rightarrow$ Auto
Use_def(Conn) $\Rightarrow$ $Stat1:\ \{p \subseteq p_0 \cup q_0 \mid \mathrm{nodes}(p) \cap \mathrm{nodes}(p_0 \cup q_0\backslash p) = \emptyset\} \subseteq \{\emptyset, p_0 \cup q_0\}\ \&\ \mathrm{HGraph}(p_0 \cup q_0)$
Use_def$(\mathrm{Conn}(p_0)) \Rightarrow$ Auto
$\langle p_0 \cup q_0, p_0 \rangle \hookrightarrow Thgraph_3(\star) \Rightarrow$ $Stat2:\ \{p \subseteq p_0 \mid \mathrm{nodes}(p) \cap \mathrm{nodes}(p_0\backslash p) = \emptyset\} \nsubseteq \{\emptyset, p_0\}\ \&\ \mathrm{Conn}(p_0 \cup q_0)\ \&\ \mathrm{nodes}(p_0) \cap \mathrm{nodes}(q_0) = \{v_0\}$
$\langle p_1 \rangle \hookrightarrow Stat2(Stat2\star) \Rightarrow$ $Stat3:\ p_1 \in \{p \subseteq p_0 \mid \mathrm{nodes}(p) \cap \mathrm{nodes}(p_0\backslash p) = \emptyset\}\ \&\ p_1 \neq p_0\ \&\ p_1 \neq \emptyset$
$\langle\ \rangle \hookrightarrow Stat3(Stat3\star) \Rightarrow$ $Stat4:\ p_0\backslash p_1 \cup p_1 = p_0\ \&\ \mathrm{nodes}(p_1) \cap \mathrm{nodes}(p_0\backslash p_1) = \emptyset$
$\langle p_0\backslash p_1, p_1 \rangle \hookrightarrow Thgraph_b \Rightarrow$ Auto
EQUAL $\langle Stat2 \rangle \Rightarrow$ $Stat5:\ \{v_0\} = (\mathrm{nodes}(p_0\backslash p_1) \cup \mathrm{nodes}(p_1)) \cap \mathrm{nodes}(q_0)$
Suppose $\Rightarrow$ $Stat6:\ \neg\langle \exists p \in \{p_1, p_0\backslash p_1\} \mid \mathrm{nodes}(p) \cap \mathrm{nodes}(q_0) = \emptyset \rangle$

$\langle p_1 \rangle \hookrightarrow Stat6 \Rightarrow$ AUTO
$\langle p_0 \backslash p_1 \rangle \hookrightarrow Stat6(Stat4\star) \Rightarrow$ false; Discharge $\Rightarrow$ $Stat7: \langle \exists p \in \{p_1, p_0 \backslash p_1\} \mid \text{nodes}(p) \cap \text{nodes}(q_0) = \emptyset \rangle$
$\langle p_2 \rangle \hookrightarrow Stat7(Stat7\star) \Rightarrow$ $Stat8: p_2 \in \{p_1, p_0 \backslash p_1\} \ \& \ \text{nodes}(p_2) \cap \text{nodes}(q_0) = \emptyset$
Suppose $\Rightarrow$ $\text{nodes}(p_2) \cap \text{nodes}(p_0 \backslash p_2) \neq \emptyset$
Suppose $\Rightarrow$ $p_2 = p_1$
EQUAL $\langle Stat4 \rangle \Rightarrow$ false; Discharge $\Rightarrow$ $p_2 = p_0 \backslash p_1 \ \& \ p_1 = p_0 \backslash (p_0 \backslash p_1)$
EQUAL $\langle Stat5 \rangle \Rightarrow$ $Stat9: \text{nodes}(p_0 \backslash p_1) \cap \text{nodes}(p_1) \neq \emptyset$
$(Stat4, Stat9\star)$Discharge $\Rightarrow$ AUTO
Suppose $\Rightarrow$ $Stat10: p_2 \cap q_0 \neq \emptyset$
$\langle e_0 \rangle \hookrightarrow Stat10(Stat4, Stat8, Stat10\star) \Rightarrow$ $e_0 \in p_0 \ \& \ e_0 \in q_0$
$\langle p_0 \cup q_0, p_0 \rangle \hookrightarrow Thgraph_3(Stat1, Stat1\star) \Rightarrow$ $\text{HGraph}(p_0)$
$\langle p_0, e_0, v_0 \rangle \hookrightarrow Thgraph_2(Stat10\star) \Rightarrow$ $e_0 \subseteq \text{nodes}(p_0) \ \& \ e_0 \not\subseteq \{v_0\}$
$\langle p_0 \cup q_0, q_0 \rangle \hookrightarrow Thgraph_3(Stat1, Stat1\star) \Rightarrow$ $\text{HGraph}(q_0)$
$\langle q_0, e_0, v_0 \rangle \hookrightarrow Thgraph_2(Stat10\star) \Rightarrow$ $\text{nodes}(p_0) \cap \text{nodes}(q_0) \neq \{v_0\}$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO
$\langle p_0 \backslash p_2, q_0 \rangle \hookrightarrow Thgraph_b \Rightarrow$ AUTO
$(Stat8\star)$ELEM $\Rightarrow$ $Stat11: \emptyset = \text{nodes}(p_2) \cap \text{nodes}(p_0 \backslash p_2 \cup q_0) \ \& \ p_0 \backslash p_2 \cup q_0 = p_0 \cup q_0 \backslash p_2$
EQUAL $\langle Stat11 \rangle \Rightarrow$ $\emptyset = \text{nodes}(p_2) \cap \text{nodes}(p_0 \cup q_0 \backslash p_2)$
$(Stat8, Stat3, Stat4, Stat1\star)$ELEM $\Rightarrow$ $Stat12: p_2 \notin \{p \subseteq p_0 \cup q_0 \mid \text{nodes}(p) \cap \text{nodes}(p_0 \cup q_0 \backslash p) = \emptyset\} \ \& \ p_2 \subseteq p_0 \cup q_0$
$\langle p_2 \rangle \hookrightarrow Stat12(Stat11\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

‖ The following proposition corresponds to Lemma 20 of [CO14].

THEOREM hgraph$_{15}$. $\text{HGraph}(P \cup Q) \ \& \ \text{nodes}(P) \cap \text{nodes}(Q) = \{V\} \ \& \ W \in \text{nodes}(Q) \backslash \{V\} \rightarrow P \subseteq \text{filter}(P \cup Q, W)$. PROOF:
Suppose_not$(p_0, q_0, v_0, w_0) \Rightarrow$ $Stat2: p_0 \not\subseteq \text{filter}(p_0 \cup q_0, w_0) \ \& \ w_0 \in \text{nodes}(q_0) \backslash \{v_0\} \ \& \ w_0 \notin \text{nodes}(p_0) \ \& \ \text{HGraph}(p_0 \cup q_0) \ \& \ \text{nodes}(p_0) \cap \text{nodes}(q_0) = \{v_0\}$
Use_def(filter) $\Rightarrow$ $p_0 \not\subseteq \{e \backslash \{w_0\} : e \in p_0 \cup q_0 \mid \text{CardAtLeast2}(e \backslash \{w_0\})\}$
Set_monot $\Rightarrow$ $\{e \backslash \{w_0\} : e \in p_0 \mid \text{CardAtLeast2}(e \backslash \{w_0\})\} \subseteq \{e \backslash \{w_0\} : e \in p_0 \cup q_0 \mid \text{CardAtLeast2}(e \backslash \{w_0\})\}$
$(Stat2\star)$ELEM $\Rightarrow$ $Stat3: p_0 \not\subseteq \{e \backslash \{w_0\} : e \in p_0 \mid \text{CardAtLeast2}(e \backslash \{w_0\})\}$
$\langle e_1 \rangle \hookrightarrow Stat3(Stat3\star) \Rightarrow$ $Stat4: e_1 \notin \{e \backslash \{w_0\} : e \in p_0 \mid \text{CardAtLeast2}(e \backslash \{w_0\})\} \ \& \ e_1 \in p_0$
$\langle e_1 \rangle \hookrightarrow Stat4(Stat4\star) \Rightarrow$ $w_0 \in e_1 \vee \neg\text{CardAtLeast2}(e_1 \backslash \{w_0\})$
$\langle p_0 \cup q_0, p_0 \rangle \hookrightarrow Thgraph_3(Stat2\star) \Rightarrow$ $\text{HGraph}(p_0)$
$\langle p_0, e_1 \rangle \hookrightarrow Thgraph_2(Stat2\star) \Rightarrow$ $e_1 \backslash \{w_0\} \in p_0 \ \& \ \neg\text{CardAtLeast2}(e_1 \backslash \{w_0\})$
Use_def(HGraph) $\Rightarrow$ $Stat5: \langle \forall e \in p_0 \mid \text{CardAtLeast2}(e) \rangle$
$\langle e_1 \backslash \{w_0\} \rangle \hookrightarrow Stat5(Stat4\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

THEOREM hgraph$_{16}$: [Trivial connected graphs] $(\text{HGraph}(\{E\}) \ \& \ P \subseteq \{E\} \rightarrow \text{Conn}(P)) \ \& \ \text{Conn}(\emptyset)$. PROOF:
Suppose_not$(e_0, p_0) \Rightarrow$ AUTO
Suppose $\Rightarrow$ $\neg\text{Conn}(\emptyset)$
$\langle \{\{\emptyset, \{\emptyset\}\}\}, \emptyset \rangle \hookrightarrow Thgraph_3 \Rightarrow$ AUTO

$\langle\ \rangle \hookrightarrow Thgraph_0 \Rightarrow$ AUTO
Use_def(Conn) $\Rightarrow$ *Stat1* : $\{p \subseteq \emptyset \mid \text{nodes}(p) \cap \text{nodes}(\emptyset \backslash p) = \emptyset\} \not\subseteq \{\emptyset\}$
$\langle q_0 \rangle \hookrightarrow Stat1(Stat1\star) \Rightarrow$ *Stat2* : $q_0 \in \{p \subseteq \emptyset \mid \text{nodes}(p) \cap \text{nodes}(\emptyset \backslash p) = \emptyset\}$ & $q_0 \neq \emptyset$
$\langle\ \rangle \hookrightarrow Stat2(Stat2\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO
$\langle \{e_0\}, p_0 \rangle \hookrightarrow Thgraph_3 \Rightarrow$ HGraph(p_0)
Use_def(Conn) $\Rightarrow$ *Stat6* : $\{p \subseteq p_0 \mid \text{nodes}(p) \cap \text{nodes}(p_0 \backslash p) = \emptyset\} \not\subseteq \{\emptyset, p_0\}$
$\langle p_1 \rangle \hookrightarrow Stat6(Stat6\star) \Rightarrow$ *Stat7* : $p_1 \in \{p \subseteq p_0 \mid \text{nodes}(p) \cap \text{nodes}(p_0 \backslash p) = \emptyset\}$ & $p_1 \notin \{\emptyset, p_0\}$
$\langle\ \rangle \hookrightarrow Stat7(\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

‖ The following proposition corresponds to Lemma 21 of [CO14].

