On representing graphs as membership digraphs

Companion proof-scenario

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This scenario introduces various notions regarding finite graphs, in particular
connectivity and its equivalent notion 'HasSpanningTree',
and regarding finite digraphs, in particular
extensionality, weak extensionality, and acyclicity.
Moreover it relates graphs to digraphs via the 'Orientates' predicate.

We will prove that every weakly extensional, acyclic digraph can be decorated *à la* Mostowski by finite sets so that: on the one hand, its arcs mimic membership; on the other hand, no collisions arise between its vertices (otherwise stated, no two vertices are sent to the same set by the decoration).

This applies, of course, also to the special case of an extensional digraph. We will manage to have one node sent to $\emptyset$ in the decoration.

We will also see that a graph whatsoever admits an orientation which is weakly extensional and acyclic; consequently, and in view of what precedes, one can regard its edges as membership arcs deprived of their natural orientation.

To end, we will endow each connected claw-free graph G with an extensional acyclic orientation so that, through such an orientation, G will be represented as a transitive set T and the membership arcs between elements of T will correspond to the edges of G.

1 Preparatory notions

1.1 From pairs to single-valued and singleton maps

Formal definition of the ordered pair and of both of its ordered-pair component extractor functions.

DEF pair_1: [Ordered pair] $\quad [\mathsf{X},\mathsf{Y}] \;=_{\mathrm{Def}}\; \{\{\mathsf{X}\},\{\{\mathsf{X}\},\{\{\mathsf{Y}\},\mathsf{Y}\}\}\}$

DEF pair_2: [First component of ordered pair] $\quad \mathsf{P}^{[1]} \;=_{\mathrm{Def}}\; \mathbf{arb}(\mathbf{arb}(\mathsf{P}))$

DEF pair_3: [Second component of ordered pair] $\quad \mathsf{P}^{[2]} \;=_{\mathrm{Def}}\; \mathbf{arb}(\mathbf{arb}(\mathbf{arb}(\mathsf{P}\setminus\{\mathbf{arb}(\mathsf{P})\})\setminus\{\mathbf{arb}(\mathsf{P})\}))$

The formal definitions, just seen, of the pairing function and of its projections enforce that $\emptyset^{[1]} = \emptyset \;\&\; \emptyset^{[2]} = \emptyset$. We will exploit this fact later, to treat $\emptyset$ as the result of applying a single-valued map f to an operand which lies outside the domain of f.

THM pair_1: [Both projections extract $\emptyset$ from $\emptyset$] $\emptyset^{[1]} = \emptyset \;\&\; \emptyset^{[2]} = \emptyset$. PROOF:

Suppose_not() ⇒ AUTO

TELEM ⇒ $\mathbf{arb}(\emptyset) = \emptyset$

TELEM ⇒ $\mathbf{arb}(\emptyset\setminus\{\mathbf{arb}(\emptyset)\})\setminus\{\emptyset\} = \emptyset$

Use_def($\emptyset^{[1]}$) ⇒ AUTO

Use_def($\emptyset^{[2]}$) ⇒ AUTO

EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

|| The following operations are usually applied to sets of ordered pairs, here called *maps*:

DEF maps_1: [Map domain, i.e. set of first components of pairs in map] $\mathbf{dom}(F) =_{\mathrm{Def}} \{x^{[1]} : x \in F\}$

DEF maps_2: [Map restriction] $F_{|A} =_{\mathrm{Def}} \{p \in F \mid p^{[1]} \in A\}$

DEF maps_3: [Image, i.e. value, of single-valued function] $F{\upharpoonright}X =_{\mathrm{Def}} \mathbf{arb}(F_{|\{X\}})^{[2]}$

DEF maps_4: [Map range, i.e. set of second components of pairs in map] $\mathbf{range}(F) =_{\mathrm{Def}} \{p^{[2]} : p \in F\}$

DEF maps_5: [Map predicate] $\mathsf{Is\_map}(F) \leftrightarrow_{\mathrm{Def}} \langle \forall p \in F \mid p = [p^{[1]}, p^{[2]}]\rangle$

DEF maps_6: [Single-valued map predicate] $\mathsf{Svm}(F) \leftrightarrow_{\mathrm{Def}} \mathsf{Is\_map}(F) \,\&\, \langle \forall p \in F, q \in F \mid p^{[1]} = q^{[1]} \rightarrow p = q\rangle$

|| After pausing to develop a couple of ancillary THEORYs, in this section we will prove, about map-related notions, the collection of statement displayed in the two tables below:

THM range_1: [Additivity of range] $(X = Y \cup Z \rightarrow \mathbf{range}(X) = \mathbf{range}(Y) \cup \mathbf{range}(Z)) \,\&\, (\mathbf{range}(X) = \emptyset \leftrightarrow X = \emptyset)$

THM domain_1: [Additivity of domain] $(X = Y \cup Z \rightarrow \mathbf{dom}(X) = \mathbf{dom}(Y) \cup \mathbf{dom}(Z)) \,\&\, (\mathbf{dom}(X) = \emptyset \leftrightarrow X = \emptyset)$

THM domain_2: [Elements of domain, of range] $P \in S \rightarrow P^{[1]} \in \mathbf{dom}(S) \,\&\, P^{[2]} \in \mathbf{range}(S)$

THM domain_3: [Typical elements of domain, of range] $[X, Y] \in S \rightarrow X \in \mathbf{dom}(S) \,\&\, Y \in \mathbf{range}(S)$

THM restr_0: [The restriction of any (set or) map is included in it] $F_{|A} \subseteq F$

THM restr_1: [The domain of a restriction is included in the restraining set] $P \in F_{|A} \rightarrow P^{[1]} \in A$

THM restr_2: [Each pair in a map belongs to the shoot of an element of its domain] $P = [X, Y] \rightarrow (P \in F_{|\{Z\}} \leftrightarrow Z = X \,\&\, [Z, Y] \in F)$

THM restr_3: [The elements of a map shoot are pairs of fixed first component] $\mathsf{Is\_map}(F) \,\&\, P \in F_{|\{Z\}} \rightarrow P = [Z, P^{[2]}] \,\&\, P \in F$

THM image_0: [The image, under $\emptyset$, of any entity is $\emptyset$ by convention] $\mathsf{Z} = \emptyset \rightarrow \mathsf{Svm}(\mathsf{Z})\ \&\ \mathsf{Z}{\upharpoonright}\mathsf{X} = \emptyset$

THM image_1: [Each pair in a map indicates membership of a operand-image pair in the map] $\mathsf{X} \in \mathbf{dom}(\mathsf{F})\ \&\ \mathsf{Is\_map}(\mathsf{F}) \rightarrow [\mathsf{X}, \mathsf{F}{\upharpoonright}\mathsf{X}] \in \mathsf{F}_{|\{\mathsf{X}\}} \cap \mathsf{F}$

THM image_2: [Application of set union to an operand is affected only by addend which has operand in its domain] $\mathsf{X} \notin \mathbf{dom}(\mathsf{F}) \rightarrow (\mathsf{F} \cup \mathsf{G}){\upharpoonright}\mathsf{X} = \mathsf{G}{\upharpoonright}\mathsf{X}$

THM image_3: [Application of a map to an entity outside its domain yields $\emptyset$ by convention] $\mathsf{X} \notin \mathbf{dom}(\mathsf{F}) \rightarrow \mathsf{F}{\upharpoonright}\mathsf{X} = \emptyset$

THM image_4: [Meaning of application of a single-valued map] $\mathsf{Svm}(\mathsf{F})\ \&\ \mathsf{P} \in \mathsf{F} \rightarrow \mathsf{P} = \left[\mathsf{P}^{[1]}, \mathsf{F}{\upharpoonright}\mathsf{P}^{[1]}\right]$

THM image_5: [Form of a single-valued map] $\mathsf{Svm}(\mathsf{F}) \leftrightarrow \mathsf{F} = \{[\mathsf{x}, \mathsf{F}{\upharpoonright}\mathsf{x}] : \mathsf{x} \in \mathbf{dom}(\mathsf{F})\}$

THM svm_0: [The subsets of single valued maps are single valued maps] $\mathsf{Svm}(\mathsf{F})\ \&\ \mathsf{G} \subseteq \mathsf{F} \rightarrow \mathsf{Svm}(\mathsf{G})$

THM svm_1: [Union of domain-disjoint single-valued maps] $\mathsf{Svm}(\mathsf{F})\ \&\ \mathsf{Svm}(\mathsf{G})\ \&\ \mathbf{dom}(\mathsf{F}) \cap \mathbf{dom}(\mathsf{G}) = \emptyset \rightarrow \mathsf{Svm}(\mathsf{F} \cup \mathsf{G})$

THM $\mathsf{singletonMap}_0$: [Domain and range of a singleton] $\mathbf{dom}(\{\mathsf{P}\}) = \left\{\mathsf{P}^{[1]}\right\}\ \&\ \mathbf{range}(\{\mathsf{P}\}) = \left\{\mathsf{P}^{[2]}\right\}$

THM $\mathsf{singletonMap}_1$: [Singleton maps are single valued] $\big(\mathsf{F} \subseteq \{[\mathsf{X}, \mathsf{Y}]\} \rightarrow \mathsf{Svm}(\mathsf{F})\big)\ \&\ \mathbf{dom}(\{[\mathsf{X}, \mathsf{Y}]\}) = \{\mathsf{X}\}\ \&\ \mathbf{range}(\{[\mathsf{X}, \mathsf{Y}]\}) = \{\mathsf{Y}\}$

THM $\mathsf{singletonMap}_2$: [Singleton map application] $\mathsf{F} = \{[\mathsf{X}, \mathsf{Y}]\} \rightarrow \mathsf{F}{\upharpoonright}\mathsf{X} = \mathsf{Y}$

THM $\mathsf{singletonMap}_3$: [Transplant of singleton sub-map] $\mathsf{Svm}(\mathsf{F})\ \&\ [\mathsf{X}, \mathsf{Y}] \in \mathsf{F}\ \&\ \mathsf{Z} \notin \mathbf{dom}(\mathsf{F})\ \&\ \mathsf{G} = \mathsf{F} \setminus \{[\mathsf{X}, \mathsf{Y}]\} \cup \{[\mathsf{Z}, \mathsf{Y}]\} \rightarrow$
$\mathsf{Svm}(\mathsf{G})\ \&\ \mathbf{dom}(\mathsf{G}) = \mathbf{dom}(\mathsf{F}) \setminus \{\mathsf{X}\} \cup \{\mathsf{Z}\}\ \&\ \mathbf{range}(\mathsf{G}) = \mathbf{range}(\mathsf{F})$

The following THEORY , devoid of inner assumptions, indicates a convenient rewriting of the domain of a map: it will not be called explicitly, but is involved in some uses of the TELEM inference mechanism.

THEORY $\mathsf{isSvm}\big(\mathsf{s}_0, \mathsf{f}(\mathsf{X}), \mathsf{P}(\mathsf{X})\big)$
END isSvm

ENTER_THEORY isSvm

THM isSvm_0. $\mathsf{Svm}\big(\{[\mathsf{x}, \mathsf{f}(\mathsf{x})] : \mathsf{x} \in \mathsf{s}_0 \mid \mathsf{P}(\mathsf{x})\}\big)$. PROOF:
Suppose_not() $\Rightarrow$ AUTO
Suppose $\Rightarrow$ *Stat1* : $\neg\big\langle \forall \mathsf{q} \in \{[\mathsf{x}, \mathsf{f}(\mathsf{x})] : \mathsf{x} \in \mathsf{s}_0 \mid \mathsf{P}(\mathsf{x})\} \mid \mathsf{q} = \left[\mathsf{q}^{[1]}, \mathsf{q}^{[2]}\right] \big\rangle$
$\langle \mathsf{p}_0 \rangle \hookrightarrow$ *Stat1(Stat1⋆)* $\Rightarrow$ *Stat2* : $\mathsf{p}_0 \in \{[\mathsf{x}, \mathsf{f}(\mathsf{x})] : \mathsf{x} \in \mathsf{s}_0 \mid \mathsf{P}(\mathsf{x})\}\ \&\ \mathsf{p}_0 \neq \left[\mathsf{p}_0{}^{[1]}, \mathsf{p}_0{}^{[2]}\right]$
$\langle \mathsf{x}_0 \rangle \hookrightarrow$ *Stat2(Stat2)* $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO
Use_def(Is_map) $\Rightarrow$ $\mathsf{Is\_map}\big(\{[\mathsf{x}, \mathsf{f}(\mathsf{x})] : \mathsf{x} \in \mathsf{s}_0\}\big)$
Use_def(Svm) $\Rightarrow$ *Stat3* : $\neg\big\langle \forall \mathsf{r} \in \{[\mathsf{x}, \mathsf{f}(\mathsf{x})] : \mathsf{x} \in \mathsf{s}_0 \mid \mathsf{P}(\mathsf{x})\}, \mathsf{q} \in \{[\mathsf{x}, \mathsf{f}(\mathsf{x})] : \mathsf{x} \in \mathsf{s}_0 \mid \mathsf{P}(\mathsf{x})\} \mid \mathsf{r}^{[1]} = \mathsf{q}^{[1]} \rightarrow \mathsf{r} = \mathsf{q} \big\rangle$
$\langle \mathsf{p}_1, \mathsf{q}_1 \rangle \hookrightarrow$ *Stat3(Stat3⋆)* $\Rightarrow$ *Stat4* : $\mathsf{p}_1, \mathsf{q}_1 \in \{[\mathsf{x}, \mathsf{f}(\mathsf{x})] : \mathsf{x} \in \mathsf{s}_0 \mid \mathsf{P}(\mathsf{x})\}\ \&\ \mathsf{p}_1{}^{[1]} = \mathsf{q}_1{}^{[1]}\ \&\ \mathsf{p}_1 \neq \mathsf{q}_1$
$\langle \mathsf{x}_1, \mathsf{x}_2 \rangle \hookrightarrow$ *Stat4(Stat4⋆)* $\Rightarrow$ $\mathsf{p}_1 = [\mathsf{x}_1, \mathsf{f}(\mathsf{x}_1)]\ \&\ \mathsf{q}_1 = [\mathsf{x}_2, \mathsf{f}(\mathsf{x}_2)]$

TELEM $\Rightarrow$ $[x_1, f(x_1)]^{[1]} = x_1$ & $[x_2, f(x_2)]^{[1]} = x_2$
EQUAL(*Stat4*) $\Rightarrow$ false; Discharge $\Rightarrow$ QED

THM isSvm$_1$. $\mathbf{dom}(\{[x, f(x)] : x \in s_0 \mid P(x)\}) = \{x \in s_0 \mid P(x)\}$ & $\{x \in s_0 \mid \text{true}\} = s_0$. PROOF:
Suppose_not() $\Rightarrow$ AUTO
SIMPLF $\Rightarrow$ $\{x \in s_0 \mid \text{true}\} = s_0$
Use_def(**dom**) $\Rightarrow$ $\{p^{[1]} : p \in \{[x, f(x)] : x \in s_0 \mid P(x)\}\} \neq \{x \in s_0 \mid P(x)\}$
SIMPLF $\Rightarrow$ *Stat1* : $\left\{[x, f(x)]^{[1]} : x \in s_0 \mid P(x)\right\} \neq \{x \in s_0 \mid P(x)\}$
$\langle x_0 \rangle \hookrightarrow$ *Stat1*(*Stat1*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ QED

THM isSvm$_2$. $\mathbf{range}(\{[x, f(x)] : x \in s_0 \mid P(x)\}) = \{f(x) : x \in s_0 \mid P(x)\}$. PROOF:
Suppose_not() $\Rightarrow$ AUTO
Use_def(**range**) $\Rightarrow$ $\{p^{[2]} : p \in \{[x, f(x)] : x \in s_0 \mid P(x)\}\} \neq \{f(x) : x \in s_0 \mid P(x)\}$
SIMPLF $\Rightarrow$ *Stat1* : $\left\{[x, f(x)]^{[2]} : x \in s_0 \mid P(x)\right\} \neq \{f(x) : x \in s_0 \mid P(x)\}$
$\langle x_0 \rangle \hookrightarrow$ *Stat1*(*Stat1*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ QED

ENTER_THEORY Set_theory
DISPLAY isSvm

THEORY isSvm$(s_0, f(X), P(X))$
$\Rightarrow$
Svm$(\{[x, f(x)] : x \in s_0 \mid P(x)\})$
$\mathbf{dom}(\{[x, f(x)] : x \in s_0 \mid P(x)\}) = \{x \in s_0 \mid P(x)\}$ & $\{x \in s_0 \mid \text{true}\} = s_0$
$\mathbf{range}(\{[x, f(x)] : x \in s_0 \mid P(x)\}) = \{f(x) : x \in s_0 \mid P(x)\}$
END isSvm

Our next THEORY captures certain properties such as additivity (and, a fortiori, monotonicity) enjoyed by any global function which, much like **dom** and **range**, is characterized by a pointwise definition.

THEORY pointwise$(g(Y), f(X))$
$\langle \forall x \mid f(x) = \{g(y) : y \in x\} \rangle$
END pointwise

ENTER_THEORY pointwise

THM pointwise$_0$: [Pointwise defined functions are monotonic] $S \subseteq T \rightarrow f(S) \subseteq f(T)$. PROOF:
Suppose_not(s_0, t_0) $\Rightarrow$ AUTO

Assump ⇒ $\mathit{Stat0}:\ \langle \forall x \mid f(x) = \{g(y) : y \in x\}\rangle \ \&\ \langle \forall x \mid f(x) = \{g(y) : y \in x\}\rangle$
$\langle s_0, t_0\rangle \hookrightarrow \mathit{Stat0}(\star)$ ⇒ $\mathit{Stat1}:\ \{g(x) : x \in s_0\} \not\subseteq \{g(x) : x \in t_0\}$
$\langle c_0\rangle \hookrightarrow \mathit{Stat1}(\star)$ ⇒ $\mathit{Stat2}:\ c_0 \in \{g(x) : x \in s_0\} \ \&\ c_0 \notin \{g(x) : x \in t_0\} \ \&\ s_0 \subseteq t_0$
$\langle x_0, x_0\rangle \hookrightarrow \mathit{Stat2}(\mathit{Stat2}\star)$ ⇒ false; Discharge ⇒ QED

THM pointwise$_1$: [Pointwise defined functions are additive] $X = Y \cup Z \to f(X) = f(Y) \cup f(Z)$. PROOF:
Suppose_not(x_0, y_0, z_0) ⇒ AUTO
$\langle y_0, x_0\rangle \hookrightarrow T\text{pointwise}_0$ ⇒ AUTO
$\langle z_0, x_0\rangle \hookrightarrow T\text{pointwise}_0$ ⇒ $\mathit{Stat1}:\ f(x_0) \not\subseteq f(y_0) \cup f(z_0)$
$\langle u_0\rangle \hookrightarrow \mathit{Stat1}(\mathit{Stat1}\star)$ ⇒ $\mathit{Stat2}:\ u_0 \in f(x_0) \ \&\ u_0 \notin f(y_0) \cup f(z_0)$
Assump ⇒ $\mathit{Stat3}:\ \langle \forall x \mid f(x) = \{g(y) : y \in x\}\rangle \ \&\ \langle \forall x \mid f(x) = \{g(y) : y \in x\}\rangle \ \&\ \langle \forall x \mid f(x) = \{g(y) : y \in x\}\rangle$
$\langle x_0, y_0, z_0\rangle \hookrightarrow \mathit{Stat3}(\mathit{Stat2}\star)$ ⇒ $\mathit{Stat4}:\ u_0 \in \{g(x) : x \in x_0\} \ \&\ u_0 \notin \{g(y) : y \in y_0\} \ \&\ u_0 \notin \{g(z) : z \in z_0\}$
$\langle x_1, x_1, x_1\rangle \hookrightarrow \mathit{Stat4}(\star)$ ⇒ false; Discharge ⇒ QED

THM pointwise$_2$: [Pointwise defined functions send $\emptyset$ to $\emptyset$] $X = \emptyset \leftrightarrow f(X) = \emptyset$. PROOF:
Suppose_not(x_0) ⇒ AUTO
Suppose ⇒ $\mathit{Stat1}:\ \{g(p) : p \in x_0\} \neq \emptyset \ \&\ x_0 = \emptyset$
$\langle p_0\rangle \hookrightarrow \mathit{Stat1}(\mathit{Stat1}\star)$ ⇒ false; Discharge ⇒ AUTO
Assump ⇒ $\mathit{Stat2}:\ \langle \forall x \mid f(x) = \{g(y) : y \in x\}\rangle$
$\langle x_0\rangle \hookrightarrow \mathit{Stat2}(\star)$ ⇒ $\mathit{Stat3}:\ x_0 \neq \emptyset \ \&\ \{g(p) : p \in x_0\} = \emptyset$
$\langle p_1, p_1\rangle \hookrightarrow \mathit{Stat3}(\mathit{Stat3}\star)$ ⇒ false; Discharge ⇒ QED

ENTER_THEORY Set_theory
DISPLAY pointwise

THEORY pointwise$\big(g(Y), f(X)\big)$
$\langle \forall x \mid f(x) = \{g(y) : y \in x\}\rangle$
⇒
$\langle \forall s, t \mid S \subseteq T \to f(S) \subseteq f(T)\rangle$
$\langle \forall x, y, z \mid x = y \cup z \to f(x) = f(y) \cup f(z)\rangle$
$\langle \forall x \mid x = \emptyset \leftrightarrow f(x) = \emptyset\rangle$
END pointwise

> Having now developed a THEORY referring to any global function which has a pointwise definition, we readily apply it to the range and domain functions.