THEOREM hgraph$_{17}$. HGraph(G) & $\langle \forall v \in \text{nodes}(G) \mid \text{filter}(G, v) \cap G = \emptyset \rangle \to \langle \forall v \in \text{nodes}(G) \mid \text{Conn}(\text{filter}(G, v)) \rangle$. PROOF:
Suppose_not$(g_0) \Rightarrow$ *Stat1* : $\langle \forall v \in \text{nodes}(g_0) \mid \text{filter}(g_0, v) \cap g_0 = \emptyset \rangle$ & *Stat2* : $\neg\langle \forall v \in \text{nodes}(g_0) \mid \text{Conn}(\text{filter}(g_0, v)) \rangle$ & HGraph(g_0)
Suppose $\Rightarrow$ *Stat4* : $\neg\langle \forall v \in \text{nodes}(g_0) \mid \text{contains}(g_0, v) = g_0 \rangle$
$\langle g_0, v_0 \rangle \hookrightarrow Thgraph_8(Stat4\star) \Rightarrow$ contains$(g_0, v_0) \subseteq g_0$
Use_def(contains(g_0, v_0)) $\Rightarrow$ AUTO
$\langle v_0 \rangle \hookrightarrow Stat4(Stat4\star) \Rightarrow$ *Stat5* : $g_0 \not\subseteq \{e \in g_0 \mid v_0 \in e\}$ & $v_0 \in \text{nodes}(g_0)$
$\langle v_0 \rangle \hookrightarrow Stat1(Stat5\star) \Rightarrow$ filter$(g_0, v_0) \cap g_0 = \emptyset$
Use_def(filter(g_0, v_0)) $\Rightarrow$ AUTO
$\langle e_0 \rangle \hookrightarrow Stat5(Stat5\star) \Rightarrow$ *Stat6* : $e_0 \notin \{e \in g_0 \mid v_0 \in e\}$ & $e_0 \notin \{e \backslash \{v_0\} : e \in g_0 \mid \text{CardAtLeast2}(e \backslash \{v_0\})\}$ & $e_0 \in g_0$
$\langle e_0, e_0 \rangle \hookrightarrow Stat6(Stat6\star) \Rightarrow$ $e_0 = e_0 \backslash \{v_0\}$ & $\neg$CardAtLeast2$(e_0 \backslash \{v_0\})$
EQUAL $\langle Stat6 \rangle \Rightarrow$ $\neg$CardAtLeast2(e_0)
Use_def(HGraph) $\Rightarrow$ *Stat7* : $\langle \forall e \in g_0 \mid \text{CardAtLeast2}(e) \rangle$
$\langle e_0 \rangle \hookrightarrow Stat7(Stat6\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO
Loc_def $\Rightarrow$ $e_1 = \text{edgeOf}(w, g_0)$
$\langle w \rangle \hookrightarrow Stat2(Stat1\star) \Rightarrow$ *Stat8* : $w \in \text{nodes}(g_0)$ & $\neg$Conn(filter(g_0, w))
$\langle w, g_0 \rangle \hookrightarrow Thgraph_e(Stat1\star) \Rightarrow$ *Stat9* : $e_1 \in g_0$
Suppose $\Rightarrow$ *Stat10* : $g_0 \neq \{e_1\}$
$\langle e_2 \rangle \hookrightarrow Stat10(Stat1\star) \Rightarrow$ *Stat11* : $e_2 \neq e_1$ & $\langle \forall v \in \text{nodes}(g_0) \mid \text{contains}(g_0, v) = g_0 \rangle$ & $e_2 \in g_0$ & HGraph(g_0)
$\langle v_1, v_1 \rangle \hookrightarrow Stat11(Stat11\star) \Rightarrow$ $(v_1 \in e_2 \leftrightarrow v_1 \notin e_1)$ & $(v_1 \in \text{nodes}(g_0) \to \text{contains}(g_0, v_1) = g_0)$
$\langle g_0, e_1, v_1 \rangle \hookrightarrow Thgraph_2 \Rightarrow$ AUTO
Use_def(contains(g_0, v_1)) $\Rightarrow$ AUTO
$\langle g_0, e_2, v_1 \rangle \hookrightarrow Thgraph_2(Stat9\star) \Rightarrow$ *Stat12* : $e_1, e_2 \in \{e : e \in g_0 \mid v_1 \in e\}$
$\langle e_3, e_4 \rangle \hookrightarrow Stat12(Stat11\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO
Use_def(filter) $\Rightarrow$ *Stat13* : filter$(g_0, w) = \{e \backslash \{w\} : e \in g_0 \mid \text{CardAtLeast2}(e \backslash \{w\})\}$ & $g_0 = \{e_1\}$
EQUAL $\langle Stat1, Stat13 \rangle \Rightarrow$ filter$(g_0, w) = \{e \backslash \{w\} : e \in \{e_1\} \mid \text{CardAtLeast2}(e \backslash \{w\})\}$ & HGraph$(\{e_1\})$
Suppose $\Rightarrow$ $\neg$CardAtLeast2$(e_1 \backslash \{w\})$
Suppose $\Rightarrow$ *Stat14* : $\{e \backslash \{w\} : e \in \{e_1\} \mid \text{CardAtLeast2}(e \backslash \{w\})\} \neq \emptyset$
$\langle e_5 \rangle \hookrightarrow Stat14(Stat14\star) \Rightarrow$ $e_5 = e_1$ & CardAtLeast2$(e_5 \backslash \{w\})$

EQUAL $\langle Stat13 \rangle \Rightarrow$ false; Discharge $\Rightarrow$ AUTO
$\langle e_1, \text{filter}(g_0, w)\rangle \hookrightarrow Thgraph_{16}(Stat8\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO
Suppose $\Rightarrow$ $Stat15: \{e \backslash \{w\} : e \in \{e_1\} \mid \text{CardAtLeast2}(e \backslash \{w\})\} \neq \{e_1 \backslash \{w\}\}$
$\langle e_6\rangle \hookrightarrow Stat15(Stat15\star) \Rightarrow$ $e_6 \in \{e \backslash \{w\} : e \in \{e_1\} \mid \text{CardAtLeast2}(e \backslash \{w\})\} \neq e_6 = e_1 \backslash \{w\}$
Suppose $\Rightarrow$ $Stat16: e_6 \notin \{e \backslash \{w\} : e \in \{e_1\} \mid \text{CardAtLeast2}(e \backslash \{w\})\}$
$\langle e_1\rangle \hookrightarrow Stat16(Stat13\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO
$(Stat15\star)$ELEM $\Rightarrow$ $Stat17: e_6 \in \{e \backslash \{w\} : e \in \{e_1\} \mid \text{CardAtLeast2}(e \backslash \{w\})\} \ \& \ e_6 \neq e_1 \backslash \{w\}$
$\langle e_7\rangle \hookrightarrow Stat17(Stat17\star) \Rightarrow$ false; Discharge $\Rightarrow$ $\{e \backslash \{w\} : e \in \{e_1\} \mid \text{CardAtLeast2}(e \backslash \{w\})\} = \{e_1 \backslash \{w\}\}$
$\langle g_0, w\rangle \hookrightarrow Thgraph_8(Stat1, Stat1\star) \Rightarrow$ $\text{HGraph}(\text{filter}(g_0, w))$
EQUAL $\langle Stat13 \rangle \Rightarrow$ $\text{HGraph}(\{e_1 \backslash \{w\}\})$
$\langle e_1 \backslash \{w\}, \text{filter}(g_0, w)\rangle \hookrightarrow Thgraph_{16}(Stat8\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

|| The following proposition corresponds to Lemma 2 of [CO14].

THEOREM hgraph$_{18}$: [A node W other than V gets lost when V is removed from a graph iff $\{V, W\}$ is the only edge to which V belongs] $\text{HGraph}(G) \rightarrow (W \in \text{lost}(G, V) \leftrightarrow \text{contains}(G, W) = \{\{V, W\}\})$. PROOF:
Suppose_not$(g_0, v_1, v_0) \Rightarrow$ AUTO
Use_def(lost) $\Rightarrow$ $Stat0: \text{HGraph}(g_0) \ \& \ \text{lost}(g_0, v_0) = \text{nodes}(g_0) \backslash \left(\text{nodes}(\text{filter}(g_0, v_0)) \cup \{v_0\}\right)$
Use_def(contains) $\Rightarrow$ $\text{contains}(g_0, v_1) = \{e \in g_0 \mid v_1 \in e\}$
$\langle g_0, v_0\rangle \hookrightarrow Thgraph_8(Stat0\star) \Rightarrow$ $Stat1: \text{nodes}(\text{filter}(g_0, v_0)) \subseteq \text{nodes}(g_0) \backslash \{v_0\} \ \& \ \text{HGraph}(\text{filter}(g_0, v_0))$
Use_def(filter) $\Rightarrow$ $\text{filter}(g_0, v_0) = \{e \backslash \{v_0\} : e \in g_0 \mid \text{CardAtLeast2}(e \backslash \{v_0\})\}$
Suppose $\Rightarrow$ $Stat2: \text{contains}(g_0, v_1) \neq \{\{v_0, v_1\}\} \ \& \ v_1 \in \text{lost}(g_0, v_0)$
Loc_def $\Rightarrow$ $e_0 = \text{edgeOf}(v_1, g_0)$
$\langle v_1, g_0\rangle \hookrightarrow Thgraph_e(Stat0\star) \Rightarrow$ $Stat3: v_1 \in e_0 \ \& \ e_0 \in g_0 \ \& \ v_1 \notin \text{nodes}(\text{filter}(g_0, v_0)) \ \& \ v_1 \neq v_0$
Suppose $\Rightarrow$ $Stat4: e_0 \notin \{e \in g_0 \mid v_1 \in e\}$
$\langle e_0\rangle \hookrightarrow Stat4(Stat3\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO
Suppose $\Rightarrow$ $v_0 \notin e_0$
Suppose $\Rightarrow$ $Stat5: e_0 \notin \{e \backslash \{v_0\} : e \in g_0 \mid \text{CardAtLeast2}(e \backslash \{v_0\})\}$
$\langle e_0\rangle \hookrightarrow Stat5(Stat3\star) \Rightarrow$ $\neg\text{CardAtLeast2}(e_0 \backslash \{v_0\}) \ \& \ e_0 \backslash \{v_0\} = e_0$
$\langle g_0, e_0, \emptyset\rangle \hookrightarrow Thgraph_2(Stat0\star) \Rightarrow$ $\text{CardAtLeast2}(e_0)$
EQUAL $\langle Stat5 \rangle \Rightarrow$ false; Discharge $\Rightarrow$ $Stat6: e_0 \in \text{filter}(g_0, v_0)$
$\langle \text{filter}(g_0, v_0), e_0, \emptyset\rangle \hookrightarrow Thgraph_2(Stat1\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO
Suppose $\Rightarrow$ $e_0 = \{v_0, v_1\}$
$\langle e_1\rangle \hookrightarrow Stat2(Stat0\star) \Rightarrow$ $Stat7: e_1 \in \{e \in g_0 \mid v_1 \in e\} \ \& \ e_1 \neq \{v_0, v_1\}$
$\langle \ \rangle \hookrightarrow Stat7(Stat7\star) \Rightarrow$ $e_1 \in g_0 \ \& \ v_1 \in e_1$
$\langle \text{filter}(g_0, v_0), e_1, v_1\rangle \hookrightarrow Thgraph_2(Stat1\star) \Rightarrow$ $Stat8: e_1 \notin \{e \backslash \{v_0\} : e \in g_0 \mid \text{CardAtLeast2}(e \backslash \{v_0\})\}$
$\langle g_0, e_1, v_1\rangle \hookrightarrow Thgraph_2(Stat0\star) \Rightarrow$ $\text{CardAtLeast2}(e_1) \ \& \ e_1 \not\subseteq \{v_1\}$
Suppose $\Rightarrow$ $e_1 \backslash \{v_0\} = e_1$
EQUAL $\Rightarrow$ $\text{CardAtLeast2}(e_1 \backslash \{v_0\})$