THM range$_1$: [Additivity of range] $(X = Y \cup Z \to \mathbf{range}(X) = \mathbf{range}(Y) \cup \mathbf{range}(Z)) \ \&\ (\mathbf{range}(X) = \emptyset \leftrightarrow X = \emptyset)$. PROOF:
Suppose_not(x_0, y_0, z_0) ⇒ AUTO
Suppose ⇒ $\mathit{Stat0}:\ \neg\langle \forall x \mid \mathbf{range}(x) = \{y^{[2]} : y \in x\}\rangle$

$\langle f_0 \rangle \hookrightarrow Stat0(Stat0\star) \Rightarrow \quad Stat1: \mathbf{range}(f_0) \neq \{y^{[2]} : y \in f_0\}$
Use_def(**range**) ⇒ false; Discharge ⇒ AUTO
APPLY ⟨ ⟩ pointwise$(g(Y) \mapsto Y^{[2]}, f(X) \mapsto \mathbf{range}(X)) \Rightarrow \quad Stat2: \langle \forall x, y, z \mid x = y \cup z \rightarrow \mathbf{range}(x) = \mathbf{range}(y) \cup \mathbf{range}(z) \rangle \;\&\; \langle \forall x \mid \mathbf{range}(x) = \emptyset \leftrightarrow x = \emptyset \rangle$
$\langle x_0, y_0, z_0, x_0 \rangle \hookrightarrow Stat2(\star) \Rightarrow$ false; Discharge ⇒ QED

THM domain$_1$: [Additivity of domain] $(X = Y \cup Z \rightarrow \mathbf{dom}(X) = \mathbf{dom}(Y) \cup \mathbf{dom}(Z)) \;\&\; (\mathbf{dom}(X) = \emptyset \leftrightarrow X = \emptyset)$. PROOF:
Suppose_not$(x_0, y_0, z_0) \Rightarrow$ AUTO
Suppose ⇒ $Stat0: \neg \langle \forall x \mid \mathbf{dom}(x) = \{y^{[1]} : y \in x\} \rangle$
$\langle f_0 \rangle \hookrightarrow Stat0(Stat0\star) \Rightarrow \quad Stat1: \mathbf{dom}(f_0) \neq \{y^{[1]} : y \in f_0\}$
Use_def(**dom**) ⇒ false; Discharge ⇒ AUTO
APPLY ⟨ ⟩ pointwise$(g(Y) \mapsto Y^{[1]}, f(X) \mapsto \mathbf{dom}(X)) \Rightarrow \quad Stat2: \langle \forall x, y, z \mid x = y \cup z \rightarrow \mathbf{dom}(x) = \mathbf{dom}(y) \cup \mathbf{dom}(z) \rangle \;\&\; \langle \forall x \mid \mathbf{dom}(x) = \emptyset \leftrightarrow x = \emptyset \rangle$
$\langle x_0, y_0, z_0, x_0 \rangle \hookrightarrow Stat2(\star) \Rightarrow$ false; Discharge ⇒ QED

THM domain$_2$: [Elements of domain, of range] $P \in S \rightarrow P^{[1]} \in \mathbf{dom}(S) \;\&\; P^{[2]} \in \mathbf{range}(S)$. PROOF:
Suppose_not$(p_0, s) \Rightarrow$ AUTO
Use_def(**dom**(s)) ⇒ AUTO
Use_def(**range**) ⇒ $Stat1: p_0{}^{[1]} \notin \{p^{[1]} : p \in s\} \vee p_0{}^{[2]} \notin \{q^{[2]} : q \in s\}$
$\langle p_0, p_0 \rangle \hookrightarrow Stat1(\star) \Rightarrow$ false; Discharge ⇒ QED

THM domain$_3$: [Typical elements of domain, of range] $[X, Y] \in S \rightarrow X \in \mathbf{dom}(S) \;\&\; Y \in \mathbf{range}(S)$. PROOF:
Suppose_not$(x_0, y_0, s) \Rightarrow$ AUTO
$\langle [x_0, y_0], s \rangle \hookrightarrow T$domain$_2 \Rightarrow$ false; Discharge ⇒ QED

‖ We now review a few basic properties of the restriction operation.

THM restr$_0$: [The restriction of any (set or) map is included in it] $F_{|A} \subseteq F$. PROOF:
Suppose_not$(f, a) \Rightarrow$ AUTO
Set_monot ⇒ $\{p : p \in f \mid p^{[1]} \in a\} \subseteq \{p : p \in f\}$
Use_def(|) ⇒ false; Discharge ⇒ QED

THM restr$_1$: [The domain of a restriction is included in the restraining set] $P \in F_{|A} \rightarrow P^{[1]} \in A$. PROOF:
Suppose_not$(p_0, f, a) \Rightarrow$ AUTO
Use_def(|) ⇒ $Stat1: p_0 \in \{p \in f \mid p^{[1]} \in a\}$
$\langle \rangle \hookrightarrow Stat1(\star) \Rightarrow$ false; Discharge ⇒ QED

THM restr$_2$: [Each pair in a map belongs to the shoot of an element of its domain] $P = [X,Y] \rightarrow (P \in F_{|\{Z\}} \leftrightarrow Z = X \,\&\, [Z,Y] \in F)$. PROOF:

Suppose_not$(p_0, x_0, y_0, f, z_0) \Rightarrow$ $Stat0: \; p_0 = [x_0, y_0] \,\&\, (p_0 \in f_{|\{z_0\}} \neq z_0 = x_0 \,\&\, [z_0, y_0] \in f)$

Use_def$(|) \Rightarrow$ $f_{|\{z_0\}} = \{p \in f \mid p^{[1]} \in \{z_0\}\}$

Suppose $\Rightarrow$ $Stat1: \; p_0 \in \{p \in f \mid p^{[1]} \in \{z_0\}\}$

$\langle\,\rangle \hookrightarrow Stat1(Stat1\star) \Rightarrow$ $p_0{}^{[1]} = z_0 \,\&\, p_0 \in f$

TELEM $\Rightarrow$ $[x_0, y_0]^{[1]} = x_0$

EQUAL $\Rightarrow$ $[x_0, y_0] \in f$

$(Stat0\star)$Discharge $\Rightarrow$ AUTO

$(Stat0\star)$ELEM $\Rightarrow$ $Stat2: \; p_0 \notin \{p \in f \mid p^{[1]} \in \{z_0\}\} \,\&\, z_0 = x_0 \,\&\, [z_0, y_0] \in f$

$\langle [z_0, y_0] \rangle \hookrightarrow Stat2(Stat2) \Rightarrow$ $p_0 \neq [z_0, y_0]$

EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

THM restr$_3$: [The elements of a map shoot are pairs of fixed first component] $\mathsf{Is\_map}(F) \,\&\, P \in F_{|\{Z\}} \rightarrow P = [Z, P^{[2]}] \,\&\, P \in F$. PROOF:

Suppose_not$(f, p_0, x_0) \Rightarrow$ AUTO

Use_def(Is_map) $\Rightarrow$ $Stat1: \; \langle \forall p \in f \mid p = [p^{[1]}, p^{[2]}] \rangle$

$\langle f, \{x_0\} \rangle \hookrightarrow T\mathsf{restr}_0(\star) \Rightarrow$ $p_0 \in f$

$\langle p_0, f, \{x_0\} \rangle \hookrightarrow T\mathsf{restr}_1(\star) \Rightarrow$ $p_0{}^{[1]} = x_0$

$\langle p_0 \rangle \hookrightarrow Stat1(Stat1) \Rightarrow$ $p_0 = [x_0, p_0{}^{[2]}]$

Discharge $\Rightarrow$ QED

|| We then review a few basic properties of the application operation.

THM image$_0$: [The image, under $\emptyset$, of any entity is $\emptyset$ by convention] $Z = \emptyset \rightarrow \mathsf{Svm}(Z) \,\&\, Z \upharpoonright X = \emptyset$. PROOF:

Suppose_not$(z_0, x_0) \Rightarrow$ AUTO

Suppose $\Rightarrow$ $\neg\mathsf{Svm}(z_0)$

Use_def(Is_map(z_0)) $\Rightarrow$ AUTO

Use_def(Svm) $\Rightarrow$ $Stat1: \; \neg(\langle \forall p \in z_0 \mid p = [p^{[1]}, p^{[2]}] \rangle \,\&\, \langle \forall p \in z_0, q \in z_0 \mid p^{[1]} = q^{[1]} \rightarrow p = q \rangle)$

$\langle p_0, p_1, p_2 \rangle \hookrightarrow Stat1(\star) \Rightarrow$ false; Discharge $\Rightarrow$ $z_0 = \emptyset \,\&\, z_0 \upharpoonright x_0 \neq \emptyset$

EQUAL $\Rightarrow$ $Stat2: \; \emptyset \upharpoonright x_0 \neq \emptyset$

$\langle \emptyset, \{x_0\} \rangle \hookrightarrow T\mathsf{restr}_0(Stat2\star) \Rightarrow$ AUTO

$T\mathsf{pair}_1(Stat2) \Rightarrow$ $\emptyset^{[2]} = \emptyset \,\&\, \mathbf{arb}(\emptyset_{|\{x_0\}}) = \emptyset$

Use_def$(\upharpoonright) \Rightarrow$ $\emptyset \upharpoonright x_0 = \mathbf{arb}(\emptyset_{|\{x_0\}})^{[2]}$

EQUAL$(Stat2) \Rightarrow$ false; Discharge $\Rightarrow$ QED

THM image$_1$: [Each pair in a map indicates membership of a operand-image pair in the map] $X \in \mathbf{dom}(F) \,\&\, \mathsf{Is\_map}(F) \rightarrow [X, F \upharpoonright X] \in F_{|\{X\}} \cap F$. PROOF:

Suppose_not$(x_0, f) \Rightarrow$ $Stat0: \; x_0 \in \mathbf{dom}(f) \,\&\, \mathsf{Is\_map}(f) \,\&\, [x_0, f \upharpoonright x_0] \notin f_{|\{x_0\}} \cap f$

Use_def$(\mathbf{dom}(f)) \Rightarrow$ AUTO

Use_def(Is_map) $\Rightarrow$ *Stat1* : $\langle \forall p \in f \mid p = [p^{[1]}, p^{[2]}] \rangle \;\&\; x_0 \in \{p^{[1]} : p \in f\}$
$\langle p_0, p_0 \rangle \hookrightarrow$ *Stat1*(*Stat1*) $\Rightarrow$ $p_0 = [p_0{}^{[1]}, p_0{}^{[2]}] \;\&\; p_0, [x_0, p_0{}^{[2]}] \in f \;\&\; x_0 = p_0{}^{[1]}$
Loc_def $\Rightarrow$ $a = \mathbf{arb}(f_{|\{x_0\}})$
$\langle p_0, p_0{}^{[1]}, p_0{}^{[2]}, f, x_0 \rangle \hookrightarrow Trestr_2$(*Stat1*) $\Rightarrow$ *Stat2* : $a \in f_{|\{x_0\}}$
$\langle a, f, \{x_0\} \rangle \hookrightarrow Trestr_1$(*Stat2*$\star$) $\Rightarrow$ $a^{[1]} = x_0$
$\langle f, \{x_0\} \rangle \hookrightarrow Trestr_0$(*Stat2*$\star$) $\Rightarrow$ $a \in f$
$\langle a \rangle \hookrightarrow$ *Stat1*(*Stat2*$\star$) $\Rightarrow$ $a = [a^{[1]}, a^{[2]}]$
Use_def($\upharpoonright$) $\Rightarrow$ *Stat3* : $f \upharpoonright x_0 = \mathbf{arb}(f_{|\{x_0\}})^{[2]}$
EQUAL(*Stat1*) $\Rightarrow$ *Stat4* : $[x_0, f \upharpoonright x_0] \in f_{|\{x_0\}} \;\&\; [x_0, f \upharpoonright x_0] \in f$
(*Stat4*, *Stat0*$\star$)Discharge $\Rightarrow$ QED

THM image$_2$: [Application of set union to an operand is affected only by addend which has operand in its domain] $X \notin \mathbf{dom}(F) \rightarrow (F \cup G) \upharpoonright X = G \upharpoonright X$. PROOF:
Suppose_not(x_0, f_0, g_0) $\Rightarrow$ AUTO

‖ For, assuming the contrary, there would exist an x_0 outside $\mathbf{dom}(f_0)$ such that $f_0 \cup g_{0|\{x_0\}} \not\subseteq g_{0|\{x_0\}} \ldots$

Use_def($\upharpoonright$) $\Rightarrow$ $(f_0 \cup g_0) \upharpoonright x_0 = \mathbf{arb}\big((f_0 \cup g_0)_{|\{x_0\}}\big)^{[2]} \;\&\; g_0 \upharpoonright x_0 = \mathbf{arb}\big(g_{0|\{x_0\}}\big)^{[2]}$
Suppose $\Rightarrow$ $(f_0 \cup g_0)_{|\{x_0\}} = g_{0|\{x_0\}}$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO
Set_monot $\Rightarrow$ $\{p \in g_0 \mid p^{[1]} \in \{x_0\}\} \subseteq \{p \in f_0 \cup g_0 \mid p^{[1]} \in \{x_0\}\}$
Use_def($|$) $\Rightarrow$ *Stat1* : $\{p \in f_0 \cup g_0 \mid p^{[1]} \in \{x_0\}\} \not\subseteq \{p \in g_0 \mid p^{[1]} \in \{x_0\}\}$

‖ ... which, taking the definition of domain into account leads us to a contradiction.

Use_def($\mathbf{dom}(f_0)$) $\Rightarrow$ AUTO
$\langle e \rangle \hookrightarrow$ *Stat1*($\star$) $\Rightarrow$ *Stat2* : $e \in \{p : p \in f_0 \cup g_0 \mid p^{[1]} \in \{x_0\}\} \;\&\; e \notin \{p : p \in g_0 \mid p^{[1]} \in \{x_0\}\} \;\&\; x_0 \notin \{p^{[1]} : p \in f_0\}$
$\langle p_0, p_0, p_0 \rangle \hookrightarrow$ *Stat2*(*Stat2*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ QED

THM image$_3$: [Application of a map to an entity outside its domain yields $\emptyset$ by convention] $X \notin \mathbf{dom}(F) \rightarrow F \upharpoonright X = \emptyset$. PROOF:
Suppose_not(x_0, f) $\Rightarrow$ AUTO
$\langle x_0, f, \emptyset \rangle \hookrightarrow Timage_2(\star)$ $\Rightarrow$ $(f \cup \emptyset) \upharpoonright x_0 = \emptyset \upharpoonright x_0$
$\langle \emptyset, x_0 \rangle \hookrightarrow Timage_0(\star)$ $\Rightarrow$ $\emptyset \upharpoonright x_0 = \emptyset \;\&\; f \cup \emptyset = f$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

THM image$_4$: [Meaning of application of a single-valued map] $\mathsf{Svm}(F) \;\&\; P \in F \rightarrow P = [P^{[1]}, F \upharpoonright P^{[1]}]$. PROOF:

Suppose_not$(\mathsf{f}, \mathsf{p}_0) \Rightarrow$ AUTO

$\langle \mathsf{p}_0, \mathsf{f} \rangle \hookrightarrow T\mathsf{domain}_2(\star) \Rightarrow \quad \mathsf{p}_0{}^{[1]} \in \mathbf{dom}(\mathsf{f})$

Use_def(Svm) $\Rightarrow$ *Stat1*: $\langle \forall \mathsf{p} \in \mathsf{f}, \mathsf{q} \in \mathsf{f} \mid \mathsf{p}^{[1]} = \mathsf{q}^{[1]} \rightarrow \mathsf{p} = \mathsf{q} \rangle \;\&\; \mathsf{Is\_map}(\mathsf{f})$

$\langle \mathsf{p}_0{}^{[1]}, \mathsf{f} \rangle \hookrightarrow T\mathsf{image}_1(\star) \Rightarrow \quad [\mathsf{p}_0{}^{[1]}, \mathsf{f} \upharpoonright \mathsf{p}_0{}^{[1]}] \in \mathsf{f}$

TELEM $\Rightarrow$ $[\mathsf{p}_0{}^{[1]}, \mathsf{f} \upharpoonright \mathsf{p}_0{}^{[1]}]^{[1]} = \mathsf{p}_0{}^{[1]}$

$\langle \mathsf{p}_0, [\mathsf{p}_0{}^{[1]}, \mathsf{f} \upharpoonright \mathsf{p}_0{}^{[1]}] \rangle \hookrightarrow Stat1(\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

‖ Single valued maps can always be represented in the following convenient form.

THM image$_5$: [Form of a single-valued map] $\mathsf{Svm}(\mathsf{F}) \leftrightarrow \mathsf{F} = \{[\mathsf{x}, \mathsf{F} \upharpoonright \mathsf{x}] : \mathsf{x} \in \mathbf{dom}(\mathsf{F})\}$. PROOF:

Suppose_not(f) $\Rightarrow$ AUTO

Suppose $\Rightarrow$ $\mathsf{f} = \{[\mathsf{x}, \mathsf{f} \upharpoonright \mathsf{x}] : \mathsf{x} \in \mathbf{dom}(\mathsf{f})\} \;\&\; \neg \mathsf{Svm}(\mathsf{f})$

Use_def(Svm) $\Rightarrow$ $\neg(\mathsf{Is\_map}(\mathsf{f}) \;\&\; \langle \forall \mathsf{p} \in \mathsf{f}, \mathsf{q} \in \mathsf{f} \mid \mathsf{p}^{[1]} = \mathsf{q}^{[1]} \rightarrow \mathsf{p} = \mathsf{q} \rangle)$

Suppose $\Rightarrow$ $\neg \mathsf{Is\_map}(\mathsf{f})$

Use_def(Is_map(f)) $\Rightarrow$ AUTO

EQUAL $\Rightarrow$ *Stat1*: $\neg \langle \forall \mathsf{p} \in \{[\mathsf{x}, \mathsf{f} \upharpoonright \mathsf{x}] : \mathsf{x} \in \mathbf{dom}(\mathsf{f})\} \mid \mathsf{p} = [\mathsf{p}^{[1]}, \mathsf{p}^{[2]}] \rangle$

$\langle \mathsf{p}_0 \rangle \hookrightarrow Stat1(Stat1\star) \Rightarrow$ *Stat2*: $\mathsf{p}_0 \in \{[\mathsf{x}, \mathsf{f} \upharpoonright \mathsf{x}] : \mathsf{x} \in \mathbf{dom}(\mathsf{f})\} \;\&\; \mathsf{p}_0 \neq [\mathsf{p}_0{}^{[1]}, \mathsf{p}_0{}^{[2]}]$

$\langle \mathsf{x}_0 \rangle \hookrightarrow Stat2(Stat2) \Rightarrow$ false; Discharge $\Rightarrow$ *Stat3*: $\neg \langle \forall \mathsf{p} \in \mathsf{f}, \mathsf{q} \in \mathsf{f} \mid \mathsf{p}^{[1]} = \mathsf{q}^{[1]} \rightarrow \mathsf{p} = \mathsf{q} \rangle$

$\langle \mathsf{p}_1, \mathsf{q}_1 \rangle \hookrightarrow Stat3(\star) \Rightarrow$ *Stat4*: $\mathsf{p}_1, \mathsf{q}_1 \in \{[\mathsf{x}, \mathsf{f} \upharpoonright \mathsf{x}] : \mathsf{x} \in \mathbf{dom}(\mathsf{f})\} \;\&\; \mathsf{p}_1{}^{[1]} = \mathsf{q}_1{}^{[1]} \;\&\; \mathsf{p}_1 \neq \mathsf{q}_1$

$\langle \mathsf{x}_1, \mathsf{y}_1 \rangle \hookrightarrow Stat4(Stat4\star) \Rightarrow$ *Stat5*: $\mathsf{p}_1 = [\mathsf{x}_1, \mathsf{f} \upharpoonright \mathsf{x}_1] \;\&\; \mathsf{q}_1 = [\mathsf{y}_1, \mathsf{f} \upharpoonright \mathsf{y}_1] \;\&\; \mathsf{p}_1{}^{[1]} = \mathsf{q}_1{}^{[1]}$