$\langle e_1 \rangle \hookrightarrow Stat8(Stat7\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO
$(Stat3\star)$ELEM $\Rightarrow$ $e_1 \setminus \{v_0\} \not\subseteq \text{nodes}(\text{filter}(g_0, v_0))$
$\langle \text{filter}(g_0, v_0), e_1 \setminus \{v_0\}, \emptyset \rangle \hookrightarrow Thgraph_2(Stat1\star) \Rightarrow$ $Stat9: e_1 \setminus \{v_0\} \notin \{e \setminus \{v_0\} : e \in g_0 \mid \text{CardAtLeast2}(e \setminus \{v_0\})\}$
Use_def(CardAtLeast2) $\Rightarrow$ $\text{CardAtLeast2}(e_1 \setminus \{v_0\}) \leftrightarrow e_1 \setminus \{v_0\} \not\subseteq \{\mathbf{arb}(e_1 \setminus \{v_0\})\}$
$\langle e_1 \rangle \hookrightarrow Stat9(Stat3\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO
$\langle \text{filter}(g_0, v_0), e_0 \setminus \{v_0\}, \emptyset \rangle \hookrightarrow Thgraph_2(Stat1\star) \Rightarrow$ $Stat10: e_0 \setminus \{v_0\} \notin \{e \setminus \{v_0\} : e \in g_0 \mid \text{CardAtLeast2}(e \setminus \{v_0\})\}$
Use_def(CardAtLeast2) $\Rightarrow$ $\text{CardAtLeast2}(e_0 \setminus \{v_0\}) \leftrightarrow e_0 \setminus \{v_0\} \not\subseteq \{\mathbf{arb}(e_0 \setminus \{v_0\})\}$
$\langle e_0 \rangle \hookrightarrow Stat10(Stat3\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO
Loc_def $\Rightarrow$ $Stat11: e_2 = \{v_0, v_1\} \;\&\; e_3 = \text{edgeOf}(v_1, \text{filter}(g_0, v_0))$
ELEM $\Rightarrow$ $Stat12: e_2 \in \{e \in g_0 \mid v_1 \in e\} \;\&\; v_1 \notin \text{nodes}(g_0) \setminus \left(\text{nodes}(\text{filter}(g_0, v_0)) \cup \{v_0\}\right) \;\&\; \{e \in g_0 \mid v_1 \in e\} \subseteq \{e_2\}$
$\langle g_0, e_2, v_1 \rangle \hookrightarrow Thgraph_2 \Rightarrow$ AUTO
$\langle\, \rangle \hookrightarrow Stat12(Stat0\star) \Rightarrow$ $Stat13: e_2 \in g_0 \;\&\; v_0 \neq v_1 \;\&\; v_1 \in \text{nodes}(\text{filter}(g_0, v_0))$
$\langle v_1, \text{filter}(g_0, v_0) \rangle \hookrightarrow Thgraph_e(Stat1\star) \Rightarrow$ $Stat14: e_3 \in \{e \setminus \{v_0\} : e \in g_0 \mid \text{CardAtLeast2}(e \setminus \{v_0\})\} \;\&\; v_1 \in e_3$
$\langle e_4 \rangle \hookrightarrow Stat14(Stat14\star) \Rightarrow$ $e_3 = e_4 \setminus \{v_0\} \;\&\; e_4 \in g_0 \;\&\; \text{CardAtLeast2}(e_4 \setminus \{v_0\})$
Suppose $\Rightarrow$ $e_4 \neq e_2$
$(Stat12\star)$ELEM $\Rightarrow$ $Stat15: e_4 \notin \{e \in g_0 \mid v_1 \in e\}$
$\langle e_3 \rangle \hookrightarrow Stat15(Stat13\star) \Rightarrow$ false; Discharge $\Rightarrow$ $e_4 = e_2$
Use_def(CardAtLeast2) $\Rightarrow$ $\text{CardAtLeast2}(e_4 \setminus \{v_0\}) \leftrightarrow e_4 \setminus \{v_0\} \not\subseteq \{\mathbf{arb}(e_4 \setminus \{v_0\})\}$
EQUAL $\langle Stat14 \rangle \Rightarrow$ $Stat16: e_2 \setminus \{v_0\} \not\subseteq \{\mathbf{arb}(e_2 \setminus \{v_0\})\}$
$(Stat11, Stat16\star)$Discharge $\Rightarrow$ QED

|| The following proposition corresponds to Lemma 3 of [CO14].

THEOREM hgraph_{19}: [If a node belongs to exactly one edge, E, and that E is a doubleton, removal of E does not disrupt connectivity]
$\text{Conn}(G) \;\&\; \text{contains}(G, V) = \{\{V, W\}\} \rightarrow \text{Conn}(G \setminus \text{contains}(G, V))$. PROOF:
Suppose_not$(g_0, v_0, v_1) \Rightarrow$ AUTO
$\langle g_0, g_0 \setminus \text{contains}(g_0, v_0) \rangle \hookrightarrow Thgraph_3 \Rightarrow$ AUTO
Use_def(Conn) $\Rightarrow$ $Stat1: \{p \subseteq g_0 \setminus \text{contains}(g_0, v_0) \mid \text{nodes}(p) \cap \text{nodes}(g_0 \setminus \text{contains}(g_0, v_0) \setminus p) = \emptyset\} \not\subseteq \{\emptyset, g_0 \setminus \text{contains}(g_0, v_0)\} \;\&$
$\text{HGraph}(g_0) \;\&\; \{p \subseteq g_0 \mid \text{nodes}(p) \cap \text{nodes}(g_0 \setminus p) = \emptyset\} \subseteq \{\emptyset, g_0\} \;\&\; \text{contains}(g_0, v_0) = \{\{v_0, v_1\}\}$
$\langle p_0 \rangle \hookrightarrow Stat1(\star) \Rightarrow$ $Stat2: p_0 \in \{p \subseteq g_0 \setminus \text{contains}(g_0, v_0) \mid \text{nodes}(p) \cap \text{nodes}(g_0 \setminus \text{contains}(g_0, v_0) \setminus p) = \emptyset\} \;\&\; p_0 \neq \emptyset \;\&\; p_0 \neq g_0 \setminus \text{contains}(g_0, v_0)$
$\langle\, \rangle \hookrightarrow Stat2(Stat1\star) \Rightarrow$ $Stat3: p_0 \notin \{p \subseteq g_0 \mid \text{nodes}(p) \cap \text{nodes}(g_0 \setminus p) = \emptyset\} \;\&$
$g_0 \setminus p_0 = g_0 \setminus \text{contains}(g_0, v_0) \setminus p_0 \cup (g_0 \setminus p_0) \cap \text{contains}(g_0, v_0) \;\&\; p_0 \subseteq g_0 \setminus \text{contains}(g_0, v_0) \;\&\; \text{nodes}(p_0) \cap \text{nodes}(g_0 \setminus \text{contains}(g_0, v_0) \setminus p_0) = \emptyset$
$\langle p_0 \rangle \hookrightarrow Stat3(Stat2\star) \Rightarrow$ $\text{nodes}(p_0) \cap \text{nodes}(g_0 \setminus p_0) \neq \emptyset$
$\langle g_0 \setminus \text{contains}(g_0, v_0) \setminus p_0, (g_0 \setminus p_0) \cap \text{contains}(g_0, v_0) \rangle \hookrightarrow Thgraph_b \Rightarrow$ AUTO
EQUAL $\langle Stat3 \rangle \Rightarrow$ $\text{nodes}(p_0) \cap \left(\text{nodes}(g_0 \setminus \text{contains}(g_0, v_0) \setminus p_0) \cup \text{nodes}((g_0 \setminus p_0) \cap \text{contains}(g_0, v_0))\right) \neq \emptyset$
$(Stat2\star)$ELEM $\Rightarrow$ $Stat4: \text{nodes}(p_0) \cap \text{nodes}((g_0 \setminus p_0) \cap \text{contains}(g_0, v_0)) \neq \emptyset$
Suppose $\Rightarrow$ $(g_0 \setminus p_0) \cap \text{contains}(g_0, v_0) = \emptyset$
EQUAL $\langle Stat3 \rangle \Rightarrow$ $Stat5: \text{nodes}(\emptyset) \neq \emptyset$