(*Stat5*)ELEM $\Rightarrow$ $\mathsf{x}_1 = \mathsf{y}_1$

EQUAL(*Stat4*) $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat6*: $\mathsf{f} \neq \{[\mathsf{x}, \mathsf{f} \upharpoonright \mathsf{x}] : \mathsf{x} \in \mathbf{dom}(\mathsf{f})\} \;\&\; \mathsf{Svm}(\mathsf{f})$

$\langle \mathsf{p}_2 \rangle \hookrightarrow Stat6(Stat6\star) \Rightarrow$ $\mathsf{p}_2 \in \mathsf{f} \neq \mathsf{p}_2 \in \{[\mathsf{x}, \mathsf{f} \upharpoonright \mathsf{x}] : \mathsf{x} \in \mathbf{dom}(\mathsf{f})\}$

Use_def(**dom**) $\Rightarrow$ $\mathsf{p}_2 \in \mathsf{f} \neq \mathsf{p}_2 \in \{[\mathsf{x}, \mathsf{f} \upharpoonright \mathsf{x}] : \mathsf{x} \in \{\mathsf{p}^{[1]} : \mathsf{p} \in \mathsf{f}\}\}$

SIMPLF $\Rightarrow$ $\mathsf{p}_2 \in \mathsf{f} \neq \mathsf{p}_2 \in \{[\mathsf{p}^{[1]}, \mathsf{f} \upharpoonright \mathsf{p}^{[1]}] : \mathsf{p} \in \mathsf{f}\}$

Suppose $\Rightarrow$ *Stat7*: $\mathsf{p}_2 \in \{[\mathsf{p}^{[1]}, \mathsf{f} \upharpoonright \mathsf{p}^{[1]}] : \mathsf{p} \in \mathsf{f}\}$

$\langle \mathsf{p}_3 \rangle \hookrightarrow Stat7(Stat6\star) \Rightarrow$ $\mathsf{p}_3 \in \mathsf{f} \;\&\; [\mathsf{p}_3{}^{[1]}, \mathsf{f} \upharpoonright \mathsf{p}_3{}^{[1]}] \notin \mathsf{f}$

$\langle \mathsf{f}, \mathsf{p}_3 \rangle \hookrightarrow T\mathsf{image}_4(Stat6\star) \Rightarrow$ false; Discharge $\Rightarrow$ *Stat13*: $\mathsf{p}_2 \notin \{[\mathsf{p}^{[1]}, \mathsf{f} \upharpoonright \mathsf{p}^{[1]}] : \mathsf{p} \in \mathsf{f}\} \;\&\; \mathsf{p}_2 \in \mathsf{f}$

$\langle \mathsf{p}_2 \rangle \hookrightarrow Stat13(Stat13\star) \Rightarrow$ $\mathsf{p}_2 \neq [\mathsf{p}_2{}^{[1]}, \mathsf{f} \upharpoonright \mathsf{p}_2{}^{[1]}]$

$\langle \mathsf{f}, \mathsf{p}_2 \rangle \hookrightarrow T\mathsf{image}_4(Stat6\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

THM svm$_0$: [The subsets of single valued maps are single valued maps] $\mathsf{Svm}(\mathsf{F}) \;\&\; \mathsf{G} \subseteq \mathsf{F} \rightarrow \mathsf{Svm}(\mathsf{G})$. PROOF:

Suppose_not(f, g) $\Rightarrow$ AUTO

Use_def(Svm(f)) $\Rightarrow$ AUTO

Suppose $\Rightarrow$ $\neg \mathsf{Is\_map}(\mathsf{g})$

Use_def(Is_map) $\Rightarrow$ *Stat1*: $\neg \langle \forall \mathsf{p} \in \mathsf{g} \mid \mathsf{p} = [\mathsf{p}^{[1]}, \mathsf{p}^{[2]}] \rangle \;\&\; \langle \forall \mathsf{p} \in \mathsf{f} \mid \mathsf{p} = [\mathsf{p}^{[1]}, \mathsf{p}^{[2]}] \rangle$

$\langle \mathsf{p}_0, \mathsf{p}_0 \rangle \hookrightarrow Stat1(\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO

Use_def(Svm) $\Rightarrow$ *Stat2*: $\neg\langle\forall p\in g,q\in g\,|\,p^{[1]}=q^{[1]}\to p=q\rangle\ \&\ \langle\forall p\in f,q\in f\,|\,p^{[1]}=q^{[1]}\to p=q\rangle\ \&\ g\subseteq f$
$\langle p_1,q_1,p_1,q_1\rangle\hookrightarrow$*Stat2*(*Stat2*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ QED

|| The union of domain-disjoint single valued maps is a single valued map.

THM svm$_1$: [Union of domain-disjoint single-valued maps] $\mathsf{Svm}(F)\ \&\ \mathsf{Svm}(G)\ \&\ \mathbf{dom}(F)\cap\mathbf{dom}(G)=\emptyset\to\mathsf{Svm}(F\cup G)$. PROOF:
Suppose_not(f, g) $\Rightarrow$ AUTO
$\langle f\rangle\hookrightarrow T$image$_5$ $\Rightarrow$ AUTO
$\langle g\rangle\hookrightarrow T$image$_5$ $\Rightarrow$ AUTO
$\langle f\cup g\rangle\hookrightarrow T$image$_5(\star)$ $\Rightarrow$ *Stat1*: $f\cup g\neq\{[z,(f\cup g)\restriction z]:\ z\in\mathbf{dom}(f\cup g)\}\ \&\ f=\{[x,f\restriction x]:\ x\in\mathbf{dom}(f)\}\ \&$
$g=\{[y,g\restriction y]:\ y\in\mathbf{dom}(g)\}\ \&\ \mathbf{dom}(f)\cap\mathbf{dom}(g)=\emptyset$
$\langle c\rangle\hookrightarrow$*Stat1*(*Stat1*$\star$) $\Rightarrow$ $c\in f\cup g\neq c\in\{[z,(f\cup g)\restriction z]:\ z\in\mathbf{dom}(f\cup g)\}$
$\langle f\cup g,f,g\rangle\hookrightarrow T$domain$_1$(*Stat1*$\star$) $\Rightarrow$ *Stat2*: $\mathbf{dom}(f\cup g)=\mathbf{dom}(f)\cup\mathbf{dom}(g)$
Suppose $\Rightarrow$ *Stat3*: $c\in\{[z,(f\cup g)\restriction z]:\ z\in\mathbf{dom}(f\cup g)\}$
$\langle z_0\rangle\hookrightarrow$*Stat3*(*Stat2*$\star$) $\Rightarrow$ $c=[z_0,(f\cup g)\restriction z_0]\ \&\ z_0\in\mathbf{dom}(f)\cup\mathbf{dom}(g)$
Suppose $\Rightarrow$ $z_0\notin\mathbf{dom}(f)$
$\langle z_0,f,g\rangle\hookrightarrow T$image$_2$(*Stat1*$\star$) $\Rightarrow$ *Stat4*: $c\notin\{[y,g\restriction y]:\ y\in\mathbf{dom}(g)\}\ \&\ z_0\in\mathbf{dom}(g)\ \&\ (f\cup g)\restriction z_0=g\restriction z_0$
EQUAL(*Stat3*) $\Rightarrow$ $c=[z_0,g\restriction z_0]$
$\langle z_0\rangle\hookrightarrow$*Stat4*(*Stat4*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO
$\langle z_0,g,f\rangle\hookrightarrow T$image$_2$(*Stat1*$\star$) $\Rightarrow$ *Stat5*: $c\notin\{[x,f\restriction x]:\ x\in\mathbf{dom}(f)\}\ \&\ z_0\in\mathbf{dom}(f)\ \&\ (g\cup f)\restriction z_0=f\restriction z_0$
EQUAL(*Stat3*) $\Rightarrow$ $c=[z_0,f\restriction z_0]$
$\langle z_0\rangle\hookrightarrow$*Stat5*(*Stat5*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO
(*Stat1*$\star$)ELEM $\Rightarrow$ *Stat6*: $c\notin\{[z,(f\cup g)\restriction z]:\ z\in\mathbf{dom}(f\cup g)\}\ \&\ c\in\{[x,f\restriction x]:\ x\in\mathbf{dom}(f)\}\cup\{[y,g\restriction y]:\ y\in\mathbf{dom}(g)\}$
Suppose $\Rightarrow$ *Stat7*: $c\in\{[x,f\restriction x]:\ x\in\mathbf{dom}(f)\}$
$\langle f\cup g,f,f\cup g\rangle\hookrightarrow T$domain$_1$ $\Rightarrow$ AUTO
$\langle x_0\rangle\hookrightarrow$*Stat7*$(\star)$ $\Rightarrow$ $c=[x_0,f\restriction x_0]\ \&\ x_0\notin\mathbf{dom}(g)\ \&\ x_0\in\mathbf{dom}(f\cup g)$
$\langle x_0,g,f\rangle\hookrightarrow T$image$_2$(*Stat7*$\star$) $\Rightarrow$ $(g\cup f)\restriction x_0=f\restriction x_0\ \&\ g\cup f=f\cup g$
EQUAL(*Stat7*) $\Rightarrow$ $c=[x_0,(f\cup g)\restriction x_0]$
$\langle x_0\rangle\hookrightarrow$*Stat6*(*Stat7*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat8*: $c\in\{[y,g\restriction y]:\ y\in\mathbf{dom}(g)\}$
$\langle f\cup g,g,f\cup g\rangle\hookrightarrow T$domain$_1$ $\Rightarrow$ AUTO
$\langle y_0\rangle\hookrightarrow$*Stat8*$(\star)$ $\Rightarrow$ $c=[y_0,g\restriction y_0]\ \&\ y_0\notin\mathbf{dom}(f)\ \&\ y_0\in\mathbf{dom}(f\cup g)$
$\langle y_0,f,g\rangle\hookrightarrow T$image$_2$(*Stat8*$\star$) $\Rightarrow$ $(f\cup g)\restriction y_0=g\restriction y_0$
EQUAL(*Stat8*) $\Rightarrow$ $c=[y_0,(f\cup g)\restriction y_0]$
$\langle y_0\rangle\hookrightarrow$*Stat6*(*Stat8*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ QED

THM singletonMap$_0$: [Domain and range of a singleton] $\mathbf{dom}(\{P\})=\{P^{[1]}\}\ \&\ \mathbf{range}(\{P\})=\{P^{[2]}\}$. PROOF:
Suppose_not(p_0) $\Rightarrow$ $\mathbf{dom}(\{p_0\})\neq\{p_0{}^{[1]}\}\vee\mathbf{range}(\{p_0\})\neq\{p_0{}^{[2]}\}$
Use_def($\mathbf{range}(\{p_0\})$) $\Rightarrow$ AUTO
Use_def($\mathbf{dom}$) $\Rightarrow$ *Stat1*: $\{p^{[1]}:\ p\in\{p_0\}\}\neq\{p_0{}^{[1]}\}\vee\{p^{[2]}:\ p\in\{p_0\}\}\neq\{p_0{}^{[2]}\}$

SIMPLF(*Stat1*⋆) ⇒ false; Discharge ⇒ QED

THM $singletonMap_1$: [Singleton maps are single valued] $\left(F \subseteq \{[X,Y]\} \rightarrow Svm(F)\right)$ & $\mathbf{dom}(\{[X,Y]\}) = \{X\}$ & $\mathbf{range}(\{[X,Y]\}) = \{Y\}$. PROOF:

Suppose_not(f_0, x_0, y_0) ⇒ AUTO

$\langle [x_0, y_0] \rangle \hookrightarrow T singletonMap_0$ ⇒ AUTO

Suppose ⇒ *Stat1* : $\neg\langle \forall p \in f_0, q \in f_0 \mid p^{[1]} = q^{[1]} \rightarrow p = q \rangle$

$\langle p_0, q_0 \rangle \hookrightarrow Stat1(Stat1\star)$ ⇒ $p_0, q_0 \in f_0$ & $p_0 \neq q_0$

Discharge ⇒ AUTO

Use_def$(Svm(f_0))$ ⇒ AUTO

Use_def(Is_map) ⇒ *Stat2* : $\neg\langle \forall p \in f_0 \mid p = [p^{[1]}, p^{[2]}] \rangle$ & $f_0 \subseteq \{[x_0, y_0]\}$

$\langle p_3 \rangle \hookrightarrow Stat2(Stat2)$ ⇒ false; Discharge ⇒ QED

THM $singletonMap_2$: [Singleton map application] $F = \{[X,Y]\} \rightarrow F{\upharpoonright}X = Y$. PROOF:

Suppose_not(f_0, x_0, y_0) ⇒ AUTO

Use_def(↾) ⇒ $f_0{\upharpoonright}x_0 = \mathbf{arb}(f_{0|\{x_0\}})^{[2]}$

Use_def(|) ⇒ *Stat1* : $f_{0|\{x_0\}} = \{p \in f_0 \mid p^{[1]} \in \{x_0\}\}$

Suppose ⇒ *Stat2* : $[x_0, y_0] \notin \{p \in f_0 \mid p^{[1]} \in \{x_0\}\}$

TELEM ⇒ $[x_0, y_0]^{[1]} \in \{x_0\}$

$\langle [x_0, y_0] \rangle \hookrightarrow Stat2(\star)$ ⇒ false; Discharge ⇒ AUTO

(*Stat1*)ELEM ⇒ $\mathbf{arb}(\{p \in f_0 \mid p^{[1]} \in \{x_0\}\}) \in \{p \in f_0 \mid p^{[1]} \in \{x_0\}\}$

EQUAL ⇒ *Stat3* : $\mathbf{arb}(f_{0|\{x_0\}}) \in \{p \in f_0 \mid p^{[1]} \in \{x_0\}\}$

$\langle\, \rangle \hookrightarrow Stat3(\star)$ ⇒ $\mathbf{arb}(f_{0|\{x_0\}}) = [x_0, y_0]$ & $[x_0, y_0]^{[1]} = x_0$

TELEM ⇒ $[x_0, y_0]^{[2]} = y_0$

EQUAL ⇒ false; Discharge ⇒ QED

THM $singletonMap_3$: [Transplant of singleton sub-map] $Svm(F)$ & $[X,Y] \in F$ & $Z \notin \mathbf{dom}(F)$ & $G = F \setminus \{[X,Y]\} \cup \{[Z,Y]\} \rightarrow$ $Svm(G)$ & $\mathbf{dom}(G) = \mathbf{dom}(F) \setminus \{X\} \cup \{Z\}$ & $\mathbf{range}(G) = \mathbf{range}(F)$. PROOF:

Suppose_not$(f_0, x_0, y_0, z_0, g_0)$ ⇒ AUTO

ELEM ⇒ *Stat0* : $Svm(g_0)$ & $\mathbf{dom}(g_0) = \mathbf{dom}(f_0) \setminus \{x_0\} \cup \{z_0\} \rightarrow \mathbf{range}(g_0) \neq \mathbf{range}(f_0)$

$\langle \{[x_0, y_0]\}, x_0, y_0 \rangle \hookrightarrow T singletonMap_1(\star)$ ⇒ $\mathbf{dom}(\{[x_0, y_0]\}) = \{x_0\}$ & $\mathbf{range}(\{[x_0, y_0]\}) = \{y_0\}$ & $f_0 \setminus \{[x_0, y_0]\} \cup \{[x_0, y_0]\} = f_0$

$\langle f_0 \setminus \{[x_0, y_0]\} \cup \{[x_0, y_0]\}, f_0 \setminus \{[x_0, y_0]\}, \{[x_0, y_0]\} \rangle \hookrightarrow T domain_1(\star)$ ⇒ $\mathbf{dom}(f_0 \setminus \{[x_0, y_0]\} \cup \{[x_0, y_0]\}) = \mathbf{dom}(f_0 \setminus \{[x_0, y_0]\}) \cup \{x_0\}$

EQUAL ⇒ $\mathbf{dom}(f_0) = \mathbf{dom}(f_0 \setminus \{[x_0, y_0]\}) \cup \{x_0\}$

Suppose ⇒ *Stat1* : $x_0 \in \mathbf{dom}(f_0 \setminus \{[x_0, y_0]\})$

Use_def(**dom**) ⇒ *Stat2* : $x_0 \in \{p^{[1]} : p \in f_0 \setminus \{[x_0, y_0]\}\}$

$\langle p_0 \rangle \hookrightarrow Stat2(Stat2)$ ⇒ $p_0 \in f_0$ & $p_0 \neq [x_0, y_0]$ & $p_0^{[1]} = [x_0, y_0]^{[1]}$

$\mathsf{Use\_def(Svm)} \Rightarrow$ $\quad Stat3: \langle \forall p \in f_0, q \in f_0 \mid p^{[1]} = q^{[1]} \rightarrow p = q \rangle$
$\langle p_0, [x_0, y_0] \rangle \hookrightarrow Stat3(\star) \Rightarrow$ $\quad$ false; $\quad$ Discharge $\Rightarrow$ $\quad$ AUTO
$\langle \{[z_0, y_0]\}, z_0, y_0 \rangle \hookrightarrow T\mathsf{singletonMap}_1(\star) \Rightarrow$ $\quad \mathsf{Svm}(\{[z_0, y_0]\})\ \&\ \mathbf{dom}(\{[z_0, y_0]\}) = \{z_0\}\ \&\ \mathbf{range}(\{[z_0, y_0]\}) = \{y_0\}$
$\langle f_0 \backslash \{[x_0, y_0]\} \cup \{[z_0, y_0]\}, f_0 \backslash \{[x_0, y_0]\}, \{[z_0, y_0]\} \rangle \hookrightarrow T\mathsf{domain}_1(\star) \Rightarrow$ $\quad \mathbf{dom}(f_0 \backslash \{[x_0, y_0]\} \cup \{[z_0, y_0]\}) = \mathbf{dom}(f_0) \backslash \{x_0\} \cup \{z_0\}$
$\langle f_0, f_0 \backslash \{[x_0, y_0]\} \rangle \hookrightarrow T\mathsf{svm}_0(\star) \Rightarrow$ $\quad \mathsf{Svm}(f_0 \backslash \{[x_0, y_0]\})$
$\langle f_0, f_0 \backslash \{[x_0, y_0]\}, f_0 \rangle \hookrightarrow T\mathsf{domain}_1(\star) \Rightarrow$ $\quad z_0 \notin \mathbf{dom}(f_0 \backslash \{[x_0, y_0]\})$
$\langle f_0 \backslash \{[x_0, y_0]\}, \{[z_0, y_0]\} \rangle \hookrightarrow T\mathsf{svm}_1(\star) \Rightarrow$ $\quad \mathsf{Svm}(f_0 \backslash \{[x_0, y_0]\} \cup \{[z_0, y_0]\})$
$\langle f_0 \backslash \{[x_0, y_0]\} \cup \{[z_0, y_0]\}, f_0 \backslash \{[x_0, y_0]\}, \{[z_0, y_0]\} \rangle \hookrightarrow T\mathsf{range}_1(\star) \Rightarrow$ $\quad \mathbf{range}(f_0 \backslash \{[x_0, y_0]\} \cup \{[z_0, y_0]\}) = \mathbf{range}(f_0 \backslash \{[x_0, y_0]\}) \cup \{y_0\}$
$\langle f_0 \backslash \{[x_0, y_0]\} \cup \{[x_0, y_0]\}, f_0 \backslash \{[x_0, y_0]\}, \{[x_0, y_0]\} \rangle \hookrightarrow T\mathsf{range}_1(\star) \Rightarrow$ $\quad \mathbf{range}(f_0 \backslash \{[x_0, y_0]\} \cup \{[x_0, y_0]\}) = \mathbf{range}(f_0 \backslash \{[x_0, y_0]\}) \cup \{y_0\}$
EQUAL $\Rightarrow$ $\quad$ false; $\quad$ Discharge $\Rightarrow$ $\quad$ QED

1.2 Basic laws on the power-set global operation

DEF $\mathcal{P}$: [Family of all subsets of a given set] $\quad \mathcal{P}S \quad =_{\text{Def}} \quad \{x : x \subseteq S\}$

About the power set operation, in this section we will prove the collection of statement displayed here:

THM pow_0: [Characterization of powerset; also: no set equals its own powerset] $(X \supseteq Y \leftrightarrow Y \in \mathcal{P}X)\ \&\ X \neq \mathcal{P}X$

THM pow_1: [Monotonicity of powerset] $S \supseteq X \rightarrow \mathcal{P}X \cup \{\emptyset, X\} \subseteq \mathcal{P}S$

THM pow_2: [Powerset of null set and of singletons] $\mathcal{P}\emptyset = \{\emptyset\}\ \&\ \mathcal{P}\{X\} = \{\emptyset, \{X\}\}$

Our next theorem characterizes the powerset formation operation in more usable terms than the very definition of this construct. It also proves that no set can equal is own powerset (else it should belong to itself, against the acyclicity of membership).

THM pow_0: [Characterization of powerset; also: no set equals its own powerset] $(X \supseteq Y \leftrightarrow Y \in \mathcal{P}X)\ \&\ X \neq \mathcal{P}X$. PROOF:
$\mathsf{Suppose\_not}(x_0, y_0) \Rightarrow$ $\quad$ AUTO

We begin by excluding the possibility that $x_0 = \mathcal{P}x_0$:

Suppose $\Rightarrow$ $\quad Stat0: x_0 \notin \{y : y \subseteq x_0\}$
$\langle x_0 \rangle \hookrightarrow Stat0 \Rightarrow$ $\quad$ false; $\quad$ Discharge $\Rightarrow$ $\quad$ AUTO
$\mathsf{Use\_def}(\mathcal{P}x_0) \Rightarrow$ $\quad$ AUTO

Arguing by contradiction, if x_0, y_0 constituted a counterexample, then either one of the literals $x_0 \supseteq y_0$ and $y_0 \in \{y : y \subseteq x_0\}$ would be true and the other one would be false.

EQUAL $\Rightarrow$ $\quad Stat1: x_0 \supseteq y_0 \neq y_0 \in \{y : y \subseteq x_0\}$

If it is the second that is true then, via a substitution in the setformer, we would contradict the falsity of the first.

Suppose ⇒ $\mathit{Stat2}: \ y_0 \in \{y : y \subseteq x_0\}$
$\langle y_1 \rangle \hookrightarrow \mathit{Stat2}(\mathit{Stat1}\star) \Rightarrow$ false; Discharge ⇒ $\mathit{Stat3}: \ y_0 \notin \{y : y \subseteq x_0\}$

But then the literals $x_0 \supseteq y_0$ and $y_0 \notin \{y : y \subseteq x_0\}$ should hold together, which gives us a contradiction if we replace the bounded variable y of the setformer by y_0.

$\langle y_0 \rangle \hookrightarrow \mathit{Stat3}(\mathit{Stat1}\star) \Rightarrow$ false; Discharge ⇒ QED

THM pow_1: [Monotonicity of powerset] $S \supseteq X \to \mathcal{P}X \cup \{\emptyset, X\} \subseteq \mathcal{P}S$. PROOF:
Suppose_not(s_0, x_0) ⇒ AUTO
Set_monot ⇒ $\{x : x \subseteq x_0\} \subseteq \{x : x \subseteq s_0\}$
Use_def$(\mathcal{P})$ ⇒ $\mathit{Stat1}: \ \emptyset \notin \{x : x \subseteq s_0\} \vee x_0 \notin \{x : x \subseteq s_0\}$
$\langle \emptyset, x_0 \rangle \hookrightarrow \mathit{Stat1} \Rightarrow$ false; Discharge ⇒ QED

THM pow_2: [Powerset of null set and of singletons] $\mathcal{P}\emptyset = \{\emptyset\} \ \& \ \mathcal{P}\{X\} = \{\emptyset, \{X\}\}$. PROOF:
Suppose_not(x_0) ⇒ AUTO
Suppose ⇒ $\mathcal{P}\emptyset \neq \{\emptyset\}$
$\langle \emptyset, \emptyset \rangle \hookrightarrow T\mathsf{pow}_1 \Rightarrow$ $\mathit{Stat0}: \ \mathcal{P}\emptyset \nsubseteq \{\emptyset\}$
$\langle y_0 \rangle \hookrightarrow \mathit{Stat0}(\mathit{Stat0}\star) \Rightarrow$ $\mathit{Stat1}: \ y_0 \in \mathcal{P}\emptyset \ \& \ y_0 \notin \{\emptyset\}$
$\langle \emptyset, y_0 \rangle \hookrightarrow T\mathsf{pow}_0(\mathit{Stat1}\star) \Rightarrow$ false; Discharge ⇒ $\mathcal{P}\{x_0\} \neq \{\emptyset, \{x_0\}\}$
$\langle \{x_0\}, \{x_0\} \rangle \hookrightarrow T\mathsf{pow}_1 \Rightarrow$ $\mathit{Stat2}: \ \mathcal{P}\{x_0\} \nsubseteq \{\emptyset, \{x_0\}\}$
$\langle y_1 \rangle \hookrightarrow \mathit{Stat2} \Rightarrow$ $\mathit{Stat3}: \ y_1 \in \mathcal{P}\{x_0\} \ \& \ y_1 \notin \{\emptyset, \{x_0\}\}$
$\langle \{x_0\}, y_1 \rangle \hookrightarrow T\mathsf{pow}_0(\mathit{Stat3}\star) \Rightarrow$ false; Discharge ⇒ QED

1.3 Basic laws on the finiteness property

Traditionally, finiteness is defined through the notion of cardinality of a set: a set is finite if its cardinality precedes the first infinite ordinal. As a shortcut, to begin developing an acceptable formal treatment of finiteness without much preparatory work, we adopt here the following definition (reminiscent of Tarski's 1924 paper "Sur les ensembles fini"): a set F is *finite* if every non-null family of subsets of F owns an inclusion-minimal element. This notion can be specified very succinctly in terms of the powerset operator.

DEF Fin: [Finiteness property] $\mathsf{Finite}(F) \leftrightarrow_{\mathrm{Def}} \langle \forall g \in \mathcal{P}(\mathcal{P}F) \setminus \{\emptyset\}, \exists m \mid g \cap \mathcal{P}m = \{m\} \rangle$

About finitude, in this section we will develop two useful THEORYs—une particularly important for arguments based on finite induction—; moreover, we will prove the collection of statement displayed here:

THM fin_0: [Monotonicity of finiteness] $\mathsf{Y} \supseteq \mathsf{X}\ \&\ \mathsf{Finite}(\mathsf{Y}) \rightarrow \mathsf{Finite}(\mathsf{X})$

THM fin_1: [Finiteness of the union of a finite set with a singleton] $\mathsf{Finite}(\mathsf{F}) \rightarrow \mathsf{Finite}(\mathsf{F} \cup \{\mathsf{X}\})$

THM fin_2: [Singletons are finite] $\mathsf{Finite}(\{\mathsf{X}\})\ \&\ \mathsf{Finite}(\emptyset)$

THM $\mathsf{part\_whole}_0$: [The domain operation applied to a single-valued map preserves finiteness] $\mathsf{Svm}(\mathsf{F}) \rightarrow \big(\mathsf{Finite}(\mathsf{F}) \leftrightarrow \mathsf{Finite}(\mathbf{dom}(\mathsf{F}))\big)$

THM $\mathsf{part\_whole}_1$: [The part is smaller than the whole] $\mathsf{Svm}(\mathsf{H})\ \&\ \mathsf{Finite}(\mathsf{H})\ \&\ \mathbf{range}(\mathsf{H}) \supseteq \mathbf{dom}(\mathsf{H}) \rightarrow \mathbf{range}(\mathsf{H}) = \mathbf{dom}(\mathsf{H})$

THM fin_0: [Monotonicity of finiteness] $\mathsf{Y} \supseteq \mathsf{X}\ \&\ \mathsf{Finite}(\mathsf{Y}) \rightarrow \mathsf{Finite}(\mathsf{X})$. PROOF:

Suppose_not$(\mathsf{y}_0, \mathsf{x}_0) \Rightarrow$ AUTO

$\langle \mathsf{y}_0, \mathsf{x}_0 \rangle \hookrightarrow T\mathsf{pow}_1(\star) \Rightarrow$ $\mathcal{P}\mathsf{y}_0 \supseteq \mathcal{P}\mathsf{x}_0$

Use_def(Finite) $\Rightarrow$ *Stat1*: $\neg\langle \forall \mathsf{g} \in \mathcal{P}(\mathcal{P}\mathsf{x}_0) \setminus \{\emptyset\}, \exists \mathsf{m} \mid \mathsf{g} \cap \mathcal{P}\mathsf{m} = \{\mathsf{m}\} \rangle\ \&\ \langle \forall \mathsf{g}' \in \mathcal{P}(\mathcal{P}\mathsf{y}_0) \setminus \{\emptyset\}, \exists \mathsf{m} \mid \mathsf{g}' \cap \mathcal{P}\mathsf{m} = \{\mathsf{m}\} \rangle$

$\langle \mathcal{P}\mathsf{y}_0, \mathcal{P}\mathsf{x}_0 \rangle \hookrightarrow T\mathsf{pow}_1(\star) \Rightarrow$ $\mathcal{P}(\mathcal{P}\mathsf{y}_0) \supseteq \mathcal{P}(\mathcal{P}\mathsf{x}_0)$

$\langle \mathsf{g}_0, \mathsf{g}_0 \rangle \hookrightarrow$ *Stat1(Stat1⋆)* $\Rightarrow$ $\neg\langle \exists \mathsf{m} \mid \mathsf{g}_0 \cap \mathcal{P}\mathsf{m} = \{\mathsf{m}\} \rangle\ \&\ \langle \exists \mathsf{m} \mid \mathsf{g}_0 \cap \mathcal{P}\mathsf{m} = \{\mathsf{m}\} \rangle$

Discharge $\Rightarrow$ QED

THEORY $\mathsf{finiteInduction}(\mathsf{s}_0, \mathsf{P}(\mathsf{S}))$

$\mathsf{Finite}(\mathsf{s}_0)\ \&\ \mathsf{P}(\mathsf{s}_0)$

END finiteInduction

ENTER_THEORY finiteInduction

THM $\mathsf{finiteInduction}_0$. $\langle \exists \mathsf{m} \mid \{\mathsf{s} \subseteq \mathsf{s}_0 \mid \mathsf{P}(\mathsf{s})\} \cap \mathcal{P}\mathsf{m} = \{\mathsf{m}\} \rangle$. PROOF:

Suppose_not() $\Rightarrow$ AUTO

Assump $\Rightarrow$ $\mathsf{Finite}(\mathsf{s}_0)\ \&\ \mathsf{P}(\mathsf{s}_0)$

Use_def(Finite) $\Rightarrow$ *Stat1*: $\langle \forall \mathsf{g} \in \mathcal{P}(\mathcal{P}\mathsf{s}_0) \setminus \{\emptyset\}, \exists \mathsf{m} \mid \mathsf{g} \cap \mathcal{P}\mathsf{m} = \{\mathsf{m}\} \rangle$

$\langle \{\mathsf{s} \subseteq \mathsf{s}_0 \mid \mathsf{P}(\mathsf{s})\} \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ $\{\mathsf{s} \subseteq \mathsf{s}_0 \mid \mathsf{P}(\mathsf{s})\} \notin \mathcal{P}(\mathcal{P}\mathsf{s}_0) \setminus \{\emptyset\}$

Suppose $\Rightarrow$ *Stat2*: $\mathsf{s}_0 \notin \{\mathsf{s} \subseteq \mathsf{s}_0 \mid \mathsf{P}(\mathsf{s})\}$

$\langle \mathsf{s}_0 \rangle \hookrightarrow$ *Stat2* $\Rightarrow$ false; Discharge $\Rightarrow$ $\{\mathsf{s} \subseteq \mathsf{s}_0 \mid \mathsf{P}(\mathsf{s})\} \notin \mathcal{P}(\mathcal{P}\mathsf{s}_0)$

Use_def($\mathcal{P}$) $\Rightarrow$ *Stat3*: $\{\mathsf{s} \subseteq \mathsf{s}_0 \mid \mathsf{P}(\mathsf{s})\} \notin \{\mathsf{y} : \mathsf{y} \subseteq \{\mathsf{z} : \mathsf{z} \subseteq \mathsf{s}_0\}\}$

$\langle \{\mathsf{s} \subseteq \mathsf{s}_0 \mid \mathsf{P}(\mathsf{s})\} \rangle \hookrightarrow$ *Stat3* $\Rightarrow$ *Stat4*: $\{\mathsf{s} \subseteq \mathsf{s}_0 \mid \mathsf{P}(\mathsf{s})\} \not\subseteq \{\mathsf{z} : \mathsf{z} \subseteq \mathsf{s}_0\}$

$\langle \mathsf{s}_1 \rangle \hookrightarrow$ *Stat4* $\Rightarrow$ *Stat5*: $\mathsf{s}_1 \in \{\mathsf{s} : \mathsf{s} \subseteq \mathsf{s}_0 \mid \mathsf{P}(\mathsf{s})\}\ \&\ \mathsf{s}_1 \notin \{\mathsf{z} : \mathsf{z} \subseteq \mathsf{s}_0\}$

$\langle \mathsf{s}, \mathsf{s}_1 \rangle \hookrightarrow$ *Stat5(Stat5⋆)* $\Rightarrow$ false; Discharge $\Rightarrow$ QED

APPLY $\langle \mathsf{v1}_\Theta : \mathsf{fin}_\Theta \rangle$ Skolem$\Rightarrow$

THM finiteInduction$_1$. $\{s \subseteq s_0 \mid P(s)\} \cap \mathcal{P}fin_\Theta = \{fin_\Theta\}$.

THM finiteInduction$_2$: [Minimal finite set satisfying P] $S \subseteq fin_\Theta \to Finite(S)$ & $(P(S) \leftrightarrow S = fin_\Theta)$. PROOF:
Suppose_not(s_1) ⇒ AUTO
$\langle\,\rangle \hookrightarrow T$finiteInduction$_1$ ⇒ $\{s \subseteq s_0 \mid P(s)\} \cap \mathcal{P}fin_\Theta = \{fin_\Theta\}$ & *Stat1* : $fin_\Theta \in \{s \subseteq s_0 \mid P(s)\}$
$\langle\,\rangle \hookrightarrow$ *Stat1* ⇒ $fin_\Theta \subseteq s_0$ & $P(fin_\Theta)$
Assump ⇒ $Finite(s_0)$
$\langle s_0, fin_\Theta\rangle \hookrightarrow T fin_0$ ⇒ $Finite(fin_\Theta)$
$\langle fin_\Theta, s_1\rangle \hookrightarrow T fin_0$ ⇒ $P(s_1) \neq s_1 = fin_\Theta$
Suppose ⇒ $s_1 = fin_\Theta$
EQUAL ⇒ false; Discharge ⇒ $s_1 \notin \{s \subseteq s_0 \mid P(s)\} \cap \mathcal{P}fin_\Theta$ & $P(s_1)$
Suppose ⇒ $s_1 \notin \mathcal{P}fin_\Theta$
Use_def$(\mathcal{P})$ ⇒ *Stat2* : $s_1 \notin \{y : y \subseteq fin_\Theta\}$
$\langle s_1\rangle \hookrightarrow$ *Stat2* ⇒ false; Discharge ⇒ *Stat3* : $s_1 \notin \{s \subseteq s_0 \mid P(s)\}$
$\langle s_1\rangle \hookrightarrow$ *Stat3* ⇒ false; Discharge ⇒ QED

ENTER_THEORY Set_theory

DISPLAY finiteInduction

THEORY finiteInduction$(s_0, P(S))$
$Finite(s_0)$ & $P(s_0)$
⇒ (fin_Θ)
$\langle \forall S \mid S \subseteq fin_\Theta \to Finite(S)$ & $(P(S) \leftrightarrow S = fin_\Theta)\rangle$
END finiteInduction

THM fin$_1$: [Finiteness of the union of a finite set with a singleton] $Finite(F) \to Finite(F \cup \{X\})$. PROOF:
Suppose_not(f_0, x_0) ⇒ AUTO

> Arguing by contradiction, suppose that f_0 and x_0 are such that f_0 is finite but $f_0 \cup \{x_0\}$ is not. A nonnull familiy g_0 of subsets of $f_0 \cup \{x_0\}$ must then exist none of whose elements is minimal. On the other hand $\{y \setminus \{x_0\} : y \in g_0\}$, which is also nonnull but consists entirely of subsets of f_0, must have a minimal element $m_0 = y_0 \setminus \{x_0\}$, with $y_0 \in g_0$.

Use_def(Finite) ⇒ *Stat0* : $\neg\langle \forall g \in \mathcal{P}(\mathcal{P}(f_0 \cup \{x_0\})) \setminus \{\emptyset\}, \exists m \mid g \cap \mathcal{P}m = \{m\}\rangle$ & *Stat1* : $\langle \forall h \in \mathcal{P}(\mathcal{P}f_0) \setminus \{\emptyset\}, \exists m \mid h \cap \mathcal{P}m = \{m\}\rangle$
$\langle g_0\rangle \hookrightarrow$ *Stat0(Stat0)* ⇒ *Stat2* : $\neg\langle \exists m \mid g_0 \cap \mathcal{P}m = \{m\}\rangle$ & $g_0 \in \mathcal{P}(\mathcal{P}(f_0 \cup \{x_0\}))$ & $g_0 \neq \emptyset$
Loc_def ⇒ *Stat3* : $h_0 = \{y \setminus \{x_0\} : y \in g_0\}$
Suppose ⇒ $h_0 \notin \mathcal{P}(\mathcal{P}f_0) \setminus \{\emptyset\}$
Suppose ⇒ *Stat4* : $\{y \setminus \{x_0\} : y \in g_0\} = \emptyset$
$\langle$**arb**$(g_0)\rangle \hookrightarrow$ *Stat4(Stat2, Stat2)* ⇒ false; Discharge ⇒ AUTO

$\mathsf{Use\_def}(\mathcal{P}) \Rightarrow$ *Stat5*: $h_0 \notin \{h : h \subseteq \{k : k \subseteq f_0\}\}$
$\langle h_0 \rangle \hookrightarrow$ *Stat5*(*Stat5*$\star$) $\Rightarrow$ *Stat6*: $h_0 \not\subseteq \{k : k \subseteq f_0\}$
$\langle k_0 \rangle \hookrightarrow$ *Stat6*(*Stat3*$\star$) $\Rightarrow$ *Stat7*: $k_0 \in \{y \setminus \{x_0\} : y \in g_0\}$ & $k_0 \notin \{k : k \subseteq f_0\}$
$\langle y_1, k_0 \rangle \hookrightarrow$ *Stat7*(*Stat7*$\star$) $\Rightarrow$ $y_1 \in g_0$ & $y_1 \not\subseteq f_0 \cup \{x_0\}$
$\mathsf{Use\_def}(\mathcal{P}) \Rightarrow$ *Stat8*: $g_0 \in \{h : h \subseteq \{k : k \subseteq f_0 \cup \{x_0\}\}\}$
$\langle h_1 \rangle \hookrightarrow$ *Stat8*(*Stat7*$\star$) $\Rightarrow$ *Stat9*: $y_1 \in \{k : k \subseteq f_0 \cup \{x_0\}\}$
$\langle k_1 \rangle \hookrightarrow$ *Stat9*(*Stat7*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO
$\langle h_0, m_0 \rangle \hookrightarrow$ *Stat1*(*Stat3*$\star$) $\Rightarrow$ *Stat10*: $m_0 \in \{y \setminus \{x_0\} : y \in g_0\}$ & $h_0 \cap \mathcal{P}m_0 = \{m_0\}$
$\langle y_0 \rangle \hookrightarrow$ *Stat10*(*Stat10*$\star$) $\Rightarrow$ *Stat11*: $m_0 = y_0 \setminus \{x_0\}$ & $y_0 \in g_0$

We will reach the desired contradiction by showing that either m_0 or $y_0 = m_0 \cup \{x_0\}$ is minimal in g_0. We check first that m_0 itself must be minimal when $m_0 \in g_0$.

Suppose $\Rightarrow$ $m_0 \in g_0$
$\langle m_0 \rangle \hookrightarrow$ *Stat2*(*Stat10*$\star$) $\Rightarrow$ *Stat12*: $g_0 \cap \mathcal{P}m_0 \not\subseteq \{m_0\}$
$\mathsf{Use\_def}(\mathcal{P}m_0) \Rightarrow$ AUTO
$\langle z_0 \rangle \hookrightarrow$ *Stat12*(*Stat3*$\star$) $\Rightarrow$ *Stat13*: $z_0 \in \{h : h \subseteq m_0\}$ & $z_0 \notin \{y \setminus \{x_0\} : y \in g_0\}$ & $z_0 \in g_0$
$\langle h_2, z_0 \rangle \hookrightarrow$ *Stat13*(*Stat11*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO

Suppose next that $m_0 \notin g_0$; we will reach a contradiction by showing that y_0 is minimal in g_0.