$\langle \emptyset \rangle \hookrightarrow Thgraph_d(Stat5\star) \Rightarrow$ false; Discharge $\Rightarrow$ $Stat6:\ \{v_0, v_1\} \in g_0\ \&\ \{v_0, v_1\} \notin p_0$
Suppose $\Rightarrow$ $v_0 \in nodes(p_0)$
Set_monot $\Rightarrow$ $\{e \in p_0 \mid v_0 \in e\} \subseteq \{e \in g_0 \mid v_0 \in e\}$
Use_def(contains) $\Rightarrow$ $\{e \in p_0 \mid v_0 \in e\} \subseteq contains(g_0, v_0)$
Suppose $\Rightarrow$ $Stat7:\ \{v_0, v_1\} \in \{e \in p_0 \mid v_0 \in e\}$
$\langle\ \rangle \hookrightarrow Stat7(Stat6\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO
$(Stat1\star)$ELEM $\Rightarrow$ $Stat8:\ \{e \in p_0 \mid v_0 \in e\} = \emptyset$
$\langle v_0, p_0 \rangle \hookrightarrow Thgraph_e(Stat6\star) \Rightarrow$ $v_0 \in edgeOf(v_0, p_0)\ \&\ edgeOf(v_0, p_0) \in p_0$
$\langle edgeOf(v_0, p_0) \rangle \hookrightarrow Stat8(Stat8\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO
Suppose $\Rightarrow$ $v_1 \notin nodes(p_0)$
$\langle \{v_0, v_1\} \rangle \hookrightarrow Thgraph_d \Rightarrow$ AUTO
EQUAL $\langle Stat1 \rangle \Rightarrow$ $nodes(contains(g_0, v_0)) = \{v_0, v_1\}$
$\langle (g_0 \backslash p_0) \cap contains(g_0, v_0), contains(g_0, v_0) \rangle \hookrightarrow Thgraph_a(Stat4\star) \Rightarrow$ false; Discharge $\Rightarrow$ $Stat9:\ v_1 \in nodes(p_0)$
Suppose $\Rightarrow$ $nodes(p_0 \cup contains(g_0, v_0)) \neq nodes(p_0) \cup \{v_0\}$
$\langle p_0, contains(g_0, v_0) \rangle \hookrightarrow Thgraph_b \Rightarrow$ AUTO
$\langle \{v_0, v_1\} \rangle \hookrightarrow Thgraph_d(Stat9\star) \Rightarrow$ $nodes(p_0) \cup nodes(\{\{v_0, v_1\}\}) = nodes(p_0) \cup \{v_0\}$
EQUAL $\langle Stat1 \rangle \Rightarrow$ false
Discharge $\Rightarrow$ $nodes(p_0 \cup contains(g_0, v_0)) = nodes(p_0) \cup \{v_0\}\ \&\ nodes(g_0 \backslash contains(g_0, v_0) \backslash p_0) \cap (nodes(p_0) \cup \{v_0\}) = nodes(g_0 \backslash contains(g_0, v_0) \backslash p_0) \cap \{v_0\}\ \&$
$g_0 \backslash (p_0 \cup contains(g_0, v_0)) = g_0 \backslash contains(g_0, v_0) \backslash p_0$
EQUAL $\langle Stat6 \rangle \Rightarrow$ $Stat11:\ nodes\Big(g_0 \backslash (p_0 \cup contains(g_0, v_0))\Big) \cap nodes(p_0 \cup contains(g_0, v_0)) = nodes(g_0 \backslash contains(g_0, v_0) \backslash p_0) \cap \{v_0\}$
Use_def$(contains(g_0, v_0)) \Rightarrow$ AUTO
Suppose $\Rightarrow$ $v_0 \in nodes(g_0 \backslash contains(g_0, v_0) \backslash p_0)$
$\langle g_0 \backslash contains(g_0, v_0) \backslash p_0, g_0 \backslash contains(g_0, v_0) \rangle \hookrightarrow Thgraph_a(Stat11\star) \Rightarrow$ $v_0 \in nodes(g_0 \backslash contains(g_0, v_0))$
$\langle v_0, g_0 \backslash contains(g_0, v_0) \rangle \hookrightarrow Thgraph_e(Stat11\star) \Rightarrow$ $Stat12:\ edgeOf(v_0, g_0 \backslash contains(g_0, v_0)) \notin \{e \in g_0 \mid v_0 \in e\}\ \&\ edgeOf(v_0, g_0 \backslash contains(g_0, v_0)) \in g_0\ \&$
$v_0 \in edgeOf(v_0, g_0 \backslash contains(g_0, v_0))$
$\langle edgeOf(v_0, g_0 \backslash contains(g_0, v_0)) \rangle \hookrightarrow Stat12(Stat12\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO
$(Stat11\star)$ELEM $\Rightarrow$ $nodes(p_0 \cup contains(g_0, v_0)) \cap nodes\Big(g_0 \backslash (p_0 \cup contains(g_0, v_0))\Big) = \emptyset$
Suppose $\Rightarrow$ $Stat13:\ p_0 \cup contains(g_0, v_0) \notin \{p \subseteq g_0 \mid nodes(p) \cap nodes(g_0 \backslash p) = \emptyset\}$
$\langle p_0 \cup contains(g_0, v_0) \rangle \hookrightarrow Stat13(Stat11\star) \Rightarrow$ $p_0 \cup contains(g_0, v_0) \not\subseteq g_0$
$\langle g_0, v_0 \rangle \hookrightarrow Thgraph_8(Stat2\star) \Rightarrow$ false; Discharge $\Rightarrow$ $Stat14:$ AUTO
$(Stat1, Stat2, Stat6, Stat14\star)$ELEM $\Rightarrow$ false; Discharge $\Rightarrow$ QED

|| The following proposition corresponds to Lemma 4 of [CO14].

THEOREM $hgraph_{20}$: [Removal of a node that belongs to exactly one edge does not disrupt connectivity] $Conn(G)\ \&\ contains(G, V) = \{E\} \rightarrow Conn(filter(G, V))$. PROOF:
Suppose_not$(g_0, v_0, e_0) \Rightarrow$ AUTO
Use_def$(contains(g_0, v_0)) \Rightarrow$ AUTO
Use_def(filter) $\Rightarrow$ $Stat1: e_0 \in \{e \in g_0 \mid v_0 \in e\}\ \&\ \{e \in g_0 \mid v_0 \in e\} \subseteq \{e_0\}\ \&\ Conn(g_0)\ \&\ \neg Conn(filter(g_0, v_0))\ \&\ filter(g_0, v_0) = \{e \backslash \{v_0\} :\ e \in g_0 \mid CardAtLeast2(e \backslash \{v_0\})\}$

$\langle\,\rangle \hookrightarrow Stat1(Stat1\star) \Rightarrow \quad Stat2:\ e_0 \in g_0\ \&\ v_0 \in e_0$
Use_def(Conn) $\Rightarrow \quad \mathsf{HGraph}(g_0)\ \&\ \{p \subseteq g_0 \mid \mathsf{nodes}(p) \cap \mathsf{nodes}(g_0 \backslash p) = \emptyset\} \subseteq \{\emptyset, g_0\}$
$\langle g_0, e_0, v_0\rangle \hookrightarrow Thgraph_2(Stat1\star) \Rightarrow \quad Stat3:\ e_0 \not\subseteq \{v_0\}$
Suppose $\Rightarrow \quad \neg \mathsf{CardAtLeast2}(e_0 \backslash \{v_0\})$
Use_def(CardAtLeast2) $\Rightarrow \quad Stat4:\ e_0 \backslash \{v_0\} \subseteq \{\mathbf{arb}(e_0 \backslash \{v_0\})\}$
$\langle v_1\rangle \hookrightarrow Stat3(Stat2\star) \Rightarrow \quad e_0 = \{v_0, v_1\}$
Suppose $\Rightarrow \quad Stat5:\ \{e \backslash \{v_0\} : e \in g_0 \mid \mathsf{CardAtLeast2}(e \backslash \{v_0\})\} \neq g_0 \backslash \{e_0\}$
$\langle e'\rangle \hookrightarrow Stat5 \Rightarrow \quad$ AUTO
Suppose $\Rightarrow \quad Stat6:\ e' \in \{e \backslash \{v_0\} : e \in g_0 \mid \mathsf{CardAtLeast2}(e \backslash \{v_0\})\}$
$\langle e_1\rangle \hookrightarrow Stat6(Stat5\star) \Rightarrow \quad e' = e_1 \backslash \{v_0\}\ \&\ e_1 \in g_0\ \&\ \mathsf{CardAtLeast2}(e_1 \backslash \{v_0\})\ \&\ e' \notin g_0 \backslash \{e_0\}$
Suppose $\Rightarrow \quad e_1 = e_0$
EQUAL $\langle Stat3\rangle \Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow \quad$ AUTO
$(Stat1\star)$ELEM $\Rightarrow \quad Stat7:\ e_1 \notin \{e \in g_0 \mid v_0 \in e\}$
$\langle e_1\rangle \hookrightarrow Stat7(Stat6\star) \Rightarrow \quad e_1 = e_0$
EQUAL $\langle Stat3\rangle \Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow \quad$ AUTO
$(Stat1\star)$ELEM $\Rightarrow \quad Stat8:\ e' \notin \{e \in g_0 \mid v_0 \in e\}\ \&\ e' \notin \{e \backslash \{v_0\} : e \in g_0 \mid \mathsf{CardAtLeast2}(e \backslash \{v_0\})\}\ \&\ e' \in g_0 \backslash \{e_0\}$
$\langle e', e'\rangle \hookrightarrow Stat8(Stat8\star) \Rightarrow \quad e' \backslash \{v_0\} = e'\ \&\ \neg \mathsf{CardAtLeast2}(e' \backslash \{v_0\})$
$\langle g_0, e', \emptyset\rangle \hookrightarrow Thgraph_2(Stat2) \Rightarrow \quad \mathsf{CardAtLeast2}(e')$
EQUAL $\langle Stat8\rangle \Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow \quad \mathsf{filter}(g_0, v_0) = g_0 \backslash \{e_0\}$
$\langle g_0, v_0, v_1\rangle \hookrightarrow Thgraph_{19}(Stat1\star) \Rightarrow \quad \mathsf{contains}(g_0, v_0) = \{\{v_0, v_1\}\} \rightarrow \mathsf{Conn}(g_0 \backslash \mathsf{contains}(g_0, v_0))$
EQUAL $\Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow \quad$ AUTO
Suppose $\Rightarrow \quad Stat9:\ \{e \backslash \{v_0\} : e \in g_0 \mid \mathsf{CardAtLeast2}(e \backslash \{v_0\})\} \neq g_0 \backslash \{e_0\} \cup \{e_0 \backslash \{v_0\}\}$
$\langle eq\rangle \hookrightarrow Stat9 \Rightarrow \quad$ AUTO
Suppose $\Rightarrow \quad Stat10:\ eq \in \{e \backslash \{v_0\} : e \in g_0 \mid \mathsf{CardAtLeast2}(e \backslash \{v_0\})\}$
$\langle e_2\rangle \hookrightarrow Stat10(Stat9\star) \Rightarrow \quad eq = e_2 \backslash \{v_0\}\ \&\ e_2 \in g_0\ \&\ \mathsf{CardAtLeast2}(e_2 \backslash \{v_0\})\ \&\ eq \notin g_0 \backslash \{e_0\} \cup \{e_0 \backslash \{v_0\}\}$
Suppose $\Rightarrow \quad e_2 = e_0$
EQUAL $\langle Stat3\rangle \Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow \quad$ AUTO
$(Stat1\star)$ELEM $\Rightarrow \quad Stat11:\ e_2 \notin \{e \in g_0 \mid v_0 \in e\}$
$\langle e_2\rangle \hookrightarrow Stat11(Stat10\star) \Rightarrow \quad e_2 = e_0$
EQUAL $\langle Stat3\rangle \Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow \quad Stat13:\ eq \notin \{e \backslash \{v_0\} : e \in g_0 \mid \mathsf{CardAtLeast2}(e \backslash \{v_0\})\}\ \&\ eq \notin \{e \in g_0 \mid v_0 \in e\}\ \&\ eq \in g_0 \backslash \{e_0\} \cup \{e_0 \backslash \{v_0\}\}$
Suppose $\Rightarrow \quad eq = e_0 \backslash \{v_0\}$
$\langle e_0\rangle \hookrightarrow Stat13(Stat2\star) \Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow \quad$ AUTO
$\langle eq, eq\rangle \hookrightarrow Stat13(Stat13\star) \Rightarrow \quad eq \backslash \{v_0\} = eq\ \&\ \neg \mathsf{CardAtLeast2}(eq \backslash \{v_0\})$
$\langle g_0, eq, \emptyset\rangle \hookrightarrow Thgraph_2(Stat2) \Rightarrow \quad \mathsf{CardAtLeast2}(eq)$
EQUAL $\langle Stat13\rangle \Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow \quad Stat14:\ \mathsf{filter}(g_0, v_0) = g_0 \backslash \{e_0\} \cup \{e_0 \backslash \{v_0\}\}$
$\langle g_0, v_0\rangle \hookrightarrow Thgraph_8 \Rightarrow \quad$ AUTO
Use_def(Conn) $\Rightarrow \quad Stat15:\ \{p \subseteq \mathsf{filter}(g_0, v_0) \mid \mathsf{nodes}(p) \cap \mathsf{nodes}(\mathsf{filter}(g_0, v_0) \backslash p) = \emptyset\} \not\subseteq \{\emptyset, \mathsf{filter}(g_0, v_0)\}$
$\langle p_0\rangle \hookrightarrow Stat15(Stat15\star) \Rightarrow \quad Stat16:\ p_0 \in \{p \subseteq \mathsf{filter}(g_0, v_0) \mid \mathsf{nodes}(p) \cap \mathsf{nodes}(\mathsf{filter}(g_0, v_0) \backslash p) = \emptyset\}\ \&\ p_0 \neq \emptyset\ \&\ p_0 \neq \mathsf{filter}(g_0, v_0)$
$\langle\,\rangle \hookrightarrow Stat16(Stat16\star) \Rightarrow \quad p_0 \subseteq \mathsf{filter}(g_0, v_0)\ \&\ \mathsf{nodes}(p_0) \cap \mathsf{nodes}(\mathsf{filter}(g_0, v_0) \backslash p_0) = \emptyset$