Suppose $\Rightarrow$ $y_0 \notin \mathcal{P}y_0$
$\mathsf{Use\_def}(\mathcal{P}) \Rightarrow$ *Stat13a*: $y_0 \notin \{s : s \subseteq y_0\}$
$\langle y_0 \rangle \hookrightarrow$ *Stat13a*(*Stat13a*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO
$\langle y_0 \rangle \hookrightarrow$ *Stat2*(*Stat11*$\star$) $\Rightarrow$ *Stat14*: $g_0 \cap \mathcal{P}y_0 \not\subseteq \{y_0\}$
$\mathsf{Use\_def}(\mathcal{P}y_0) \Rightarrow$ AUTO
$\langle z_1 \rangle \hookrightarrow$ *Stat14*(*Stat11*$\star$) $\Rightarrow$ *Stat15*: $z_1 \in \{h : h \subseteq y_0\}$ & $z_1 \in g_0$ & $z_1 \setminus \{x_0\} \neq y_0 \setminus \{x_0\}$
EQUAL(*Stat10*) $\Rightarrow$ $h_0 \cap \mathcal{P}(y_0 \setminus \{x_0\}) = \{y_0 \setminus \{x_0\}\}$
Suppose $\Rightarrow$ $z_1 \setminus \{x_0\} \notin \mathcal{P}(y_0 \setminus \{x_0\})$
$\mathsf{Use\_def}(\mathcal{P}) \Rightarrow$ *Stat16*: $z_1 \setminus \{x_0\} \notin \{h : h \subseteq y_0 \setminus \{x_0\}\}$
$\langle z_1 \setminus \{x_0\} \rangle \hookrightarrow$ *Stat16*(*Stat16*$\star$) $\Rightarrow$ $z_1 \setminus \{x_0\} \not\subseteq y_0 \setminus \{x_0\}$
$\langle h_3 \rangle \hookrightarrow$ *Stat15*(*Stat16*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO
Suppose $\Rightarrow$ *Stat17*: $z_1 \setminus \{x_0\} \notin \{y \setminus \{x_0\} : y \in g_0\}$
$\langle z_1 \rangle \hookrightarrow$ *Stat17*(*Stat15*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ $z_1 \setminus \{x_0\} \in h_0$
(*Stat15*$\star$)Discharge $\Rightarrow$ QED

THM fin_2: [Singletons are finite] $\mathsf{Finite}(\{X\})$ & $\mathsf{Finite}(\emptyset)$. PROOF:
$\mathsf{Suppose\_not}(x_0) \Rightarrow$ AUTO
$\langle \{x_0\}, \emptyset \rangle \hookrightarrow T\mathsf{fin}_0$ $\Rightarrow$ $\neg\mathsf{Finite}(\{x_0\})$

Use_def(Finite) $\Rightarrow$ $\quad$ *Stat1* : $\neg\langle\forall g \in \mathcal{P}(\mathcal{P}\{x_0\})\setminus\{\emptyset\}, \exists m \mid g \cap \mathcal{P}m = \{m\}\rangle$
$\langle g_0\rangle\hookrightarrow$*Stat1* $\Rightarrow$ $\quad$ *Stat2* : $\neg\langle\exists m \mid g_0 \cap \mathcal{P}m = \{m\}\rangle \;\&\; g_0 \in \mathcal{P}(\mathcal{P}\{x_0\})\setminus\{\emptyset\}$
Use_def($\mathcal{P}$) $\Rightarrow$ $\quad$ *Stat3* : $g_0 \in \{y : y \subseteq \mathcal{P}\{x_0\}\}$
$\langle x_0\rangle\hookrightarrow T\mathsf{pow}_2$ $\Rightarrow$ $\quad$ AUTO
$\langle y_0\rangle\hookrightarrow$*Stat3*(*Stat2*$\star$) $\Rightarrow$ $\quad$ *Stat4* : $g_0 \neq \emptyset \;\&\; g_0 \subseteq \{\emptyset, \{x_0\}\}$
Suppose $\Rightarrow$ $\quad$ $\emptyset \in g_0$
$\langle\emptyset\rangle\hookrightarrow$*Stat2*(*Stat3*$\star$) $\Rightarrow$ $\quad$ false; $\quad$ Discharge $\Rightarrow$ $\quad$ $g_0 = \{\{x_0\}\}$
$\langle\{x_0\}\rangle\hookrightarrow$*Stat2*(*Stat3*$\star$) $\Rightarrow$ $\quad$ false; $\quad$ Discharge $\Rightarrow$ $\quad$ QED

THEORY finiteImage(s_0, f(X))
Finite(s_0)
END finiteImage

ENTER_THEORY finiteImage

THM finiteImage. Finite($\{f(x) : x \in s_0\}$). PROOF:
Suppose_not() $\Rightarrow$ $\quad$ AUTO

> We can prove the claim by means of finite induction. Arguing by contradiction, let us assume that s_0 has, via the global function $f(X)$, infinite image; then take an s_1 which is finite and minimal (w. r. t. inclusion) and has, much like s_0, infinite image $\{f(x) : x \in s_1\}$. As one sees easily, $s_1 \neq \emptyset$; hence, if we remove an element a from s_1, we find that $\{f(x) : x \in s_1\setminus\{a\}\}$ is finite in consequence of the supposed minimality of s_1. Since the union of two finite sets is finite, we get the finiteness of $\{f(x) : x \in s_1\setminus\{a\}\} \cup \{f\}(a)$, which hence must differ from $\{f(x) : x \in s_1\}$.

Assump $\Rightarrow$ $\quad$ Finite(s_0)
APPLY $\langle \mathsf{fin}_\Theta : s_1\rangle$ finiteInduction$\left(s_0 \mapsto s_0, P(S) \mapsto \neg\mathrm{Finite}(\{f(x) : x \in S\})\right)$ $\Rightarrow$ $\quad$ *Stat1* : $\langle\forall s \mid s \subseteq s_1 \rightarrow \mathrm{Finite}(s) \;\&\; \left(\neg\mathrm{Finite}(\{f(x) : x \in s\}) \leftrightarrow s = s_1\right)\rangle$
$\langle s_1\rangle\hookrightarrow$*Stat1* $\Rightarrow$ $\quad$ $\neg$Finite($\{f(x) : x \in s_1\}$)
Loc_def $\Rightarrow$ $\quad$ *Stat0* : $a = \mathbf{arb}(s_1)$
$\langle f(a)\rangle\hookrightarrow T\mathsf{fin}_2$ $\Rightarrow$ $\quad$ Finite($\{f\}(a)$) & Finite($\emptyset$)
Suppose $\Rightarrow$ $\quad$ $s_1 = \emptyset$
ELEM $\Rightarrow$ $\quad$ $\{f(x) : x \in \emptyset\} = \emptyset$
EQUAL $\Rightarrow$ $\quad$ false; $\quad$ Discharge $\Rightarrow$ $\quad$ AUTO
(*Stat0*)ELEM $\Rightarrow$ $\quad$ *Stat2* : $s_1\setminus\{a\} \subseteq s_1 \;\&\; s_1\setminus\{a\} \neq s_1$
Suppose $\Rightarrow$ $\quad$ $\{f(x) : x \in s_1\} = \{f(x) : x \in s_1\setminus\{a\}\} \cup \{f\}(a)$
$\langle s_1\setminus\{a\}\rangle\hookrightarrow$*Stat1*(*Stat2*$\star$) $\Rightarrow$ $\quad$ Finite($\{f(x) : x \in s_1\setminus\{a\}\}$)
$\langle\{f(x) : x \in s_1\setminus\{a\}\}, f(a)\rangle\hookrightarrow T\mathsf{fin}_1$(*Stat1*$\star$) $\Rightarrow$ $\quad$ Finite($\{f(x) : x \in s_1\setminus\{a\}\} \cup \{f\}(a)$)
EQUAL(*Stat1*) $\Rightarrow$ $\quad$ false; $\quad$ Discharge $\Rightarrow$ $\quad$ AUTO

‖ On the other hand, $\{f(x) : x \in s_1 \setminus \{a\}\} \cup \{f\}(a)$ and $\{f(x) : x \in s_1\}$ must be equal: in fact $a \in s_1$, and therefore $f(a) \in \{f(x) : x \in s_1\}$; moreover, by monotonicity, $\{f(x) : x \in s_1 \setminus \{a\}\} \subseteq \{f(x) : x \in s_1\}$ and ...

Set_monot ⇒ $\{f(x) : x \in s_1 \setminus \{a\}\} \subseteq \{f(x) : x \in s_1\}$
Suppose ⇒ *Stat3*: $f(a) \notin \{f(x) : x \in s_1\}$
$\langle a \rangle \hookrightarrow$ *Stat3(Stat2, Stat2⋆)* ⇒ false; Discharge ⇒ *Stat4*: $\{f(x) : x \in s_1\} \not\subseteq \{f(x) : x \in s_1 \setminus \{a\}\} \cup \{f\}(a)$

‖ ... one easily sees that $\{f(x) : x \in s_1\} \subseteq \{f(x) : x \in s_1 \setminus \{a\}\} \cup \{f\}(a)$, ...

$\langle b \rangle \hookrightarrow$ *Stat4(Stat4⋆)* ⇒ *Stat5*: $b \in \{f(x) : x \in s_1\}$ & $b \notin \{f(x) : x \in s_1 \setminus \{a\}\} \cup \{f\}(a)$
$\langle x_0 \rangle \hookrightarrow$ *Stat5(Stat5⋆)* ⇒ $f(x_0) \notin \{f(x) : x \in s_1 \setminus \{a\}\}$ & $x_0 \in s_1$ & $f(x_0) \neq f(a)$
Suppose ⇒ $x_0 = a$
EQUAL(*Stat5*) ⇒ false; Discharge ⇒ *Stat6*: $f(x_0) \notin \{f(x) : x \in s_1 \setminus \{a\}\}$ & $x_0 \neq a$ & $x_0 \in s_1$
$\langle x_0 \rangle \hookrightarrow$ *Stat6(Stat6⋆)* ⇒ false

‖ which leads us to the sought contradiction.

Discharge ⇒ QED

ENTER_THEORY Set_theory
DISPLAY finiteImage

THEORY finiteImage$(s_0, f(X))$
Finite(s_0)
⇒
Finite$(\{f(x) : x \in s_0\})$
END finiteImage

THM part_whole$_0$: [The domain operation applied to a single-valued map preserves finiteness] $\mathsf{Svm}(F) \to (\mathsf{Finite}(F) \leftrightarrow \mathsf{Finite}(\mathbf{dom}(F)))$. PROOF:
Suppose_not(f_1) ⇒ AUTO
Suppose ⇒ Finite(f_1)
APPLY ⟨ ⟩ finiteImage$(s_0 \mapsto f_1, f(X) \mapsto X^{[1]})$ ⇒ Finite$(\{x^{[1]} : x \in f_1\})$
Use_def(**dom**) ⇒ false; Discharge ⇒ AUTO
$\langle f_1 \rangle \hookrightarrow$ *Timage*$_5$ ⇒ $f_1 = \{[x, f_1 \upharpoonright x] : x \in \mathbf{dom}(f_1)\}$
APPLY ⟨ ⟩ finiteImage$(s_0 \mapsto \mathbf{dom}(f_1), f(X) \mapsto [x, f_1 \upharpoonright x])$ ⇒ Finite$(\{[x, f_1 \upharpoonright x] : x \in \mathbf{dom}(f_1)\})$
EQUAL ⇒ false; Discharge ⇒ QED

THM part_whole$_1$: [The part is smaller than the whole] $\mathsf{Svm}(H)$ & $\mathsf{Finite}(H)$ & $\mathbf{range}(H) \supseteq \mathbf{dom}(H) \to \mathbf{range}(H) = \mathbf{dom}(H)$. PROOF+:
Suppose_not(h_2) ⇒ AUTO

For, assuming that a counterexample h_1 to the claim exists, a minimal counterexample h_0 would be provided by the finite induction principle.

APPLY $\langle \mathsf{fin}_\Theta : h_0 \rangle$ finiteInduction$\Big(s_0 \mapsto h_2, P(S) \mapsto (\mathsf{Svm}(S) \,\&\, \mathbf{range}(S) \supseteq \mathbf{dom}(S) \,\&\, \mathbf{range}(S) \neq \mathbf{dom}(S))\Big) \Rightarrow$

Stat1 : $\langle \forall S \mid S \subseteq h_0 \rightarrow \mathsf{Finite}(S) \,\&\, (\mathsf{Svm}(S) \,\&\, \mathbf{range}(S) \supseteq \mathbf{dom}(S) \,\&\, \mathbf{range}(S) \neq \mathbf{dom}(S) \leftrightarrow S = h_0)\rangle$

$\langle h_0 \rangle \hookrightarrow$ *Stat1(Stat1⋆)* $\Rightarrow$ *Stat2* : $\mathbf{range}(h_0) \neq \mathbf{dom}(h_0) \,\&\, \mathsf{Finite}(h_0) \,\&\, \mathsf{Svm}(h_0) \,\&\, \mathbf{range}(h_0) \supseteq \mathbf{dom}(h_0)$

Let us pick a p_0 in h_0 such that $p_0^{[2]} \notin \mathbf{dom}(h_0)$ and put $h_1 = h_0 \setminus \{p_0\}$. By Theorem svm_0, also h_1 is a function; moreover $\mathbf{range}(h_1) = \mathbf{range}(h_0) \setminus \{p_0^{[2]}\}$, $\mathbf{range}(h_1) \supseteq \mathbf{dom}(h_0)$, and $\mathbf{dom}(h_0) \supseteq \mathbf{dom}(h_1)$ holds, and hence by the minimality assumption we have $\mathbf{range}(h_1) = \mathbf{dom}(h_1)$. By inserting p_0 back into h_1, we get $\mathbf{range}(h_1) \cup \{p_0^{[2]}\} = \mathbf{range}(h_0)$, $\mathbf{range}(h_0) \supseteq \mathbf{dom}(h_0)$, and $\mathbf{dom}(h_0) = \mathbf{dom}(h_1) \cup \{p_0^{[1]}\}$ and therefore either $p_0^{[1]} = p_0^{[2]}$ or $p_0^{[1]} \in \mathbf{range}(h_1)$; however both cases must be rejected: the former in light of the fact $p_0^{[2]} \notin \mathbf{dom}(h_0)$, because $p_0 \in h_0$; the latter in view of the single-valuedness of h_0 since $\mathbf{range}(h_1) = \mathbf{dom}(h_1)$ and $\mathbf{dom}(h_1) \subseteq \mathbf{dom}(h_0)$.

Use_def($\mathbf{range}(h_0)$) $\Rightarrow$ AUTO

$\langle y_0 \rangle \hookrightarrow$ *Stat2(Stat2⋆)* $\Rightarrow$ *Stat3* : $y_0 \in \{p^{[2]} : p \in h_0\} \,\&\, y_0 \notin \mathbf{dom}(h_0)$

$\langle p_0 \rangle \hookrightarrow$ *Stat3(Stat3⋆)* $\Rightarrow$ *Stat4* : $p_0 \in h_0 \,\&\, p_0^{[2]} = y_0$

Loc_def $\Rightarrow$ $h_1 = h_0 \setminus \{p_0\}$

Suppose $\Rightarrow$ *Stat5* : $\mathbf{range}(h_1) \not\supseteq \mathbf{dom}(h_1)$

$\langle h_0, h_1, h_0 \rangle \hookrightarrow T\mathsf{domain}_1$ *(Stat3⋆)* $\Rightarrow$ $\mathbf{dom}(h_1) \subseteq \mathbf{dom}(h_0)$

$\langle y_1 \rangle \hookrightarrow$ *Stat5(Stat2⋆)* $\Rightarrow$ $y_1 \neq y_0 \,\&\, y_1 \in \mathbf{range}(h_0) \,\&\, y_1 \notin \mathbf{range}(h_1)$

Use_def($\mathbf{range}$) $\Rightarrow$ *Stat6* : $y_1 \in \{p^{[2]} : p \in h_0\} \,\&\, y_1 \notin \{p^{[2]} : p \in h_1\}$

$\langle p_2, p_2 \rangle \hookrightarrow$ *Stat6(Stat4)* $\Rightarrow$ false

Discharge $\Rightarrow$ AUTO

$\langle h_1 \rangle \hookrightarrow$ *Stat1(Stat2⋆)* $\Rightarrow$ $\mathbf{range}(h_1) = \mathbf{dom}(h_1)$

$\langle p_0 \rangle \hookrightarrow T\mathsf{singletonMap}_0$ *(Stat4⋆)* $\Rightarrow$ $\mathbf{range}(\{p_0\}) = \{y_0\} \,\&\, \mathbf{dom}(\{p_0\}) = \{p_0^{[1]}\}$

$\langle h_0, h_1, \{p_0\} \rangle \hookrightarrow T\mathsf{range}_1$ *(Stat2⋆)* $\Rightarrow$ $\mathbf{range}(h_1) \cup \{y_0\} \supseteq \mathbf{dom}(h_0)$

$\langle h_0, h_1, \{p_0\} \rangle \hookrightarrow T\mathsf{domain}_1$ *(Stat4⋆)* $\Rightarrow$ $p_0^{[1]} = y_0 \vee p_0^{[1]} \in \mathbf{dom}(h_1)$

Use_def($\mathbf{dom}$) $\Rightarrow$ *Stat9* : $y_0 \notin \{p^{[1]} : p \in h_0\} \,\&\, \mathbf{dom}(h_1) = \{q^{[1]} : q \in h_1\}$

$\langle p_0 \rangle \hookrightarrow$ *Stat9(Stat4⋆)* $\Rightarrow$ *Stat10* : $p_0^{[1]} \in \{p^{[1]} : p \in h_1\}$

$\langle p_1 \rangle \hookrightarrow$ *Stat10(Stat4⋆)* $\Rightarrow$ $p_1 \neq p_0 \,\&\, p_1, p_0 \in h_0 \,\&\, p_1^{[1]} = p_0^{[1]}$

Use_def(Svm) $\Rightarrow$ *Stat11* : $\langle \forall p \in h_0, q \in h_0 \mid p^{[1]} = q^{[1]} \rightarrow p = q \rangle$

$\langle p_1, p_0 \rangle \hookrightarrow$ *Stat11(Stat10⋆)* $\Rightarrow$ false; Discharge $\Rightarrow$ QED

2 Rudiments about (finite) graphs

2.1 Cartesian product

DEF cartesianProduct: [Cartesian product] $S \times T =_{\text{Def}} \{[x, y] : x \in S, y \in T\}$

> About Cartesian product, in this section we will prove the collection of statement displayed here:

THM cartesian$_0$. $X \in S \times T \leftrightarrow X = \left[X^{[1]}, X^{[2]}\right] \ \& \ X^{[1]} \in S \ \& \ X^{[2]} \in T$

THM cartesian$_1$. $S \neq \emptyset \rightarrow \mathbf{range}(S \times T) = T$

THM cartesian$_2$. $T \neq \emptyset \rightarrow \mathbf{dom}(S \times T) = S$

THM cartesian$_3$: [Annichilator of Cartesian product] $S \times \emptyset = \emptyset \ \& \ \emptyset \times S = \emptyset$

THM cartesian$_4$: [Domain of Cartesian square] $\mathbf{dom}(S \times S) = S$

THM cartesian$_5$: [Monotonicity of Cartesian product] $S \subseteq S' \ \& \ T \subseteq T' \rightarrow S \times T \subseteq S' \times T'$

THM cartesian$_6$: [All subsets of a Cartesian product are maps] $F \subseteq S \times T \rightarrow \mathsf{Is\_map}(F)$

THM cartesian$_0$. $X \in S \times T \leftrightarrow X = \left[X^{[1]}, X^{[2]}\right] \ \& \ X^{[1]} \in S \ \& \ X^{[2]} \in T$. PROOF:

Suppose_not(p_0, s_0, t_0) $\Rightarrow$ AUTO

Use_def($\times$) $\Rightarrow$ $s_0 \times t_0 = \{[x, y] : x \in s_0, y \in t_0\}$

Suppose $\Rightarrow$ *Stat1* : $p_0 \in \{[x, y] : x \in s_0, y \in t_0\}$

$\langle x_0, y_0 \rangle \hookrightarrow$ *Stat1(Stat1)* $\Rightarrow$ $p_0 = \left[p_0^{[1]}, p_0^{[2]}\right] \ \& \ p_0^{[1]} \in s_0 \ \& \ p_0^{[2]} \in t_0$

Discharge $\Rightarrow$ *Stat2* : $p_0 \notin \{[x, y] : x \in s_0, y \in t_0\}$

$\langle p_0^{[1]}, p_0^{[2]} \rangle \hookrightarrow$ *Stat2*($\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ QED

THM cartesian$_1$. $S \neq \emptyset \rightarrow \mathbf{range}(S \times T) = T$. PROOF:

Suppose_not(s_0, t_0) $\Rightarrow$ *Stat1* : $s_0 \neq \emptyset \ \& \ \mathbf{range}(s_0 \times t_0) \neq t_0$

Use_def($\mathbf{range}(s_0 \times t_0)$) $\Rightarrow$ AUTO

$\langle a_0, b_0 \rangle \hookrightarrow$ *Stat1*($\star$) $\Rightarrow$ *Stat2* : $a_0 \in s_0 \ \& \ (b_0 \in \left\{u^{[2]} : u \in s_0 \times t_0\right\} \neq b_0 \in t_0)$

Use_def($\times$) $\Rightarrow$ $b_0 \in \left\{u^{[2]} : u \in \{[x, y] : x \in s_0, y \in t_0\}\right\} \neq b_0 \in t_0$