Loc_def ⇒ $p_1 = \textbf{if } e_0 \backslash \{v_0\} \in p_0 \textbf{ then } filter(g_0, v_0) \backslash p_0 \textbf{ else } p_0 \textbf{ fi}$
Suppose ⇒ $nodes(p_1) \cap nodes(filter(g_0, v_0) \backslash p_1) \neq \emptyset$
Suppose ⇒ $p_1 = p_0$
EQUAL $\langle Stat16 \rangle$ ⇒ false; Discharge ⇒ AUTO
$(Stat16\star)$ELEM ⇒ $p_1 = filter(g_0, v_0) \backslash p_0 \;\&\; filter(g_0, v_0) \backslash p_1 = p_0$
EQUAL $\langle Stat16 \rangle$ ⇒ $nodes(filter(g_0, v_0) \backslash p_0) \cap nodes(p_0) \neq \emptyset$
$(Stat16\star)$Discharge ⇒ AUTO
$(Stat16\star)$ELEM ⇒ $Stat17: \; p_1 \neq \emptyset \;\&\; p_1 \neq filter(g_0, v_0) \;\&\; p_1 \subseteq filter(g_0, v_0) \;\&\; nodes(p_1) \cap nodes(filter(g_0, v_0) \backslash p_1) = \emptyset$
Suppose ⇒ $v_0 \in nodes(g_0 \backslash \{e_0\})$
Loc_def ⇒ $e_4 = edgeOf(v_0, g_0 \backslash \{e_0\})$
$\langle v_0, g_0 \backslash \{e_0\} \rangle \hookrightarrow Thgraph_e(Stat17\star)$ ⇒ $Stat18: \; v_0 \in e_4 \;\&\; e_4 \in g_0 \backslash \{e_0\}$
$(Stat1, Stat18\star)$ELEM ⇒ $Stat19: \; e_4 \notin \{e \in g_0 \mid v_0 \in e\}$
$\langle e_4 \rangle \hookrightarrow Stat19(Stat18\star)$ ⇒ false; Discharge ⇒ AUTO
Suppose ⇒ $nodes(filter(g_0, v_0) \backslash p_1) \neq nodes(g_0 \backslash p_1) \backslash \{v_0\}$
$(Stat2, Stat2\star)$ELEM ⇒ $e_0 \in g_0 \;\&\; v_0 \in e_0$
$(Stat14\star)$ELEM ⇒ $g_0 \backslash \{e_0\} \cup \{e_0 \backslash \{v_0\}\} \backslash p_1 = g_0 \backslash \{e_0\} \backslash p_1 \cup \{e_0 \backslash \{v_0\}\} \;\&\; g_0 \backslash \{e_0\} \backslash p_1 \cup \{e_0\} = g_0 \backslash p_1$
$\langle g_0 \backslash \{e_0\} \backslash p_1, \{e_0 \backslash \{v_0\}\} \rangle \hookrightarrow Thgraph_b$ ⇒ AUTO
$\langle e_0 \backslash \{v_0\} \rangle \hookrightarrow Thgraph_d$ ⇒ AUTO
$\langle e_0 \rangle \hookrightarrow Thgraph_d$ ⇒ AUTO
$\langle g_0 \backslash \{e_0\} \backslash p_1, g_0 \backslash \{e_0\} \rangle \hookrightarrow Thgraph_a(Stat17\star)$ ⇒ $nodes(g_0 \backslash \{e_0\} \backslash p_1) \cup (nodes(\{e_0\}) \backslash \{v_0\}) = nodes(g_0 \backslash \{e_0\} \backslash p_1) \cup nodes(\{e_0\}) \backslash \{v_0\}$
$\langle g_0 \backslash \{e_0\} \backslash p_1, \{e_0\} \rangle \hookrightarrow Thgraph_b$ ⇒ AUTO
EQUAL $\langle Stat14 \rangle$ ⇒ false; Discharge ⇒ AUTO
$\langle p_1, filter(g_0, v_0) \rangle \hookrightarrow Thgraph_a(Stat3\star)$ ⇒ $nodes(p_1) \cap nodes(g_0 \backslash p_1) = \emptyset$
$(Stat2\star)$ELEM ⇒ $Stat20: \; p_1 \notin \{p \subseteq g_0 \mid nodes(p) \cap nodes(g_0 \backslash p) = \emptyset\} \;\&\; p_1 \subseteq g_0 \;\&\; p_1 \neq g_0$
$\langle p_1 \rangle \hookrightarrow Stat20(Stat17\star)$ ⇒ false; Discharge ⇒ QED

> The following proposition reflects an initial remark, or 'prelude', inside the proof of Theorem 2 of [CO14].

THEOREM $hgraph_{21}$: [Prelude to Theorem $hgraph_{22}$] $Conn(G) \;\&\; W \in nodes(G) \;\&\; \neg\langle \exists v \in nodes(G) \mid Conn(filter(G, v)) \rangle \rightarrow$
$lost(G, W) = \emptyset \;\&\; nodes(G) = nodes(filter(G, W)) \cup \{W\}$. PROOF:
Suppose_not(g_0, w_0) ⇒ AUTO
Use_def(Conn) ⇒ $Stat1: \; \neg\langle \exists v \in nodes(g_0) \mid Conn(filter(g_0, v)) \rangle \;\&\; HGraph(g_0) \;\&$
$\{p \subseteq g_0 \mid nodes(p) \cap nodes(g_0 \backslash p) = \emptyset\} \subseteq \{\emptyset, g_0\} \;\&\; w_0 \in nodes(g_0) \;\&\; \neg\Big(lost(g_0, w_0) = \emptyset \;\&\; nodes(g_0) = nodes(filter(g_0, w_0)) \cup \{w_0\}\Big)$
Suppose ⇒ $Stat2: \; \langle \exists v \mid lost(g_0, v) \neq \emptyset \rangle$
$\langle v_0 \rangle \hookrightarrow Stat2(Stat2\star)$ ⇒ $Stat3: \; lost(g_0, v_0) \neq \emptyset$
$\langle v_1 \rangle \hookrightarrow Stat3(Stat3\star)$ ⇒ $v_1 \in lost(g_0, v_0)$
$\langle g_0, v_1, v_0 \rangle \hookrightarrow Thgraph_{18}(Stat1\star)$ ⇒ $contains(g_0, v_1) = \{\{v_0, v_1\}\}$
$\langle g_0, v_1, \{v_0, v_1\} \rangle \hookrightarrow Thgraph_{20}(\star)$ ⇒ $Conn(filter(g_0, v_1))$

$\mathsf{Use\_def}(\mathsf{lost}(g_0, v_0)) \Rightarrow$ AUTO
$\langle v_1 \rangle \hookrightarrow Stat1(Stat3\star) \Rightarrow$ false; Discharge $\Rightarrow$ $Stat4:\ \neg\langle \exists v \mid \mathsf{lost}(g_0, v) \neq \emptyset \rangle$
$\langle w_0 \rangle \hookrightarrow Stat4(Stat4\star) \Rightarrow$ $\mathsf{lost}(g_0, w_0) = \emptyset$
Suppose $\Rightarrow$ $Stat5:\ \neg\langle \forall v \in \mathsf{nodes}(g_0) \mid \mathsf{nodes}(g_0) = \mathsf{nodes}(\mathsf{filter}(g_0, v)) \cup \{v\} \rangle$
$\langle v_3 \rangle \hookrightarrow Stat5(Stat5\star) \Rightarrow$ $v_3 \in \mathsf{nodes}(g_0)\ \&\ \mathsf{nodes}(g_0) \neq \mathsf{nodes}(\mathsf{filter}(g_0, v_3)) \cup \{v_3\}$
$\mathsf{Use\_def}(\mathsf{lost}(g_0, v_3)) \Rightarrow$ AUTO
$\langle g_0, v_3 \rangle \hookrightarrow \mathsf{Thgraph}_8 \Rightarrow$ AUTO
$\langle v_3 \rangle \hookrightarrow Stat4(Stat4\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO
$(Stat4\star)$ELEM $\Rightarrow$ $Stat6:\ \langle \forall v \in \mathsf{nodes}(g_0) \mid \mathsf{nodes}(g_0) = \mathsf{nodes}(\mathsf{filter}(g_0, v)) \cup \{v\} \rangle$
$\langle w_0 \rangle \hookrightarrow Stat6(Stat1\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

‖ The following proposition corresponds to Theorem 2 of [CO14].

THEOREM hgraph_{22}: [Every non-null connected graph has a vertex whose removal does not disrupt connectivity] $\mathsf{Conn}(G)\ \&\ G \neq \emptyset \rightarrow \langle \exists v \in \mathsf{nodes}(G) \mid \mathsf{Conn}(\mathsf{filter}(G, v)) \rangle$. PROOF:
$\mathsf{Suppose\_not}(g_0) \Rightarrow$ AUTO

‖ Arguing by contradiction, suppose that a connected non-null graph g_0 exists such that $\mathsf{Conn}(\mathsf{filter}(g_0, v))$ does not node for any of its nodes v. Recall that g_0 is finite and has a non-null set of nodes.

$\mathsf{Use\_def}(\mathsf{Conn}) \Rightarrow$ $Stat1:\ \neg\langle \exists v \in \mathsf{nodes}(g_0) \mid \mathsf{Conn}(\mathsf{filter}(g_0, v)) \rangle\ \&\ \mathsf{HGraph}(g_0)\ \&\ \{p \subseteq g_0 \mid \mathsf{nodes}(p) \cap \mathsf{nodes}(g_0 \backslash p) = \emptyset\} \subseteq \{\emptyset, g_0\}\ \&\ g_0 \neq \emptyset$
$\langle g_0 \rangle \hookrightarrow \mathsf{Thgraph}_1(Stat1\star) \Rightarrow$ $Stat2:\ \mathsf{nodes}(g_0) \neq \emptyset\ \&\ \mathsf{Finite}(g_0)$

‖ Consider the set $\{p \subseteq g_0 \mid \mathsf{Conn}(p)\ \&\ \langle \exists v \in \mathsf{nodes}(g_0) \mid p \subseteq \mathsf{filter}(g_0, v) \rangle\}$ of all connected subgraphs of g_0 which are included in $\mathsf{filter}(g_0, v)$ for some node v of g_0. This set is non-null, because $\emptyset$ belongs to it.

Suppose $\Rightarrow$ $Stat3:\ \{p \subseteq g_0 \mid \mathsf{Conn}(p)\ \&\ \langle \exists v \in \mathsf{nodes}(g_0) \mid p \subseteq \mathsf{filter}(g_0, v) \rangle\} = \emptyset$

‖ In fact, $\emptyset$ is a connected graph. Moreover, it is included in g_0 and satisfies whichever inclusion of the form $\emptyset \subseteq X$.

$\langle \emptyset \rangle \hookrightarrow \mathsf{Thgraph}_{16}(Stat3\star) \Rightarrow$ $\mathsf{Conn}(\emptyset)$
$\langle \emptyset \rangle \hookrightarrow Stat3(Stat3\star) \Rightarrow$ $Stat4:\ \neg\langle \exists v \in \mathsf{nodes}(g_0) \mid \emptyset \subseteq \mathsf{filter}(g_0, v) \rangle$
$\langle v_0 \rangle \hookrightarrow Stat2(Stat4\star) \Rightarrow$ $v_0 \in \mathsf{nodes}(g_0)$
$\langle v_0 \rangle \hookrightarrow Stat4(Stat4\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO

‖ The finiteness of g_0 yields the finiteness of $\mathcal{P}g_0$ and, consequently, the finiteness of any subset of $\mathcal{P}g_0$: in particular, the set of subgraphs of g_0 introduced above is finite; therefore it has an inclusion-maximal element m_0.

$\langle g_0 \rangle \hookrightarrow T\mathsf{fin}_5(\mathit{Stat2}\star) \Rightarrow$ $\mathsf{Finite}(\mathcal{P} g_0)$
Set_monot $\Rightarrow$ $\{p \subseteq g_0 \mid \mathsf{Conn}(p) \,\&\, \langle \exists v \in \mathsf{nodes}(g_0) \mid p \subseteq \mathsf{filter}(g_0, v) \rangle \} \subseteq \{x : x \subseteq g_0\}$
Use_def($\mathcal{P}$) $\Rightarrow$ $\mathsf{Finite}(\{p \subseteq g_0 \mid \mathsf{Conn}(p) \,\&\, \langle \exists v \in \mathsf{nodes}(g_0) \mid p \subseteq \mathsf{filter}(g_0, v) \rangle \})$
$\langle \{p \subseteq g_0 \mid \mathsf{Conn}(p) \,\&\, \langle \exists v \in \mathsf{nodes}(g_0) \mid p \subseteq \mathsf{filter}(g_0, v) \rangle \} \rangle \hookrightarrow T\mathsf{fin}_4(\mathit{Stat2}\star) \Rightarrow$
$\mathit{Stat5}: \langle \exists m \in \{p \subseteq g_0 \mid \mathsf{Conn}(p) \,\&\, \langle \exists v \in \mathsf{nodes}(g_0) \mid p \subseteq \mathsf{filter}(g_0, v) \rangle \} \mid \{x \in \{p \subseteq g_0 \mid \mathsf{Conn}(p) \,\&\, \langle \exists v \in \mathsf{nodes}(g_0) \mid p \subseteq \mathsf{filter}(g_0, v) \rangle \} \setminus \{m\} \mid m \subseteq x\} = \emptyset \rangle$

That is, the relationship $m_0 \in \{p \subseteq g_0 \mid \mathsf{Conn}(p) \,\&\, \langle \exists v \in \mathsf{nodes}(g_0) \mid p \subseteq \mathsf{filter}(g_0, v) \rangle \}$ holds, and no alike relation holds for a set m strictly included in m_0. Plainly, m_0 is a connected subgraph of g_0 and there is a node w_0 such that m_0 is also a subgraph of $\mathsf{filter}(g_0, v)$.

$\langle m_0 \rangle \hookrightarrow \mathit{Stat5}(\mathit{Stat5}\star) \Rightarrow$ $\mathit{Stat7}: m_0 \in \{p \subseteq g_0 \mid \mathsf{Conn}(p) \,\&\, \langle \exists v \in \mathsf{nodes}(g_0) \mid p \subseteq \mathsf{filter}(g_0, v) \rangle \} \,\&$
$\mathit{Stat8}: \{x \in \{p \subseteq g_0 \mid \mathsf{Conn}(p) \,\&\, \langle \exists v \in \mathsf{nodes}(g_0) \mid p \subseteq \mathsf{filter}(g_0, v) \rangle \} \setminus \{m_0\} \mid m_0 \subseteq x\} = \emptyset$
$\langle\ \rangle \hookrightarrow \mathit{Stat7}(\mathit{Stat7}\star) \Rightarrow$ $\mathit{Stat9}: \langle \exists v \in \mathsf{nodes}(g_0) \mid m_0 \subseteq \mathsf{filter}(g_0, v) \rangle \,\&\, m_0 \subseteq g_0 \,\&\, \mathsf{Conn}(m_0)$
$\langle w_0 \rangle \hookrightarrow \mathit{Stat9}(\mathit{Stat9}\star) \Rightarrow$ $w_0 \in \mathsf{nodes}(g_0) \,\&\, m_0 \subseteq g_0 \cap \mathsf{filter}(g_0, w_0)$

Notice that $m_0 = \emptyset$ cannot hold; for, otherwise, it would turn out that $\mathsf{filter}(g_0, v) \cap g_0 = \emptyset$ for every node v. But then, by Theorem hgraph_{17}, each graph of the form $\mathsf{filter}(g_0, v)$, with v a node of g_0, would be connected; however, we are working under the assumption that no such graph is connected and therefore must discard the possibility that $m_0 = \emptyset$.

Suppose $\Rightarrow$ $\mathit{Stat10}: m_0 = \emptyset$
Suppose $\Rightarrow$ $\mathit{Stat11}: \neg \langle \forall v \in \mathsf{nodes}(g_0) \mid \mathsf{filter}(g_0, v) \cap g_0 = \emptyset \rangle$
$\langle w_2 \rangle \hookrightarrow \mathit{Stat11}(\mathit{Stat11}\star) \Rightarrow$ $\mathit{Stat12}: \mathsf{filter}(g_0, w_2) \cap g_0 \neq \emptyset \,\&\, w_2 \in \mathsf{nodes}(g_0)$
$\langle e_1 \rangle \hookrightarrow \mathit{Stat12}(\mathit{Stat12}\star) \Rightarrow$ $\mathit{Stat13}: e_1 \in \mathsf{filter}(g_0, w_2) \,\&\, e_1 \in g_0$
$\langle g_0, \{e_1\} \rangle \hookrightarrow \mathsf{Thgraph}_3(\mathit{Stat1}, \mathit{Stat13}\star) \Rightarrow$ $\mathsf{HGraph}(\{e_1\})$
$\langle e_1, \{e_1\} \rangle \hookrightarrow \mathsf{Thgraph}_{16}(\mathit{Stat13}\star) \Rightarrow$ $\mathsf{Conn}(\{e_1\})$
$\langle e_1 \rangle \hookrightarrow \mathit{Stat8}(\mathit{Stat7}, \mathit{Stat7}\star) \Rightarrow$ $\mathit{Stat14}: \{e_1\} \notin \{x \in \{p \subseteq g_0 \mid \mathsf{Conn}(p) \,\&\, \langle \exists v \in \mathsf{nodes}(g_0) \mid p \subseteq \mathsf{filter}(g_0, v) \rangle \} \setminus \{m_0\} \mid m_0 \subseteq x\}$
$\langle \{e_1\} \rangle \hookrightarrow \mathit{Stat14}(\mathit{Stat10}\star) \Rightarrow$ $\mathit{Stat15}: \{e_1\} \notin \{p \subseteq g_0 \mid \mathsf{Conn}(p) \,\&\, \langle \exists v \in \mathsf{nodes}(g_0) \mid p \subseteq \mathsf{filter}(g_0, v) \rangle \}$
$\langle \{e_1\} \rangle \hookrightarrow \mathit{Stat15}(\mathit{Stat13}\star) \Rightarrow$ $\mathit{Stat16}: \neg \langle \exists v \in \mathsf{nodes}(g_0) \mid \{e_1\} \subseteq \mathsf{filter}(g_0, v) \rangle$
$\langle w_2 \rangle \hookrightarrow \mathit{Stat16}(\mathit{Stat12}\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO
$\langle g_0 \rangle \hookrightarrow \mathsf{Thgraph}_{17}(\mathit{Stat1}\star) \Rightarrow$ $\mathit{Stat17}: \langle \forall v \in \mathsf{nodes}(g_0) \mid \mathsf{Conn}(\mathsf{filter}(g_0, v)) \rangle$
$\langle v_6 \rangle \hookrightarrow \mathit{Stat2}(\mathit{Stat2}, \mathit{Stat2}\star) \Rightarrow$ $v_6 \in \mathsf{nodes}(g_0)$
$\langle v_6 \rangle \hookrightarrow \mathit{Stat17}(\mathit{Stat17}\star) \Rightarrow$ $\mathsf{Conn}(\mathsf{filter}(g_0, v_6))$
$\langle v_6 \rangle \hookrightarrow \mathit{Stat1}(\mathit{Stat17}\star) \Rightarrow$ false; Discharge $\Rightarrow$ $\mathit{Stat18}:$ AUTO

Since $\mathsf{HGraph}(\mathsf{filter}(g_0, w_0))$ is not connected, $\mathsf{filter}(g_0, w_0)$ has a strict, non-null subgraph q_0 that shares no nodes with its complementary subgraph of $\mathsf{filter}(g_0, w_0)$. Either q_0 or its complement must include m_0 by Theorem hgraph_{13}, so let q_1 be the one of these two graphs which has m_0 as a subgraph.

$\langle g_0, w_0 \rangle \hookrightarrow Thgraph_8 \;\Rightarrow\;$ AUTO
$\mathsf{Use\_def}\Big(\mathsf{Conn}(\mathsf{filter}(g_0, w_0))\Big) \Rightarrow$ AUTO
$\langle w_0 \rangle \hookrightarrow Stat1(Stat9\star) \Rightarrow$ $Stat31: \; \{p \subseteq \mathsf{filter}(g_0, w_0) \mid \mathsf{nodes}(p) \cap \mathsf{nodes}(\mathsf{filter}(g_0, w_0) \backslash p) = \emptyset\} \not\subseteq \{\emptyset, \mathsf{filter}(g_0, w_0)\}$
$\langle q_0 \rangle \hookrightarrow Stat31(Stat31\star) \Rightarrow$ $Stat32: \; q_0 \in \{p \subseteq \mathsf{filter}(g_0, w_0) \mid \mathsf{nodes}(p) \cap \mathsf{nodes}(\mathsf{filter}(g_0, w_0) \backslash p) = \emptyset\} \;\&\; q_0 \notin \{\emptyset, \mathsf{filter}(g_0, w_0)\}$
$\langle\, \rangle \hookrightarrow Stat32(Stat32\star) \Rightarrow$ $Stat33: \; q_0 \subseteq \mathsf{filter}(g_0, w_0) \;\&\; \mathsf{nodes}(q_0) \cap \mathsf{nodes}(\mathsf{filter}(g_0, w_0) \backslash q_0) = \emptyset$
$\langle q_0, \mathsf{filter}(g_0, w_0), m_0 \rangle \hookrightarrow Thgraph_{13}(Stat9\star) \Rightarrow$ $m_0 \subseteq \mathsf{filter}(g_0, w_0) \backslash q_0 \vee m_0 \subseteq q_0$
$\mathsf{Loc\_def} \Rightarrow$ $q_1 = \textbf{if } m_0 \subseteq q_0 \textbf{ then } q_0 \textbf{ else } \mathsf{filter}(g_0, w_0) \backslash q_0 \textbf{ fi}$
$(Stat32\star)\mathsf{ELEM} \Rightarrow$ $Stat35: \; q_1 \subseteq \mathsf{filter}(g_0, w_0) \;\&\; q_1 \neq \mathsf{filter}(g_0, w_0) \;\&\; m_0 \subseteq q_1$
Suppose $\Rightarrow$ $\mathsf{nodes}(q_1) \cap \mathsf{nodes}(\mathsf{filter}(g_0, w_0) \backslash q_1) \neq \emptyset$
Suppose $\Rightarrow$ $q_1 = q_0$
EQUAL $\langle Stat33 \rangle \Rightarrow$ false; Discharge $\Rightarrow$ AUTO
$(Stat33\star)\mathsf{ELEM} \Rightarrow$ $q_1 = \mathsf{filter}(g_0, w_0) \backslash q_0 \;\&\; q_0 = \mathsf{filter}(g_0, w_0) \backslash q_1 \;\&\; \mathsf{nodes}(\mathsf{filter}(g_0, w_0) \backslash q_1) \cap \mathsf{nodes}(q_1) \neq \emptyset$
EQUAL $\langle Stat33 \rangle \Rightarrow$ false; Discharge $\Rightarrow$ AUTO