Suppose $\Rightarrow$ *Stat3* : $b_0 \notin \left\{u^{[2]} : u \in \{[x, y] : x \in s_0, y \in t_0\}\right\}$

TELEM $\Rightarrow$ $b_0 = [a_0, b_0]^{[2]}$

$\langle [a_0, b_0] \rangle \hookrightarrow$ *Stat3(Stat3*$\star$) $\Rightarrow$ *Stat4* : $[a_0, b_0] \notin \{[x, y] : x \in s_0, y \in t_0\}$

$\langle a_0, b_0 \rangle \hookrightarrow$ *Stat4(Stat2*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat5* : $b_0 \in \left\{u^{[2]} : u \in \{[x, y] : x \in s_0, y \in t_0\}\right\}$

$\langle u_0 \rangle \hookrightarrow$ *Stat5(Stat5*$\star$) $\Rightarrow$ *Stat6* : $u_0 \in \{[x, y] : x \in s_0, y \in t_0\} \ \& \ b_0 = u_0^{[2]}$

$\langle x_0, y_0 \rangle \hookrightarrow$ *Stat6(Stat6)* $\Rightarrow$ $b_0 \in t_0$

(*Stat2*$\star$)Discharge $\Rightarrow$ QED

THM cartesian$_2$. $\mathsf{T} \neq \emptyset \rightarrow \mathbf{dom}(\mathsf{S} \times \mathsf{T}) = \mathsf{S}$. PROOF:
Suppose_not(t_0, s_0) $\Rightarrow$ *Stat1* : $t_0 \neq \emptyset$ & $\mathbf{dom}(s_0 \times t_0) \neq s_0$
Use_def($\mathbf{dom}(s_0 \times t_0)$) $\Rightarrow$ AUTO
$\langle a_0, b_0 \rangle \hookrightarrow$ *Stat1*$(\star)$ $\Rightarrow$ *Stat2* : $a_0 \in t_0$ & $(b_0 \in \{u^{[1]} : u \in s_0 \times t_0\} \neq b_0 \in s_0)$
Use_def($\times$) $\Rightarrow$ $b_0 \in \{u^{[1]} : u \in \{[x, y] : x \in s_0, y \in t_0\}\} \neq b_0 \in s_0$
Suppose $\Rightarrow$ *Stat3* : $b_0 \notin \{u^{[1]} : u \in \{[x, y] : x \in s_0, y \in t_0\}\}$
TELEM $\Rightarrow$ $b_0 = [b_0, a_0]^{[1]}$
$\langle [b_0, a_0] \rangle \hookrightarrow$ *Stat3*(*Stat3*$\star$) $\Rightarrow$ *Stat4* : $[b_0, a_0] \notin \{[x, y] : x \in s_0, y \in t_0\}$
$\langle b_0, a_0 \rangle \hookrightarrow$ *Stat4*(*Stat2*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat5* : $b_0 \in \{u^{[1]} : u \in \{[x, y] : x \in s_0, y \in t_0\}\}$
$\langle u_0 \rangle \hookrightarrow$ *Stat5*(*Stat5*$\star$) $\Rightarrow$ *Stat6* : $u_0 \in \{[x, y] : x \in s_0, y \in t_0\}$ & $b_0 = u_0^{[1]}$
$\langle x_0, y_0 \rangle \hookrightarrow$ *Stat6*(*Stat6*) $\Rightarrow$ $b_0 \in s_0$
(*Stat2*$\star$)Discharge $\Rightarrow$ QED

THM cartesian$_3$: [Annichilator of Cartesian product] $\mathsf{S} \times \emptyset = \emptyset$ & $\emptyset \times \mathsf{S} = \emptyset$. PROOF:
Suppose_not(s_0) $\Rightarrow$ AUTO
Suppose $\Rightarrow$ *Stat1* : $\{[x, y] : x \in s_0, y \in \emptyset\} \neq \emptyset \vee \{[x, y] : x \in \emptyset, y \in s_0\} \neq \emptyset$
$\langle x_1, y_1, x_2, y_2 \rangle \hookrightarrow$ *Stat1*(*Stat1*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO
Use_def($\times$) $\Rightarrow$ false; Discharge $\Rightarrow$ QED

THM cartesian$_4$: [Domain of Cartesian square] $\mathbf{dom}(\mathsf{S} \times \mathsf{S}) = \mathsf{S}$. PROOF:
Suppose_not(s_0) $\Rightarrow$ AUTO
$\langle s_0, s_0 \rangle \hookrightarrow T$cartesian$_2(\star)$ $\Rightarrow$ $s_0 = \emptyset$
$\langle s_0 \rangle \hookrightarrow T$cartesian$_3(\star)$ $\Rightarrow$ $s_0 \times \emptyset = \emptyset$
$\langle \emptyset \rangle \hookrightarrow T$domain$_1(\star)$ $\Rightarrow$ $\mathbf{dom}(\emptyset) = \emptyset$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

THM cartesian$_5$: [Monotonicity of Cartesian product] $\mathsf{S} \subseteq \mathsf{S}'$ & $\mathsf{T} \subseteq \mathsf{T}' \rightarrow \mathsf{S} \times \mathsf{T} \subseteq \mathsf{S}' \times \mathsf{T}'$. PROOF:
Suppose_not(s_0, s_1, t_0, t_1) $\Rightarrow$ AUTO
Use_def($\times$) $\Rightarrow$ *Stat1* : $\{[x, y] : x \in s_0, y \in t_0\} \not\subseteq \{[x, y] : x \in s_1, y \in t_1\}$
$\langle c \rangle \hookrightarrow$ *Stat1*(*Stat1*$\star$) $\Rightarrow$ *Stat2* : $c \in \{[x, y] : x \in s_0, y \in t_0\}$ & $c \notin \{[x, y] : x \in s_1, y \in t_1\}$
$\langle x, y, x, y \rangle \hookrightarrow$ *Stat2*$(\star)$ $\Rightarrow$ false; Discharge $\Rightarrow$ QED

THM cartesian$_6$: [All subsets of a Cartesian product are maps] $\mathsf{F} \subseteq \mathsf{S} \times \mathsf{T} \rightarrow$ Is_map(F). PROOF:
Suppose_not(f_0, s_0, t_0) $\Rightarrow$ AUTO

Use_def(Is_map) $\Rightarrow$ $\quad$ *Stat1* : $\neg\langle\forall p \in f_0 \,|\, p = [p^{[1]}, p^{[2]}]\rangle$
$\langle p_0\rangle \hookrightarrow Stat1(\star) \Rightarrow$ $\quad p_0 \in s_0 \times t_0 \;\&\; p_0 \neq [p_0{}^{[1]}, p_0{}^{[2]}]$
$\langle p_0, s_0, t_0\rangle \hookrightarrow T\text{cartesian}_0(\star) \Rightarrow$ $\quad$ false; $\quad$ Discharge $\Rightarrow$ $\quad$ QED

2.2 Digraphs as resulting from graphs

|| An ancillary statement:

THM vertexInduced$_0$: [Loop-freeness gets inherited] $E \subseteq \{\{x,y\} : x \in V, y \in V \setminus \{x\}\} \rightarrow E \cap \{\{x,y\} : x \in W, y \in W\} = E \cap \{\{x,y\} : x \in W, y \in W \setminus \{x\}\}$. PROOF:
Suppose_not(e_0, v_0, w_0) $\Rightarrow$ $\quad$ *Stat1* : $e_0 \cap \{\{x,y\} : x \in w_0, y \in w_0\} \neq e_0 \cap \{\{x,y\} : x \in w_0, y \in w_0 \setminus \{x\}\} \;\&\; e_0 \subseteq \{\{x,y\} : x \in v_0, y \in v_0 \setminus \{x\}\}$
Set_monot $\Rightarrow$ $\quad \{\{x,y\} : x \in w_0, y \in w_0\} \supseteq \{\{x,y\} : x \in w_0, y \in w_0 \setminus \{x\}\}$
$\langle p_0\rangle \hookrightarrow Stat1(\star) \Rightarrow$ $\quad$ *Stat2* : $p_0 \in \{\{x,y\} : x \in w_0, y \in w_0\} \;\&\; p_0 \in \{\{x,y\} : x \in v_0, y \in v_0 \setminus \{x\}\} \;\&\; p_0 \notin \{\{x,y\} : x \in w_0, y \in w_0 \setminus \{x\}\}$
$\langle x_0, y_0, x_1, y_1, x_0, y_0\rangle \hookrightarrow Stat2(Stat2\star) \Rightarrow$ $\quad$ false; $\quad$ Discharge $\Rightarrow$ $\quad$ QED

DEF orientation: [Orientation of a graph] $\quad$ Orientates(D, V, E) $\quad \leftrightarrow_{\text{Def}}$ $\quad E \cap \{\{x,y\} : x \in V, y \in V \setminus \{x\}\} = \{\{p^{[1]}, p^{[2]}\} : p \in D \,|\, p = [p^{[1]}, p^{[2]}]\}$

|| About orientations of a graph, in this section we will prove the collection of statement displayed here:

THM orientation$_0$: [Orientation does not take pseudo-edges into account] $W \supseteq V \rightarrow (\text{Orientates}(D,V,E) \leftrightarrow \text{Orientates}(D,V,E \cap \{\{x,y\} : x \in W, y \in W\}))$

THM orientation$_1$: [The void graph is trivially orientable] $V \subseteq \{S\} \rightarrow \text{Orientates}(\emptyset, V, E)$

THM orientation$_2$: [Oriented arcs are vertex pairs] $\text{Orientates}(D,V,E) \rightarrow \text{Orientates}(D \cap (V \times V), V, E)$

THM orientation$_3$: [Inward orientability of the neighbors af a new vertex]
$\text{Orientates}(D,V,E) \;\&\; W = V \cup \{S\} \;\&\; S \notin V \;\&\; D' = D \cup \{S\} \times \{x \in V \,|\, \{S,x\} \in E\} \;\&\; D = D \cap (V \times V) \rightarrow \text{Orientates}(D', W, E)$

THM orientation$_4$: [Addition of an isolated node to an orientation] $\text{Orientates}(D,V,E) \;\&\; W = V \cup \{S\} \;\&\; \{x \in V \,|\, \{S,x\} \in E\} = \emptyset \;\&\; D = D \cap (V \times V) \rightarrow$
$\text{Orientates}(D,W,E)$

THM orientation$_5$: [Addition of an isolated node to an orientation, 2] $\text{Orientates}(D,W,E) \;\&\; \{x \in V \,|\, \{S,x\} \in E\} = \emptyset \;\&\; W = V \cup \{S\} \;\&\; D = D \cap (V \times V) \rightarrow$
$S \notin \mathbf{dom}(D)$

THM orientation$_6$: [Undirected edges result from oriented arcs, each arc consisting of edges] $\text{Orientates}(D,V,E) \;\&\; [X,Y] \in D \rightarrow \{X,Y\} \in E \;\&\; \{X,Y\} \subseteq V$

THM orientation$_7$: [An undirected edge calls for an oriented arc] $\text{Orientates}(D,V,E) \;\&\; X \in V \;\&\; Y \in V \setminus \{X\} \;\&\; [X,Y] \notin D \;\&\; [Y,X] \notin D \rightarrow \{X,Y\} \notin E$

THM orientation$_8$: [No self-loops in a digraph resulting from orientating a graph] $\text{Orientates}(D,V,E) \rightarrow [U,U] \notin D$

THM orientation$_0$: [Orientation does not take pseudo-edges into account] $W \supseteq V \rightarrow (\text{Orientates}(D,V,E) \leftrightarrow \text{Orientates}(D,V,E \cap \{\{x,y\} : x \in W, y \in W\}))$. PROOF:
Suppose_not(v_1, v_0, d_0, e_0) $\Rightarrow$ $\quad$ AUTO

Suppose ⇒ *Stat1*: $\{\{x,y\} : x \in v_0, y \in v_0 \setminus \{x\}\} \not\subseteq \{\{x,y\} : x \in v_1, y \in v_1\}$
$\langle c \rangle \hookrightarrow$ *Stat1*(*Stat1*⋆) ⇒ *Stat2*: $c \in \{\{x,y\} : x \in v_0, y \in v_0 \setminus \{x\}\}$ & $c \notin \{\{x,y\} : x \in v_1, y \in v_1\}$
$\langle x_0, y_0, x_0, y_0 \rangle \hookrightarrow$ *Stat2*(⋆) ⇒ false; Discharge ⇒ AUTO
Use_def(Orientates) ⇒ $e_0 \cap \{\{x,y\} : x \in v_0, y \in v_0 \setminus \{x\}\} = \{\{p^{[1]}, p^{[2]}\} : p \in d_0 \mid p = [p^{[1]}, p^{[2]}]\} \neq$
$e_0 \cap \{\{x,y\} : x \in v_1, y \in v_1\} \cap \{\{x,y\} : x \in v_0, y \in v_0 \setminus \{x\}\} = \{\{p^{[1]}, p^{[2]}\} : p \in d_0 \mid p = [p^{[1]}, p^{[2]}]\}$
Discharge ⇒ QED

THM orientation$_1$: [The void graph is trivially orientable] $V \subseteq \{S\} \rightarrow$ Orientates$(\emptyset, V, E)$. PROOF:
Suppose_not(v_0, s_0, e_0) ⇒ AUTO
Use_def(Orientates) ⇒ *Stat1*: $e_0 \cap \{\{x,y\} : x \in v_0, y \in v_0 \setminus \{x\}\} \neq \{\{p^{[1]}, p^{[2]}\} : p \in \emptyset \mid p = [p^{[1]}, p^{[2]}]\}$
$\langle c \rangle \hookrightarrow$ *Stat1* ⇒ *Stat2*: $c \in \{\{p^{[1]}, p^{[2]}\} : p \in \emptyset \mid p = [p^{[1]}, p^{[2]}]\} \vee c \in \{\{x,y\} : x \in v_0, y \in v_0 \setminus \{x\}\}$
$\langle p_0, x_0, y_0 \rangle \hookrightarrow$ *Stat2* ⇒ false; Discharge ⇒ QED

THM orientation$_2$: [Oriented arcs are vertex pairs] Orientates$(D, V, E) \rightarrow$ Orientates$(D \cap (V \times V), V, E)$. PROOF:
Suppose_not(d_0, v_0, e_0) ⇒ AUTO
Set_monot ⇒ $\{\{p^{[1]}, p^{[2]}\} : p \in d_0 \mid p = [p^{[1]}, p^{[2]}]\} \supseteq \{\{p^{[1]}, p^{[2]}\} : p \in d_0 \cap (v_0 \times v_0) \mid p = [p^{[1]}, p^{[2]}]\}$
Use_def(Orientates) ⇒ *Stat1*: $\{\{p^{[1]}, p^{[2]}\} : p \in d_0 \mid p = [p^{[1]}, p^{[2]}]\} \not\subseteq \{\{p^{[1]}, p^{[2]}\} : p \in d_0 \cap (v_0 \times v_0) \mid p = [p^{[1]}, p^{[2]}]\}$ &
$\{\{x,y\} : x \in v_0, y \in v_0 \setminus \{x\}\} \supseteq \{\{p^{[1]}, p^{[2]}\} : p \in d_0 \mid p = [p^{[1]}, p^{[2]}]\}$
$\langle q_0 \rangle \hookrightarrow$ *Stat1*(*Stat1*⋆) ⇒ *Stat2*: $q_0 \in \{\{p^{[1]}, p^{[2]}\} : p \in d_0 \mid p = [p^{[1]}, p^{[2]}]\}$ & $q_0 \notin \{\{p^{[1]}, p^{[2]}\} : p \in d_0 \cap (v_0 \times v_0) \mid p = [p^{[1]}, p^{[2]}]\}$
Use_def(×) ⇒ $v_0 \times v_0 = \{[x,y] : x \in v_0, y \in v_0\}$
$\langle p_0, p_0 \rangle \hookrightarrow$ *Stat2*(*Stat1*⋆) ⇒ *Stat3*: $\{p_0^{[1]}, p_0^{[2]}\} \in \{\{x,y\} : x \in v_0, y \in v_0 \setminus \{x\}\}$ & $p_0 \notin \{[x,y] : x \in v_0, y \in v_0\}$ & $p_0 = [p_0^{[1]}, p_0^{[2]}]$
$\langle x_0, y_0, p_0^{[1]}, p_0^{[2]} \rangle \hookrightarrow$ *Stat3*(*Stat3*⋆) ⇒ false; Discharge ⇒ QED

THM orientation$_3$: [Inward orientability of the neighbors af a new vertex]
Orientates(D, V, E) & $W = V \cup \{S\}$ & $S \notin V$ & $D' = D \cup \{S\} \times \{x \in V \mid \{S, x\} \in E\}$ & $D = D \cap (V \times V) \rightarrow$ Orientates(D', W, E). PROOF:
Suppose_not$(d_0, v_0, e_2, v_1, x_0, d_1)$ ⇒ *Stat0*:
$d_1 = d_0 \cup \{x_0\} \times \{t \in v_0 \mid \{x_0, t\} \in e_2\}$ & $\neg$Orientates(d_1, v_1, e_2) & $x_0 \in v_1$ & $v_0 = v_1 \setminus \{x_0\}$ & Orientates(d_0, v_0, e_2) & $d_0 = d_0 \cap (v_0 \times v_0)$
(*Stat0*⋆)ELEM ⇒ *Stat2*: $v_1 = v_0 \cup \{x_0\}$
EQUAL ⇒ $\neg$Orientates$(d_0 \cup \{x_0\} \times \{t \in v_0 \mid \{x_0, t\} \in e_2\}, v_1, e_2)$
Use_def(Orientates) ⇒ *Stat3*: $e_2 \cap \{\{x,y\} : x \in v_1, y \in v_1 \setminus \{x\}\} \neq \{\{p^{[1]}, p^{[2]}\} : p \in d_0 \cup \{x_0\} \times \{t \in v_0 \mid \{x_0, t\} \in e_2\} \mid p = [p^{[1]}, p^{[2]}]\}$ &
$e_2 \cap \{\{x,y\} : x \in v_0, y \in v_0 \setminus \{x\}\} = \{\{p^{[1]}, p^{[2]}\} : p \in d_0 \mid p = [p^{[1]}, p^{[2]}]\}$
Set_monot(*Stat3*) ⇒ $\{\{p^{[1]}, p^{[2]}\} : p \in d_0 \mid p = [p^{[1]}, p^{[2]}]\} \subseteq \{\{p^{[1]}, p^{[2]}\} : p \in d_0 \cup \{x_0\} \times \{t \in v_0 \mid \{x_0, t\} \in e_2\} \mid p = [p^{[1]}, p^{[2]}]\}$
Suppose ⇒ $\{\{x,y\} : x \in v_1, y \in v_1 \setminus \{x\}\} \not\supseteq \{\{x,y\} : x \in v_0, y \in v_0 \setminus \{x\}\}$
EQUAL ⇒ *Stat4*: $\{\{x,y\} : x \in v_1, y \in v_1 \setminus \{x\}\} \not\supseteq \{\{x,y\} : x \in v_1 \setminus \{x_0\}, y \in v_1 \setminus \{x_0\} \setminus \{x\}\}$
$\langle q_2 \rangle \hookrightarrow$ *Stat4*(*Stat4*⋆) ⇒ *Stat5*: $q_2 \in \{\{x,y\} : x \in v_1 \setminus \{x_0\}, y \in v_1 \setminus \{x_0\} \setminus \{x\}\}$ & $q_2 \notin \{\{x,y\} : x \in v_1, y \in v_1 \setminus \{x\}\}$
$\langle x_3, y_3, x_3, y_3 \rangle \hookrightarrow$ *Stat5*(*Stat5*⋆) ⇒ false; (*Stat5*⋆)Discharge ⇒ AUTO

$\langle q_1\rangle \hookrightarrow Stat3(Stat3\star) \Rightarrow \quad Stat6:\ q_1 \notin \{\{p^{[1]}, p^{[2]}\} : p \in d_0 \mid p = [p^{[1]}, p^{[2]}]\}\ \&\ q_1 \notin e_2 \cap \{\{x, y\} : x \in v_0, y \in v_0 \setminus \{x\}\}\ \&$
$\quad q_1 \in e_2 \cap \{\{x, y\} : x \in v_1, y \in v_1 \setminus \{x\}\} \neq q_1 \in \{\{p^{[1]}, p^{[2]}\} : p \in d_0 \cup \{x_0\} \times \{t \in v_0 \mid \{x_0, t\} \in e_2\} \mid p = [p^{[1]}, p^{[2]}]\}$

$\text{Suppose} \Rightarrow \quad Stat7:\ q_1 \in \{\{p^{[1]}, p^{[2]}\} : p \in d_0 \cup \{x_0\} \times \{t \in v_0 \mid \{x_0, t\} \in e_2\} \mid p = [p^{[1]}, p^{[2]}]\}$