Plainly $w_0 \notin \mathsf{nodes}(q_1)$ because $w_0 \notin \mathsf{nodes}(\mathsf{filter}(g_0, w_0))$. Therefore, by Theorem hgraph_9, we have $g_0 = \mathsf{cov}(g_0, q_1) \cup \mathsf{cov}(g_0, \mathsf{filter}(g_0, w_0) \backslash q_1)$. Moreover, $\mathsf{lost}(g_0, w_0) = \emptyset$ holds by Theorem hgraph_{21}, and $\mathsf{cov}(g_0, q_1) \cap \mathsf{cov}(g_0, \mathsf{filter}(g_0, w_0) \backslash q_1) = \emptyset$ holds by Theorem hgraph_{11}. It readily follows that $\mathsf{cov}(g_0, \mathsf{filter}(g_0, w_0) \backslash q_1) = g_0 \backslash \mathsf{cov}(g_0, q_1)$. Also, $\mathsf{nodes}(\mathsf{cov}(g_0, q_1)) \cap \mathsf{nodes}\Big(\mathsf{cov}(g_0, \mathsf{filter}(g_0, w_0) \backslash q_1)\Big) = \{w_0\}$ follows from Theorem hgraph_{12}.

$\langle q_1, \mathsf{filter}(g_0, w_0) \rangle \hookrightarrow Thgraph_a(Stat18\star) \Rightarrow$ $Stat36: \; w_0 \notin \mathsf{nodes}(q_1)$
$\langle g_0, w_0, q_1 \rangle \hookrightarrow Thgraph_9 \;\Rightarrow\;$ AUTO
$\langle g_0, w_0 \rangle \hookrightarrow Thgraph_{21}(\star) \Rightarrow$ $Stat37: \; \mathsf{cov}(g_0, q_1) \cup \mathsf{cov}(g_0, \mathsf{filter}(g_0, w_0) \backslash q_1) = g_0 \;\&\; \mathsf{lost}(g_0, w_0) = \emptyset \;\&\; \mathsf{HGraph}(g_0) \;\&\; \mathsf{Conn}(g_0) \;\&\; w_0 \in \mathsf{nodes}(g_0)$
$\langle g_0, q_1, w_0 \rangle \hookrightarrow Thgraph_{11}(Stat32\star) \Rightarrow$ $\mathsf{cov}(g_0, q_1) \cap \mathsf{cov}(g_0, \mathsf{filter}(g_0, w_0) \backslash q_1) = \emptyset$
$(Stat37\star)\mathsf{ELEM} \Rightarrow$ $\mathsf{cov}(g_0, \mathsf{filter}(g_0, w_0) \backslash q_1) = g_0 \backslash \mathsf{cov}(g_0, q_1)$
$\langle g_0, q_1, w_0 \rangle \hookrightarrow Thgraph_{12}(Stat18\star) \Rightarrow$ $\mathsf{nodes}(\mathsf{cov}(g_0, q_1)) \cap \mathsf{nodes}\Big(\mathsf{cov}(g_0, \mathsf{filter}(g_0, w_0) \backslash q_1)\Big) = \{w_0\}$

We then get $\mathsf{Conn}(\mathsf{cov}(g_0, q_1))$ by means of Theorem hgraph_{14}.

EQUAL $\langle Stat37 \rangle \Rightarrow$ $\mathsf{Conn}\Big(\mathsf{cov}(g_0, q_1) \cup \mathsf{cov}(g_0, \mathsf{filter}(g_0, w_0) \backslash q_1)\Big)$
$\langle \mathsf{cov}(g_0, q_1), \mathsf{cov}(g_0, \mathsf{filter}(g_0, w_0) \backslash q_1), w_0 \rangle \hookrightarrow Thgraph_{14}(\star) \Rightarrow$ $\mathsf{Conn}(\mathsf{cov}(g_0, q_1))$

This connected set $\mathsf{cov}(g_0, q_1)$ will turn out to contradict the minimality of m_0. In the first place $\mathsf{cov}(g_0, q_1)$ includes m_0, by Theorem hgraph_3, and this inclusion is strict, because w_0 belongs to the nodes of $\mathsf{cov}(g_0, q_1)$ but does not belong to the nodes of m_0.

$\langle g_0, q_1 \rangle \hookrightarrow Thgraph_5(Stat9\star) \Rightarrow \quad m_0 \subseteq cov(g_0, q_1)$
Suppose $\Rightarrow \quad m_0 = cov(g_0, q_1)$
EQUAL $\langle Stat37 \rangle \Rightarrow \quad w_0 \in nodes(m_0) \leftrightarrow w_0 \in nodes(cov(g_0, q_1))$
$\langle m_0, q_1 \rangle \hookrightarrow Thgraph_a(Stat35\star) \Rightarrow \quad$ false; Discharge $\Rightarrow$ AUTO

In the second place, the graph $g_0 \backslash cov(g_0, q_1)$ (which equals $cov(g_0, filter(g_0, w_0) \backslash q_1)$ as already seen, is easily shown to be non-null by means of Theorem $hgraph_6$.

Suppose $\Rightarrow \quad Stat38: \ cov(g_0, filter(g_0, w_0) \backslash q_1) = \emptyset$
$\langle g_0, w_0 \rangle \hookrightarrow Thgraph_8 \Rightarrow$ AUTO
$\langle g_0, filter(g_0, w_0) \backslash q_1, filter(g_0, w_0) \backslash q_1 \rangle \hookrightarrow Thgraph_7(Stat38\star) \Rightarrow \quad \neg\big(nodes(g_0) \supseteq nodes(filter(g_0, w_0) \backslash q_1) \ \& \ HGraph(filter(g_0, w_0) \backslash q_1) \ \& \ filter(g_0, w_0) \backslash q_1 \neq \emptyset\big)$
Suppose $\Rightarrow \quad \neg HGraph(filter(g_0, w_0) \backslash q_1)$
$\langle filter(g_0, w_0), filter(g_0, w_0) \backslash q_1 \rangle \hookrightarrow Thgraph_3(Stat38\star) \Rightarrow \quad \neg HGraph(filter(g_0, w_0))$
$(Stat37\star)$Discharge $\Rightarrow$ AUTO
$\langle filter(g_0, w_0) \backslash q_1, filter(g_0, w_0) \rangle \hookrightarrow Thgraph_a(Stat35\star) \Rightarrow \quad$ false; Discharge $\Rightarrow$ AUTO
$\langle g_0, g_0 \backslash cov(g_0, q_1) \rangle \hookrightarrow Thgraph_3(Stat37\star) \Rightarrow \quad Stat39: \ HGraph(g_0 \backslash cov(g_0, q_1)) \ \& \ g_0 \backslash cov(g_0, q_1) \neq \emptyset$

Thirdly, we can find a node w_1 of g_0 such that $w_1 \neq w_0$ and $w_1 \in nodes(g_0 \backslash cov(g_0, q_1))$; then, in view of Theorem $hgraph_{15}$, we get $cov(g_0, q_1) \subseteq filter(g_0, w_1)$.

$\langle e_0 \rangle \hookrightarrow Stat39(Stat39\star) \Rightarrow \quad e_0 \in g_0 \ \& \ e_0 \notin cov(g_0, q_1)$
$\langle g_0, e_0, w_0 \rangle \hookrightarrow Thgraph_2(Stat37\star) \Rightarrow \quad Stat40: \ e_0 \not\subseteq \{w_0\}$
$\langle w_1 \rangle \hookrightarrow Stat40(Stat40\star) \Rightarrow \quad w_1 \in e_0 \ \& \ w_1 \neq w_0$
$\langle g_0 \backslash cov(g_0, q_1), e_0, w_1 \rangle \hookrightarrow Thgraph_2(Stat39\star) \Rightarrow \quad w_1 \in nodes(g_0 \backslash cov(g_0, q_1))$
$\langle g_0, q_1 \rangle \hookrightarrow Thgraph_5(Stat37, Stat37\star) \Rightarrow \quad g_0 = cov(g_0, q_1) \cup (g_0 \backslash cov(g_0, q_1))$
EQUAL $\langle Stat37 \rangle \Rightarrow \quad nodes(cov(g_0, q_1)) \cap nodes(g_0 \backslash cov(g_0, q_1)) = \{w_0\} \ \&$
$HGraph\Big(cov(g_0, q_1) \cup (g_0 \backslash cov(g_0, q_1))\Big) \ \& \ filter\Big(cov(g_0, q_1) \cup (g_0 \backslash cov(g_0, q_1)), w_1\Big) = filter(g_0, w_1)$
$\langle cov(g_0, q_1), g_0 \backslash cov(g_0, q_1), w_0, w_1 \rangle \hookrightarrow Thgraph_{15}(Stat40\star) \Rightarrow \quad cov(g_0, q_1) \subseteq filter(g_0, w_1)$

Taken together, these properties show that $cov(g_0, q_1)$ violates the minimality of m_0. This contradiction leads us to the desired conclusion.

$\langle cov(g_0, q_1) \rangle \hookrightarrow Stat8(Stat7, Stat7\star) \Rightarrow \quad Stat41: \ cov(g_0, q_1) \notin \{x \in \{p \subseteq g_0 \mid Conn(p) \ \& \ \langle \exists v \in nodes(g_0) \mid p \subseteq filter(g_0, v) \rangle\} \setminus \{m_0\} \mid m_0 \subseteq x\}$
$\langle cov(g_0, q_1) \rangle \hookrightarrow Stat41(Stat37\star) \Rightarrow \quad Stat42: \ cov(g_0, q_1) \notin \{p \subseteq g_0 \mid Conn(p) \ \& \ \langle \exists v \in nodes(g_0) \mid p \subseteq filter(g_0, v) \rangle\}$
$\langle cov(g_0, q_1) \rangle \hookrightarrow Stat42(Stat37\star) \Rightarrow \quad Stat43: \ \neg\langle \exists v \in nodes(g_0) \mid cov(g_0, q_1) \subseteq filter(g_0, v) \rangle$
$\langle g_0 \backslash cov(g_0, q_1), g_0 \rangle \hookrightarrow Thgraph_a(Stat43\star) \Rightarrow \quad nodes(g_0 \backslash cov(g_0, q_1)) \subseteq nodes(g_0)$
$\langle w_1 \rangle \hookrightarrow Stat43(Stat40\star) \Rightarrow \quad$ false; Discharge $\Rightarrow$ QED

3.6 Cut vertices; existence of a non-cut vertex in any nonnull connected hypergraph

DEF hgraph_8: [Disconnecting vertex] $\mathsf{Cutting}(G, V) \leftrightarrow_{\mathrm{Def}} G \neq \{\mathsf{edgeOf}(V, G)\} \ \& \ \neg\Big(\mathsf{Conn}\big(\mathsf{filter}(G, V)\big) \ \& \ \mathsf{lost}(G, V) = \emptyset\Big)$

‖ The following proposition corresponds to Lemma 5 of [CO14].