$\langle p_1\rangle \hookrightarrow Stat7(Stat6\star) \Rightarrow \quad Stat8:\ \{p_1{}^{[1]}, p_1{}^{[2]}\} \notin \{\{p^{[1]}, p^{[2]}\} : p \in d_0 \mid p = [p^{[1]}, p^{[2]}]\}\ \&\ p_1 \in d_0 \cup \{x_0\} \times \{t \in v_0 \mid \{x_0, t\} \in e_2\}\ \&\ p_1 = [p_1{}^{[1]}, p_1{}^{[2]}]\ \&\ q_1 = \{p_1{}^{[1]}, p_1{}^{[2]}\}$

$\langle p_1\rangle \hookrightarrow Stat8(Stat8\star) \Rightarrow \quad p_1 \in \{x_0\} \times \{t \in v_0 \mid \{x_0, t\} \in e_2\}$

$\langle p_1, \{x_0\}, \{t \in v_0 \mid \{x_0, t\} \in e_2\}\rangle \hookrightarrow T\text{cartesian}_0(Stat8\star) \Rightarrow \quad Stat9:\ p_1{}^{[2]} \in \{t \in v_0 \mid \{x_0, t\} \in e_2\}\ \&\ p_1{}^{[1]} = x_0$

$\langle\ \rangle \hookrightarrow Stat9(Stat6\star) \Rightarrow \quad Stat10:\ \{x_0, p_1{}^{[2]}\} \notin \{\{x, y\} : x \in v_1, y \in v_1 \setminus \{x\}\}\ \&\ p_1{}^{[2]} \in v_0\ \&\ \{x_0, p_1{}^{[2]}\} \in e_2$

$\langle x_0, p_1{}^{[2]}\rangle \hookrightarrow Stat10(Stat10, Stat0\star) \Rightarrow \quad \text{false}; \qquad (Stat10\star)\text{Discharge} \Rightarrow \quad Stat11:\ q_1 \in \{\{x, y\} : x \in v_1, y \in v_1 \setminus \{x\}\}\ \&\ q_1 \notin \{\{x, y\} : x \in v_0, y \in v_0 \setminus \{x\}\}\ \&$
$\quad Stat11a:\ q_1 \notin \{\{p^{[1]}, p^{[2]}\} : p \in d_0 \cup \{x_0\} \times \{t \in v_0 \mid \{x_0, t\} \in e_2\} \mid p = [p^{[1]}, p^{[2]}]\}\ \&\ q_1 \in e_2$

$\langle x_4, y_4, x_4, y_4, [x_4, y_4]\rangle \hookrightarrow Stat11(Stat11, Stat2\star) \Rightarrow \quad Stat12:$
$\quad [x_4, y_4] \notin \{x_0\} \times \{t \in v_0 \mid \{x_0, t\} \in e_2\}\ \&\ q_1 = \{x_4, y_4\}\ \&\ x_4 \neq y_4\ \&\ x_0 \in q_1\ \&\ x_4, y_4 \in v_1\ \&\ v_1 = v_0 \cup \{x_0\}\ \&\ q_1 \in e_2$

$\text{Suppose} \Rightarrow \quad x_4 = x_0$

$\langle [x_4, y_4], \{x_0\}, \{t \in v_0 \mid \{x_0, t\} \in e_2\}\rangle \hookrightarrow T\text{cartesian}_0(Stat12\star) \Rightarrow \quad Stat13:\ y_4 \notin \{t \in v_0 \mid \{x_0, t\} \in e_2\}$

$\langle y_4\rangle \hookrightarrow Stat13(Stat12\star) \Rightarrow \quad \text{false}; \qquad (Stat13\star)\text{Discharge} \Rightarrow \quad \text{AUTO}$

$\langle [y_4, x_4]\rangle \hookrightarrow Stat11a(Stat12\star) \Rightarrow \quad [y_4, x_4] \notin \{x_0\} \times \{t \in v_0 \mid \{x_0, t\} \in e_2\}$

$\langle [y_4, x_4], \{x_0\}, \{t \in v_0 \mid \{x_0, t\} \in e_2\}\rangle \hookrightarrow T\text{cartesian}_0(Stat12\star) \Rightarrow \quad Stat14:\ x_4 \notin \{t \in v_0 \mid \{x_0, t\} \in e_2\}$

$\langle x_4\rangle \hookrightarrow Stat14(Stat12\star) \Rightarrow \quad \text{false}; \qquad (Stat14\star)\text{Discharge} \Rightarrow \quad \text{QED}$

THM orientation$_4$: [Addition of an isolated node to an orientation] $\text{Orientates}(D, V, E)\ \&\ W = V \cup \{S\}\ \&\ \{x \in V \mid \{S, x\} \in E\} = \emptyset\ \&\ D = D \cap (V \times V) \rightarrow$
$\text{Orientates}(D, W, E)$. PROOF:

$\text{Suppose\_not}(d_0, v_0, e_0, w_0, s_0) \Rightarrow \quad \text{AUTO}$

$\text{Suppose} \Rightarrow \quad w_0 = v_0$

$\text{EQUAL} \Rightarrow \quad \text{false}; \qquad \text{Discharge} \Rightarrow \quad \text{AUTO}$

$\langle \{s_0\}\rangle \hookrightarrow T\text{cartesian}_3 \Rightarrow \quad \{s_0\} \times \emptyset = \emptyset\ \&\ \{x \in v_0 \mid \{s_0, x\} \in e_0\} = \emptyset$

$\text{EQUAL} \Rightarrow \quad \{s_0\} \times \{x \in v_0 \mid \{x, s_0\} \in e_0\} = \emptyset$

$\langle d_0, v_0, e_0, w_0, s_0, d_0\rangle \hookrightarrow T\text{orientation}_3(\star) \Rightarrow \quad \text{false}; \qquad \text{Discharge} \Rightarrow \quad \text{QED}$

THM orientation$_5$: [Addition of an isolated node to an orientation, 2] $\text{Orientates}(D, W, E)\ \&\ \{x \in V \mid \{S, x\} \in E\} = \emptyset\ \&\ W = V \cup \{S\}\ \&\ D = D \cap (V \times V) \rightarrow$
$S \notin \mathbf{dom}(D)$. PROOF:

$\text{Suppose\_not}(d_0, w_0, e_0, v_0, s_0) \Rightarrow \quad \text{AUTO}$

$\text{Use\_def}(\mathbf{dom}) \Rightarrow \quad Stat1:\ s_0 \in \{p^{[1]} : p \in d_0\}$

$\langle p_0\rangle \hookrightarrow Stat1(\star) \Rightarrow \quad Stat2:\ p_0{}^{[2]} \notin \{x \in v_0 \mid \{s_0, x\} \in e_0\}\ \&\ p_0 \in d_0\ \&\ p_0{}^{[1]} = s_0\ \&\ p_0 \in v_0 \times v_0$

$\langle p_0, v_0, v_0\rangle \hookrightarrow T\text{cartesian}_0(Stat2\star) \Rightarrow \quad p_0 = [p_0{}^{[1]}, p_0{}^{[2]}]\ \&\ p_0{}^{[2]} \in v_0$

$\langle p_0{}^{[2]}\rangle \hookrightarrow Stat2(Stat2\star) \Rightarrow \quad \{p_0{}^{[1]}, p_0{}^{[2]}\} \notin e_0$

$\text{Use\_def}(\text{Orientates}) \Rightarrow \quad Stat3:\ \{p_0{}^{[1]}, p_0{}^{[2]}\} \notin \{\{p^{[1]}, p^{[2]}\} : p \in d_0 \mid p = [p^{[1]}, p^{[2]}]\}$

$\langle p_0\rangle \hookrightarrow Stat3(Stat2\star) \Rightarrow \quad \text{false}; \qquad \text{Discharge} \Rightarrow \quad \text{QED}$

THM orientation$_6$: [Undirected edges result from oriented arcs, each arc consisting of edges] $\mathsf{Orientates}(D,V,E)$ & $[X,Y]\in D\rightarrow \{X,Y\}\in E$ & $\{X,Y\}\subseteq V$. PROOF:
Suppose_not$(d_0,v_0,e_0,x_0,y_0)\Rightarrow$ AUTO
Suppose $\Rightarrow$ *Stat1*: $\{x_0,y_0\}\notin\{\{p^{[1]},p^{[2]}\}: p\in d_0 \mid p=[p^{[1]},p^{[2]}]\}$
$\langle [x_0,y_0]\rangle\hookrightarrow$*Stat1*$(\star)\Rightarrow$ false; Discharge $\Rightarrow$ AUTO
Use_def(Orientates) $\Rightarrow$ *Stat2*: $\{x_0,y_0\}\in\{\{x,y\}: x\in v_0, y\in v_0\setminus\{x\}\}$ & $\{x_0,y_0\}\in e_0$
$\langle x_1,y_1\rangle\hookrightarrow$*Stat2*$(\star)\Rightarrow$ false; Discharge $\Rightarrow$ QED

THM orientation$_7$: [An undirected edge calls for an oriented arc] $\mathsf{Orientates}(D,V,E)$ & $X\in V$ & $Y\in V\setminus\{X\}$ & $[X,Y]\notin D$ & $[Y,X]\notin D\rightarrow \{X,Y\}\notin E$. PROOF:
Suppose_not$(d_0,v_0,e_0,x_0,y_0)\Rightarrow$ AUTO
Suppose $\Rightarrow$ *Stat1*: $\{x_0,y_0\}\notin\{\{x,y\}: x\in v_0, y\in v_0\setminus\{x\}\}$
$\langle x_0,y_0\rangle\hookrightarrow$*Stat1*$(\star)\Rightarrow$ false; Discharge $\Rightarrow$ AUTO
Use_def(Orientates) $\Rightarrow$ *Stat2*: $\{x_0,y_0\}\in\{\{p^{[1]},p^{[2]}\}: p\in d_0 \mid p=[p^{[1]},p^{[2]}]\}$
$\langle p_0\rangle\hookrightarrow$*Stat2*(*Stat2*$\star$) $\Rightarrow$ *Stat3*: $\{p_0^{[1]},p_0^{[2]}\}=\{x_0,y_0\}$ & $[p_0^{[1]},p_0^{[2]}]\in d_0$
Suppose $\Rightarrow$ $x_0=p_0^{[1]}$ & $y_0=p_0^{[2]}$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO
(*Stat3*$\star$)ELEM $\Rightarrow$ $y_0=p_0^{[1]}$ & $x_0=p_0^{[2]}$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

THM orientation$_8$: [No self-loops in a digraph resulting from orientating a graph] $\mathsf{Orientates}(D,V,E)\rightarrow [U,U]\notin D$. PROOF:
Suppose_not$(d_0,v_0,e_0,x_0)\Rightarrow$ AUTO
Use_def(Orientates) $\Rightarrow$ *Stat0*: $\{\{x,y\}: x\in v_0, y\in v_0\setminus\{x\}\}\supseteq\{\{p^{[1]},p^{[2]}\}: p\in d_0 \mid p=[p^{[1]},p^{[2]}]\}$ & $[x_0,x_0]\in d_0$
Suppose $\Rightarrow$ *Stat1*: $\{x_0,x_0\}\notin\{\{p^{[1]},p^{[2]}\}: p\in d_0 \mid p=[p^{[1]},p^{[2]}]\}$
$\langle [x_0,x_0]\rangle\hookrightarrow$*Stat1*(*Stat0*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat2*: $\{x_0,x_0\}\in\{\{x,y\}: x\in v_0, y\in v_0\setminus\{x\}\}$
$\langle x_1,y_1\rangle\hookrightarrow$*Stat2*(*Stat2*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ QED

2.3 Weak extensionality and extensionality

DEF extensionality$_0$: [Extensionality] $\mathsf{Extensional}(V,D) \leftrightarrow_{\mathrm{Def}} \langle\forall x\in V, y\in V, \exists z \mid ([x,z]\in D\leftrightarrow [y,z]\in D)\rightarrow x=y\rangle$

DEF extensionality$_1$: [Weak extensionality] $\mathsf{WExtensional}(V,D) =_{\mathrm{Def}} \mathsf{Extensional}\big(V\cap \mathbf{dom}(D\cap(V\times V)), D\cap(V\times V)\big)$

About extensionality-related notions concerning graphs, in this section we will prove the collection of statement displayed here:

THM weaXtensionality$_0$: [Weak extensionality has only to do with vertices] $\mathsf{WExtensional}(V,D) \leftrightarrow \mathsf{WExtensional}\big(V, D \cap (V \times V)\big)$

THM weaXtensionality$_1$: [Addition of an isolated vertex preserves weak extensionality] $\mathsf{WExtensional}(V,D)\ \&\ S \notin \mathbf{dom}(D)\ \&\ W = V \cup \{S\}\ \&\ D = D \cap (V \times V) \rightarrow \mathsf{WExtensional}(W,D)$

THM weaXtensionality$_2$: [A new vertex whose out-neighborhood comprises what was, previously, a source does not disrupt weak extensionality] $\mathsf{WExtensional}(V,D)\ \&\ D' = D \cup \{S\} \times V'\ \&\ V' \not\subseteq \mathbf{range}(D)\ \&\ D = D \cap (V \times V)\ \&\ W \supseteq V \cup V' \rightarrow \mathsf{WExtensional}(W,D')$

THM extensionality$_0$: [Extensional digraphs are weakly extensional] $\mathsf{Extensional}(V,D)\ \&\ V \times V \supseteq D \rightarrow \mathsf{WExtensional}(V,D)$

THM extensionality$_1$: [Making a new vertex the son of a set of vertices comprising a sink cannot disrupt extensionality] $V \times V \supseteq D\ \&\ X \notin V\ \&\ V \supseteq S\ \&\ \langle \exists t \in S \mid D_{|\{t\}} = \emptyset \rangle\ \&\ \mathsf{Extensional}(V,D) \rightarrow \mathsf{Extensional}(V \cup \{X\}, D \cup S \times \{X\})$

THM extensionality$_2$: [An extensional digraph has at most one sink] $\mathsf{Extensional}(V,D)\ \&\ X,Y \in V\ \&\ \langle \forall z \mid [X,z] \notin D \rangle\ \&\ \langle \forall z \mid [Y,z] \notin D \rangle \rightarrow X = Y$

THM weaXtensionality$_0$: [Weak extensionality has only to do with vertices] $\mathsf{WExtensional}(V,D) \leftrightarrow \mathsf{WExtensional}\big(V, D \cap (V \times V)\big)$. PROOF:

Suppose_not$(v_0,d_0) \Rightarrow$ AUTO

Use_def(WExtensional) $\Rightarrow$ $\mathsf{Extensional}\big(v_0 \cap \mathbf{dom}(d_0 \cap (v_0 \times v_0) \cap (v_0 \times v_0)), d_0 \cap (v_0 \times v_0) \cap (v_0 \times v_0)\big) \neq \mathsf{Extensional}\big(v_0 \cap \mathbf{dom}(d_0 \cap (v_0 \times v_0)), d_0 \cap (v_0 \times v_0)\big)$

TELEM $\Rightarrow$ $d_0 \cap (v_0 \times v_0) \cap (v_0 \times v_0) = d_0 \cap (v_0 \times v_0)$

EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

THM weaXtensionality$_1$: [Addition of an isolated vertex preserves weak extensionality] $\mathsf{WExtensional}(V,D)\ \&\ S \notin \mathbf{dom}(D)\ \&\ W = V \cup \{S\}\ \&\ D = D \cap (V \times V) \rightarrow \mathsf{WExtensional}(W,D)$. PROOF:

Suppose_not$(v_0,d_0,s_0,w_0) \Rightarrow$ *Stat0*: $\mathsf{WExtensional}(v_0,d_0)\ \&\ s_0 \notin \mathbf{dom}(d_0)\ \&\ w_0 = v_0 \cup \{s_0\}\ \&\ d_0 = d_0 \cap (v_0 \times v_0)\ \&\ \neg\mathsf{WExtensional}(w_0,d_0)$

Use_def(WExtensional) $\Rightarrow$ $\mathsf{Extensional}\big(v_0 \cap \mathbf{dom}(d_0 \cap (v_0 \times v_0)), d_0 \cap (v_0 \times v_0)\big)\ \&\ \neg\mathsf{Extensional}\big(w_0 \cap \mathbf{dom}(d_0 \cap (w_0 \times w_0)), d_0 \cap (w_0 \times w_0)\big)$

Use_def(Extensional) $\Rightarrow$ *Stat1*: $\neg\langle \forall x \in w_0 \cap \mathbf{dom}(d_0 \cap (w_0 \times w_0)), y \in w_0 \cap \mathbf{dom}(d_0 \cap (w_0 \times w_0)), \exists z \mid \big([x,z] \in d_0 \cap (w_0 \times w_0) \leftrightarrow [y,z] \in d_0 \cap (w_0 \times w_0)\big) \rightarrow x = y\rangle\ \&$ $\langle \forall x \in v_0 \cap \mathbf{dom}(d_0 \cap (v_0 \times v_0)), y \in v_0 \cap \mathbf{dom}(d_0 \cap (v_0 \times v_0)), \exists z \mid \big([x,z] \in d_0 \cap (v_0 \times v_0) \leftrightarrow [y,z] \in d_0 \cap (v_0 \times v_0)\big) \rightarrow x = y\rangle$

$\langle x_0, y_0, x_0, y_0\rangle \hookrightarrow$ *Stat1(Stat1⋆)* $\Rightarrow$ *Stat2*: $\neg\langle \exists z \mid \big([x_0,z] \in d_0 \cap (w_0 \times w_0) \leftrightarrow [y_0,z] \in d_0 \cap (w_0 \times w_0)\big) \rightarrow x_0 = y_0\rangle\ \&\ x_0 \in w_0 \cap \mathbf{dom}(d_0 \cap (w_0 \times w_0))\ \&$ $y_0 \in w_0 \cap \mathbf{dom}(d_0 \cap (w_0 \times w_0))\ \&\ \neg\Big(x_0, y_0 \in v_0 \cap \mathbf{dom}(d_0 \cap (v_0 \times v_0))\ \&\ \neg\langle \exists z \mid \big([x_0,z] \in d_0 \cap (v_0 \times v_0) \leftrightarrow [y_0,z] \in d_0 \cap (v_0 \times v_0)\big) \rightarrow x_0 = y_0\rangle\Big)$

Suppose $\Rightarrow$ *Stat3*: $d_0 \cap (v_0 \times v_0) \neq d_0 \cap (w_0 \times w_0)$

$\langle c\rangle \hookrightarrow$ *Stat3(Stat3, Stat0⋆)* $\Rightarrow$ $c \in d_0\ \&\ c \in v_0 \times v_0\ \&\ c \notin w_0 \times w_0$

Set_monot $\Rightarrow$ $\{[x,y] : x \in v_0, y \in v_0\} \subseteq \{[x,y] : x \in w_0, y \in w_0\}$

Use_def($\times$) $\Rightarrow$ false; Discharge $\Rightarrow$ $d_0 \cap (v_0 \times v_0) = d_0 \cap (w_0 \times w_0)$

EQUAL $\Rightarrow$ $\mathbf{dom}(d_0 \cap (v_0 \times v_0)) = \mathbf{dom}(d_0 \cap (w_0 \times w_0))$

Suppose $\Rightarrow$ $w_0 \cap \mathbf{dom}(d_0 \cap (w_0 \times w_0)) \neq v_0 \cap \mathbf{dom}(d_0 \cap (v_0 \times v_0))$

EQUAL $\Rightarrow$ *Stat4*: $w_0 \cap \mathbf{dom}(d_0) \neq v_0 \cap \mathbf{dom}(d_0)$

$\langle v_0 \times v_0, d_0, v_0 \times v_0\rangle \hookrightarrow T\mathsf{domain}_1(\mathit{Stat0}\star) \Rightarrow$ $\mathbf{dom}(d_0) \subseteq \mathbf{dom}(v_0 \times v_0)$

Use_def($\mathbf{dom}(v_0 \times v_0)$) $\Rightarrow$ AUTO

$\langle d\rangle \hookrightarrow$ *Stat4(Stat0⋆)* $\Rightarrow$ *Stat5*: $d \in \{p^{[1]} : p \in v_0 \times v_0\}\ \&\ d \notin v_0$

$\langle p_0 \rangle \hookrightarrow Stat5(Stat5\star) \Rightarrow$ $d = p_0^{[1]}$ & $p_0 \in v_0 \times v_0$
$\langle p_0, v_0, v_0 \rangle \hookrightarrow T\text{cartesian}_0(Stat5\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO
$(Stat0\star)$ELEM $\Rightarrow$ $Stat8: \langle \exists z \,|\, ([x_0, z] \in d_0 \cap (v_0 \times v_0) \leftrightarrow [y_0, z] \in d_0 \cap (v_0 \times v_0)) \rightarrow x_0 = y_0 \rangle$
$\langle z_0 \rangle \hookrightarrow Stat8(Stat8\star) \Rightarrow$ $[x_0, z_0] \in d_0 \cap (v_0 \times v_0) \leftrightarrow [y_0, z_0] \in d_0 \cap (v_0 \times v_0) \rightarrow x_0 = y_0$
$\langle z_0 \rangle \hookrightarrow Stat2(Stat2\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

THM weaXtensionality$_2$: [A new vertex whose out-neighborhood comprises what was, previously, a source does not disrupt weak extensionality]
$\text{WExtensional}(V, D)$ & $D' = D \cup \{S\} \times V'$ & $V' \not\subseteq \mathbf{range}(D)$ & $D = D \cap (V \times V)$ & $W \supseteq V \cup V' \rightarrow \text{WExtensional}(W, D')$. PROOF:
Suppose_not$(v_0, d_0, d_1, s_0, v_1, w_0) \Rightarrow$ AUTO

Suppose $v_0, d_0, d_1, s_0, v_1, w_0$ form a counterexample to the claim so that, among others, $v_0 \cap \mathbf{dom}(d_0), d_0$ is extensional whereas $w_0 \cap \mathbf{dom}(d_1 \cap (w_0 \times w_0)), d_1 \cap (w_0 \times w_0)$ is not.

Use_def(WExtensional) $\Rightarrow$ $\text{Extensional}(v_0 \cap \mathbf{dom}(d_0 \cap (v_0 \times v_0)), d_0 \cap (v_0 \times v_0))$ & $\neg\text{Extensional}(w_0 \cap \mathbf{dom}(d_1 \cap (w_0 \times w_0)), d_1 \cap (w_0 \times w_0))$
EQUAL $\Rightarrow$ $\text{Extensional}(v_0 \cap \mathbf{dom}(d_0), d_0)$
Use_def(Extensional) $\Rightarrow$ $Stat1: \neg\langle \forall x \in w_0 \cap \mathbf{dom}(d_1 \cap (w_0 \times w_0)), y \in w_0 \cap \mathbf{dom}(d_1 \cap (w_0 \times w_0)), \exists z \,|\, ([x, z] \in d_1 \cap (w_0 \times w_0) \leftrightarrow [y, z] \in d_1 \cap (w_0 \times w_0)) \rightarrow x = y \rangle$ &
$\langle \forall x \in v_0 \cap \mathbf{dom}(d_0), y \in v_0 \cap \mathbf{dom}(d_0), \exists z \,|\, ([x, z] \in d_0 \leftrightarrow [y, z] \in d_0) \rightarrow x = y \rangle$

Consequently, there must exist x_0, y_0 in $w_0 \cap \mathbf{dom}(d_1 \cap (w_0 \times w_0))$ which $d_1 \cap (w_0 \times w_0)$ does not differentiate; but which d_0 differentiates if they belong to $\mathbf{dom}(d_0)$.

$\langle v_0 \times v_0, d_0, v_0 \times v_0 \rangle \hookrightarrow T\text{domain}_1(\star) \Rightarrow$ $\mathbf{dom}(d_0) \subseteq \mathbf{dom}(v_0 \times v_0)$
$\langle v_0 \rangle \hookrightarrow T\text{cartesian}_4(Stat1\star) \Rightarrow$ $v_0 = \mathbf{dom}(v_0 \times v_0)$
$\langle x_0, y_0, x_0, y_0 \rangle \hookrightarrow Stat1(Stat1\star) \Rightarrow$ $Stat2:$
$\neg\langle \exists z \,|\, ([x_0, z] \in d_1 \cap (w_0 \times w_0) \leftrightarrow [y_0, z] \in d_1 \cap (w_0 \times w_0)) \rightarrow x_0 = y_0 \rangle$ & $(x_0, y_0 \in \mathbf{dom}(d_0) \rightarrow \langle \exists z \,|\, ([x_0, z] \in d_0 \leftrightarrow [y_0, z] \in d_0) \rightarrow x_0 = y_0 \rangle)$ &
$x_0, y_0 \in w_0 \cap \mathbf{dom}(d_1 \cap (w_0 \times w_0))$

Plainly, $x_0 \neq y_0$ must hold.

$\langle v_0, w_0, v_0, w_0 \rangle \hookrightarrow T\text{cartesian}_5(\star) \Rightarrow$ $Stat2a: v_0 \times v_0 \subseteq w_0 \times w_0$ & $d_0 \subseteq d_1$
Suppose $\Rightarrow$ $x_0 = y_0$
$\langle \emptyset \rangle \hookrightarrow Stat2(Stat2\star) \Rightarrow$ false; Discharge $\Rightarrow$ $Stat3: v_1 \backslash \mathbf{range}(d_0) \neq \emptyset$ & $x_0 \neq y_0$

Let z_0 be a source of d_0 which does not belong to v_1.

Use_def($\mathbf{range}(d_0)$) $\Rightarrow$ AUTO
$\langle z_0 \rangle \hookrightarrow Stat3(Stat3\star) \Rightarrow$ $Stat4: z_0 \notin \{p^{[2]} : p \in d_0\}$ & $z_0 \in v_1$

Observe that $\mathbf{dom}(d_0) \subseteq \mathbf{dom}(d_1 \cap (w_0 \times w_0))$ and $\mathbf{dom}(d_1 \cap (w_0 \times w_0)) \subseteq \mathbf{dom}(d_0) \cup \{s_0\}$, where s_0 is the new vertex.

$\langle d_1 \cap (w_0 \times w_0), d_0, d_1 \cap (w_0 \times w_0)\rangle \hookrightarrow T\mathsf{domain}_1(\star) \Rightarrow$ $Stat4a: \mathbf{dom}(d_0) \subseteq \mathbf{dom}(d_1 \cap (w_0 \times w_0))$
Suppose $\Rightarrow$ $\mathbf{dom}(d_1 \cap (w_0 \times w_0)) \not\subseteq \mathbf{dom}(d_0) \cup \{s_0\}$
$\langle v_1, \{s_0\}\rangle \hookrightarrow T\mathsf{cartesian}_2(\star) \Rightarrow$ $\mathbf{dom}(\{s_0\} \times v_1) = \{s_0\}$
$\langle d_1, d_0, \{s_0\} \times v_1\rangle \hookrightarrow T\mathsf{domain}_1(\star) \Rightarrow$ $\mathbf{dom}(d_1) = \mathbf{dom}(d_0) \cup \{s_0\}$
$\langle d_1, d_1 \cap (w_0 \times w_0), d_1\rangle \hookrightarrow T\mathsf{domain}_1(Stat2\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO

Also notice that $\{[x_0, z_0], [y_0, z_0]\} \subseteq w_0 \times w_0$.

Suppose $\Rightarrow$ $[x_0, z_0] \notin w_0 \times w_0 \lor [y_0, z_0] \notin w_0 \times w_0$
Use_def($\times$) $\Rightarrow$ $Stat5: [x_0, z_0] \notin \{[x, y] : x \in w_0, y \in w_0\} \lor [y_0, z_0] \notin \{[x, y] : x \in w_0, y \in w_0\}$
$\langle x_0, z_0, y_0, z_0\rangle \hookrightarrow Stat5(Stat4, Stat4\star) \Rightarrow$ $\neg x_0, y_0, z_0 \in w_0$
Discharge $\Rightarrow$ AUTO

We can exclude that $s_0 \in \{x_0, y_0\}$, through the following argument:

Suppose $\Rightarrow$ $Stat6: x_0 = s_0 \lor y_0 = s_0$

Let t_0 be the one, of x_0, y_0 which differs from s_0, so that $[t_0, z_0] \notin d_0$ (because z_0 is a source of d_0).

Loc_def $\Rightarrow$ $Stat7: t_0 = \mathbf{arb}(\{x_0, y_0\} \setminus \{s_0\})$
$\langle [t_0, z_0]\rangle \hookrightarrow Stat4(Stat4, Stat4\star) \Rightarrow$ $Stat9: [t_0, z_0] \notin d_0$

On the contrary, $[s_0, z_0] \in d_0$ because we have drawn z_0 from v_1. It hence follows easily that $[t_0, z_0] \notin d_1$ and $[s_0, z_0] \in d_1$.

Suppose $\Rightarrow$ $[s_0, z_0] \notin d_1$
EQUAL $\Rightarrow$ $Stat10: [s_0, z_0] \notin d_0 \cup \{s_0\} \times v_1$
$\langle [s_0, z_0], \{s_0\}, v_1\rangle \hookrightarrow T\mathsf{cartesian}_0(Stat10, Stat4) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO
$(Stat3, Stat6, Stat7)$ELEM $\Rightarrow$ $Stat11: \{x_0, y_0\} = \{s_0, t_0\}$
Suppose $\Rightarrow$ $[t_0, z_0] \in d_1$
EQUAL $\Rightarrow$ $Stat12: [t_0, z_0] \in d_0 \cup \{s_0\} \times v_1$
$\langle [t_0, z_0], \{s_0\}, v_1\rangle \hookrightarrow T\mathsf{cartesian}_0(Stat9, Stat12) \Rightarrow$ $t_0 = s_0$
$(Stat3\star)$Discharge $\Rightarrow$ AUTO

If $x_0 = s_0$, we have $[y_0, z_0] \notin d_1$ and $[x_0, z_0] \in d_1$, contradicting the fact that they x_0, y_0 are not differentiated by d_1

Suppose ⇒ $x_0 = s_0 \;\&\; y_0 = t_0$
EQUAL(*Stat6*) ⇒ $[x_0, z_0] \in d_1 \;\&\; [y_0, z_0] \notin d_1$
$\langle z_0 \rangle \hookrightarrow$ *Stat2*(*Stat4a*⋆) ⇒ false; Discharge ⇒ AUTO

‖ It is likewise untenable that $y_0 = s_0$.

(*Stat11*⋆)ELEM ⇒ $y_0 = s_0 \;\&\; x_0 = t_0$
EQUAL(*Stat6*) ⇒ $[y_0, z_0] \in d_1 \;\&\; [x_0, z_0] \notin d_1$
$\langle z_0 \rangle \hookrightarrow$ *Stat2*(*Stat4a*⋆) ⇒ false; Discharge ⇒ AUTO

‖ But then both x_0 and y_0 belong to $\mathbf{dom}(v_0)$ and hence d_0 differentiates them; i. e., there is a z_1 such that either one of the conditions $[x_0, z_1] \in d_0$, $[y_0, z_1] \in d_0$, but not both, are met.

(*Stat2*⋆)ELEM ⇒ *Stat16* : $\langle \exists z \mid ([x_0, z] \in d_0 \leftrightarrow [y_0, z] \in d_0) \rightarrow x_0 = y_0 \rangle$
$\langle z_1 \rangle \hookrightarrow$ *Stat16*(*Stat16*, *Stat3*⋆) ⇒ $[x_0, z_1] \in d_0 \neq [y_0, z_1] \in d_0$

‖ Our proof by contradiction is then completed by excluding either possibility through the following argument.

EQUAL ⇒ *Stat17* : $d_0 = d_0 \cap (v_0 \times v_0) \;\&\; d_1 = d_0 \cup \{s_0\} \times v_1$
Suppose ⇒ *Stat18* : $[x_0, z_1] \in d_0 \cap (v_0 \times v_0) \;\&\; [y_0, z_1] \notin d_0$
$\langle z_1 \rangle \hookrightarrow$ *Stat2*(*Stat18*, *Stat2a*⋆) ⇒ *Stat19* : $[y_0, z_1] \in d_1 \;\&\; [y_0, z_1] \notin d_0$
EQUAL(*Stat17*) ⇒ *Stat20* : $[y_0, z_1] \in d_0 \cup \{s_0\} \times v_1$
$\langle [y_0, z_1], \{s_0\}, v_1 \rangle \hookrightarrow$ *T*cartesian$_0$(*Stat19*, *Stat20*⋆) ⇒ $y_0 = s_0$
(*Stat4*⋆)Discharge ⇒ AUTO
(*Stat16*⋆)ELEM ⇒ *Stat21* : $[y_0, z_1] \in d_0 \cap (v_0 \times v_0) \;\&\; [x_0, z_1] \notin d_0$
$\langle z_1 \rangle \hookrightarrow$ *Stat2*(*Stat21*, *Stat2a*⋆) ⇒ $[x_0, z_1] \in d_1$
EQUAL(*Stat17*) ⇒ *Stat22* : $[x_0, z_1] \in d_0 \cup \{s_0\} \times v_1$
$\langle [x_0, z_1], \{s_0\}, v_1 \rangle \hookrightarrow$ *T*cartesian$_0$(*Stat21*, *Stat22*⋆) ⇒ $x_0 = s_0$
(*Stat4*⋆)Discharge ⇒ QED

THM extensionality$_0$: [Extensional digraphs are weakly extensional] $\mathsf{Extensional}(V, D) \;\&\; V \times V \supseteq D \rightarrow \mathsf{WExtensional}(V, D)$. PROOF:
Suppose_not(v_0, d_0) ⇒ AUTO
Use_def(WExtensional) ⇒ $\neg\mathsf{Extensional}\big(v_0 \cap \mathbf{dom}(d_0 \cap (v_0 \times v_0)), d_0 \cap (v_0 \times v_0)\big)$
ELEM ⇒ $d_0 \cap (v_0 \times v_0) = d_0$
EQUAL ⇒ *Stat1* : $\mathsf{Extensional}(v_0, d_0) \;\&\; \neg\mathsf{Extensional}(v_0 \cap \mathbf{dom}(d_0), d_0)$
Use_def(Extensional) ⇒ *Stat2* : $\neg\langle \forall x \in v_0 \cap \mathbf{dom}(d_0), y \in v_0 \cap \mathbf{dom}(d_0), \exists z \mid ([x, z] \in d_0 \leftrightarrow [y, z] \in d_0) \rightarrow x = y \rangle \;\&\;$
$\langle \forall x \in v_0, y \in v_0, \exists z \mid ([x, z] \in d_0 \leftrightarrow [y, z] \in d_0) \rightarrow x = y \rangle$
$\langle x_0, y_0, x_0, y_0 \rangle \hookrightarrow$ *Stat2*(*Stat2*⋆) ⇒ false; Discharge ⇒ QED

THM extensionality$_1$: [Making a new vertex the son of a set of vertices comprising a sink cannot disrupt extensionality]

$V \times V \supseteq D \;\&\; X \notin V \;\&\; V \supseteq S \;\&\; \langle \exists t \in S \mid D_{|\{t\}} = \emptyset \rangle \;\&\; \mathsf{Extensional}(V, D) \rightarrow \mathsf{Extensional}(V \cup \{X\}, D \cup S \times \{X\})$. PROOF:

Suppose_not$(v_0, d_0, x_0, s_0) \Rightarrow$ AUTO

Use_def(Extensional) $\Rightarrow$ *Stat1*: $\neg\langle \forall x \in v_0 \cup \{x_0\}, y \in v_0 \cup \{x_0\}, \exists z \mid ([x,z] \in d_0 \cup s_0 \times \{x_0\} \leftrightarrow [y,z] \in d_0 \cup s_0 \times \{x_0\}) \rightarrow x = y \rangle \;\&$
Stat1a: $\langle \forall x \in v_0, y \in v_0, \exists z \mid ([x,z] \in d_0 \leftrightarrow [y,z] \in d_0) \rightarrow x = y \rangle$

$\langle u_0, w_0, u_0, w_0 \rangle \hookrightarrow$ *Stat1*(*Stat1*$\star$) $\Rightarrow$ *Stat2*: $\neg\langle \exists z \mid ([u_0,z] \in d_0 \cup s_0 \times \{x_0\} \leftrightarrow [w_0,z] \in d_0 \cup s_0 \times \{x_0\}) \rightarrow u_0 = w_0 \rangle \;\&\; u_0 \in v_0 \cup \{x_0\} \;\&$
$w_0 \in v_0 \cup \{x_0\} \;\&\; (u_0, w_0 \in v_0 \rightarrow \langle \exists z \mid ([u_0,z] \in d_0 \leftrightarrow [w_0,z] \in d_0) \rightarrow u_0 = w_0 \rangle)$

ELEM $\Rightarrow$ *Stat3*: $\langle \exists t \in s_0 \mid d_{0|\{t\}} = \emptyset \rangle \;\&\; v_0 \times v_0 \supseteq d_0 \;\&\; x_0 \notin v_0 \;\&\; v_0 \supseteq s_0$

Suppose $\Rightarrow$ $u_0, w_0 \in v_0$

(*Stat2*$\star$)ELEM $\Rightarrow$ *Stat4*: $\langle \exists z \mid ([u_0,z] \in d_0 \leftrightarrow [w_0,z] \in d_0) \rightarrow u_0 = w_0 \rangle$

$\langle z_0 \rangle \hookrightarrow$ *Stat4* $\Rightarrow$ AUTO

$\langle z_0 \rangle \hookrightarrow$ *Stat2*(*Stat4*$\star$) $\Rightarrow$ *Stat5*: $([u_0,z_0] \in d_0 \neq [w_0,z_0] \in d_0) \;\&\; ([u_0,z_0] \in d_0 \cup s_0 \times \{x_0\} \leftrightarrow [w_0,z_0] \in d_0 \cup s_0 \times \{x_0\})$

Suppose $\Rightarrow$ $z_0 \notin v_0$

$\langle [u_0,z_0], v_0, v_0 \rangle \hookrightarrow T$cartesian$_0$(*Stat3*$\star$) $\Rightarrow$ $[u_0,z_0] \notin d_0$

$\langle [w_0,z_0], v_0, v_0 \rangle \hookrightarrow T$cartesian$_0$(*Stat3*$\star$) $\Rightarrow$ $[w_0,z_0] \notin d_0$

(*Stat4*$\star$)Discharge $\Rightarrow$ AUTO

Suppose $\Rightarrow$ $[u_0,z_0] \in d_0$

$\langle [w_0,z_0], s_0, \{x_0\} \rangle \hookrightarrow T$cartesian$_0$(*Stat5*$\star$) $\Rightarrow$ $z_0 = x_0$

(*Stat3*$\star$)Discharge $\Rightarrow$ AUTO

$\langle [u_0,z_0], s_0, \{x_0\} \rangle \hookrightarrow T$cartesian$_0$(*Stat5*$\star$) $\Rightarrow$ $z_0 = x_0$

(*Stat3*$\star$)Discharge $\Rightarrow$ AUTO

Suppose $\Rightarrow$ *Stat6*: $\langle \exists z \mid [x_0,z] \in d_0 \cup s_0 \times \{x_0\} \rangle$

$\langle z_1 \rangle \hookrightarrow$ *Stat6*(*Stat6*$\star$) $\Rightarrow$ $[x_0,z_1] \in d_0 \cup s_0 \times \{x_0\}$

$\langle [x_0,z_1], v_0, v_0 \rangle \hookrightarrow T$cartesian$_0$(*Stat3*$\star$) $\Rightarrow$ $[x_0,z_1] \in s_0 \times \{x_0\}$

$\langle [x_0,z_1], s_0, \{x_0\} \rangle \hookrightarrow T$cartesian$_0$(*Stat3*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat7*: $\neg\langle \exists z \mid [x_0,z] \in d_0 \cup s_0 \times \{x_0\} \rangle$

$\langle t_0 \rangle \hookrightarrow$ *Stat3*(*Stat3*, *Stat3*$\star$) $\Rightarrow$ $t_0 \in s_0 \;\&\; d_{0|\{t_0\}} = \emptyset$

Suppose $\Rightarrow$ *Stat8*: $\langle \exists z \mid [t_0,z] \in d_0 \rangle$

$\langle z_2 \rangle \hookrightarrow$ *Stat8*(*Stat8*$\star$) $\Rightarrow$ $[t_0,z_2] \in d_0$

$\langle [t_0,z_2], t_0, z_2, d_0, t_0 \rangle \hookrightarrow T$restr$_2$(*Stat7*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat9*: $\neg\langle \exists z \mid [t_0,z] \in d_0 \rangle$

$\langle [t_0,x_0], s_0, \{x_0\} \rangle \hookrightarrow T$cartesian$_0$(*Stat7*$\star$) $\Rightarrow$ $[t_0,x_0] \in s_0 \times \{x_0\}$

$\langle \emptyset \rangle \hookrightarrow$ *Stat2*(*Stat2*, *Stat2*$\star$) $\Rightarrow$ *Stat10*: $u_0 \neq w_0$

Suppose $\Rightarrow$ $u_0 \in v_0 \;\&\; w_0 = x_0$

Suppose $\Rightarrow$ *Stat11*: $\langle \exists z \mid [u_0,z] \in d_0 \rangle$

$\langle z_3 \rangle \hookrightarrow$ *Stat11*(*Stat11*$\star$) $\Rightarrow$ $[u_0,z_3] \in d_0 \cup s_0 \times \{x_0\}$

$\langle z_3 \rangle \hookrightarrow$ *Stat2*(*Stat10*$\star$) $\Rightarrow$ $[w_0,z_3] \in d_0 \cup