THEOREM hgraph_{23}: [A node that occurs in exactly one edge is not a cut vertex] $\mathsf{Conn}(G) \ \& \ \mathsf{contains}(G, V) = \{E\} \rightarrow \neg\mathsf{Cutting}(G, V)$. PROOF:

$\mathsf{Suppose\_not}(g_0, v_0, e_0) \Rightarrow$ AUTO

$\mathsf{Use\_def}(\mathsf{Cutting}) \Rightarrow$ *Stat1*: $\mathsf{Conn}(g_0) \ \& \ g_0 \neq \{\mathsf{edgeOf}(v_0, g_0)\} \ \& \ \neg\Big(\mathsf{Conn}\big(\mathsf{filter}(g_0, v_0)\big) \ \& \ \mathsf{lost}(g_0, v_0) = \emptyset\Big) \ \& \ \mathsf{contains}(g_0, v_0) = \{e_0\}$

$\langle g_0, v_0, e_0\rangle \hookrightarrow T\mathsf{hgraph}_{20}(\mathit{Stat1}\star) \Rightarrow$ *Stat2*: $\mathsf{lost}(g_0, v_0) \neq \emptyset$

$\mathsf{Use\_def}(\mathsf{Conn}(g_0)) \Rightarrow$ AUTO

$\langle g_0, w_0, v_0\rangle \hookrightarrow T\mathsf{hgraph}_{18} \Rightarrow$ AUTO

$\langle w_0\rangle \hookrightarrow \mathit{Stat2}(\mathit{Stat1}\star) \Rightarrow \mathsf{contains}(g_0, w_0) = \{\{v_0, w_0\}\}$

$\mathsf{Suppose} \Rightarrow e_0 \neq \{v_0, w_0\}$

$\mathsf{Use\_def}(\mathsf{contains}) \Rightarrow$ *Stat3*: $\{v_0, w_0\} \in \{e : e \in g_0 \mid w_0 \in e\} \ \& \ \{v_0, w_0\} \notin \{e \in g_0 \mid v_0 \in e\}$

$\langle e_1, e_1\rangle \hookrightarrow \mathit{Stat3}(\mathit{Stat3}\star) \Rightarrow \mathsf{false};$ $\mathsf{Discharge} \Rightarrow$ *Stat4*: $e_0 = \{v_0, w_0\}$

$\mathsf{Suppose} \Rightarrow \mathsf{cov}(g_0, \{e_0\}) \neq \{e_0\}$

$\langle g_0, \{e_0\}\rangle \hookrightarrow T\mathsf{hgraph}_3 \Rightarrow$ AUTO

$\langle g_0, v_0\rangle \hookrightarrow T\mathsf{hgraph}_8(\mathit{Stat1}\star) \Rightarrow \mathsf{HGraph}(\{e_0\})$

$\langle \{e_0\}, g_0\rangle \hookrightarrow T\mathsf{hgraph}_4(\mathit{Stat4}\star) \Rightarrow \{a : e \in \{e_0\}, v \in e, a \in \mathsf{contains}(g_0, v)\} \neq \{\{v_0, w_0\}\}$

$\mathsf{SIMPLF} \Rightarrow$ *Stat5*: $\{a : v \in e_0, a \in \mathsf{contains}(g_0, v)\} \neq \{\{v_0, w_0\}\}$

$\mathsf{Suppose} \Rightarrow$ *Stat6*: $\{v_0, w_0\} \notin \{a : v \in e_0, a \in \mathsf{contains}(g_0, v)\}$

$\langle w_0, \{v_0, w_0\}\rangle \hookrightarrow \mathit{Stat6}(\mathit{Stat2}\star) \Rightarrow \mathsf{false};$ $\mathsf{Discharge} \Rightarrow$ AUTO

$\langle w_1\rangle \hookrightarrow \mathit{Stat5}(\mathit{Stat5}\star) \Rightarrow$ *Stat7*: $w_1 \in \{a : v \in e_0, a \in \mathsf{contains}(g_0, v)\} \ \& \ w_1 \neq \{v_0, w_0\}$

$\langle v_1, u_1\rangle \hookrightarrow \mathit{Stat7}(\mathit{Stat4}\star) \Rightarrow v_1 \in \{v_0, w_0\} \ \& \ w_1 \in \mathsf{contains}(g_0, v_1)$

$\mathsf{Suppose} \Rightarrow v_1 = v_0$

$\mathsf{EQUAL}\ \langle \mathit{Stat1}\rangle \Rightarrow \mathsf{false};$ $\mathsf{Discharge} \Rightarrow v_1 = w_0$

$\mathsf{EQUAL}\ \langle \mathit{Stat2}\rangle \Rightarrow \mathsf{false};$ $\mathsf{Discharge} \Rightarrow$ AUTO

$\mathsf{Suppose} \Rightarrow$ *Stat8*: $\{e_0\} \notin \{p \subseteq g_0 \mid \mathsf{nodes}(p) \cap \mathsf{nodes}(g_0 \setminus p) = \emptyset\}$

$\langle g_0, \{e_0\}\rangle \hookrightarrow T\mathsf{hgraph}_6(\mathit{Stat4}\star) \Rightarrow \mathsf{nodes}(\{e_0\}) \cap \mathsf{nodes}(g_0 \setminus \{e_0\}) = \emptyset$

$\langle \{e_0\}\rangle \hookrightarrow \mathit{Stat8}(\star) \Rightarrow e_0 \notin g_0$

$\mathsf{Use\_def}(\mathsf{contains}) \Rightarrow$ *Stat9*: $e_0 \in \{e \in g_0 \mid v_0 \in e\}$

$\langle\ \rangle \hookrightarrow \mathit{Stat9}(\mathit{Stat8}\star) \Rightarrow \mathsf{false};$ $\mathsf{Discharge} \Rightarrow$ AUTO

$(\mathit{Stat1}\star)\mathsf{ELEM} \Rightarrow$ *Stat11*: $\{e_0\} = g_0$

$\langle v_0, g_0\rangle \hookrightarrow T\mathsf{hgraph}_e(\mathit{Stat10}\star) \Rightarrow$ *Stat10*: $v_0 \in \mathsf{nodes}(g_0) \rightarrow \mathsf{edgeOf}(v_0, g_0) \in g_0$

$\langle g_0, e_0\rangle \hookrightarrow T\mathsf{hgraph}_2(\mathit{Stat1}\star) \Rightarrow$ *Stat12*: $e_0 = \mathsf{edgeOf}(v_0, g_0)$

$\mathsf{EQUAL}\ \langle \mathit{Stat1}, \mathit{Stat11}, \mathit{Stat12}\rangle \Rightarrow \mathsf{false};$ $\mathsf{Discharge} \Rightarrow$ QED

> The following proof of the Corollary 1 of [CO14] also embodies the proof of the proposition named, therein, Theorem 1.

THEOREM hgraph_{24}: [Every non-null connected graph has a non-cut vertex] $\mathsf{Conn(G)}\ \&\ \mathsf{G} \neq \emptyset \rightarrow \langle \exists \mathsf{v} \in \mathsf{nodes(G)} \mid \neg \mathsf{Cutting(G, v)} \rangle$. PROOF:

$\mathsf{Suppose\_not(g_0)} \Rightarrow$ AUTO

$\langle \mathsf{g_0} \rangle \hookrightarrow T\mathsf{hgraph}_{22} \Rightarrow$ AUTO

$\mathsf{Use\_def(Conn)} \Rightarrow$ $\mathit{Stat1}: \neg\langle \exists \mathsf{v} \in \mathsf{nodes(g_0)} \mid \neg \mathsf{Cutting(g_0, v)} \rangle\ \&\ \langle \exists \mathsf{v} \in \mathsf{nodes(g_0)} \mid \mathsf{Conn}\big(\mathsf{filter(g_0, v)}\big) \rangle\ \&\ \{\mathsf{p} \subseteq \mathsf{g_0} \mid \mathsf{nodes(p)} \cap \mathsf{nodes(g_0 \setminus p)} = \emptyset\} \subseteq \{\emptyset, \mathsf{g_0}\}\ \&\ \mathsf{HGraph(g_0)}$

$\mathsf{Use\_def}\big(\mathsf{Cutting(g_0, v_0)}\big) \Rightarrow$ AUTO

$\langle \mathsf{v_0, v_0} \rangle \hookrightarrow \mathit{Stat1}(\mathit{Stat1}\star) \Rightarrow$ $\mathit{Stat2}: \mathsf{lost(g_0, v_0)} \neq \emptyset\ \&\ \mathsf{v_0} \in \mathsf{nodes(g_0)}\ \&\ \mathsf{Conn}\big(\mathsf{filter(g_0, v_0)}\big)\ \&\ \mathsf{v_0} \in \mathsf{nodes(g_0)}$

$\langle \mathsf{w_0} \rangle \hookrightarrow \mathit{Stat2}(\mathit{Stat2}\star) \Rightarrow$ $\mathsf{w_0} \in \mathsf{lost(g_0, v_0)}$

$\mathsf{Use\_def}\big(\mathsf{contains(g_0, w_0)}\big) \Rightarrow$ AUTO

$\langle \mathsf{g_0, w_0, v_0} \rangle \hookrightarrow T\mathsf{hgraph}_{18}(\star) \Rightarrow$ $\mathit{Stat3}: \{\mathsf{v_0, w_0}\} \in \{\mathsf{e} \in \mathsf{g_0} \mid \mathsf{w_0} \in \mathsf{e}\}\ \&\ \mathsf{contains(g_0, w_0)} = \{\{\mathsf{v_0, w_0}\}\}$

$\langle \mathsf{g_0, w_0}, \{\mathsf{v_0, w_0}\} \rangle \hookrightarrow T\mathsf{hgraph}_{23}(\star) \Rightarrow$ $\neg \mathsf{Cutting(g_0, w_0)}$

$\langle \mathsf{w_0} \rangle \hookrightarrow \mathit{Stat1}(\star) \Rightarrow$ $\mathsf{w_0} \notin \mathsf{nodes(g_0)}$

$\langle \mathsf{g_0}, \{\mathsf{v_0, w_0}\} \rangle \hookrightarrow T\mathsf{hgraph}_{2}(\mathit{Stat1}\star) \Rightarrow$ $\{\mathsf{v_0, w_0}\} \notin \mathsf{g_0}$

$\langle\ \rangle \hookrightarrow \mathit{Stat3}(\mathit{Stat3}\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED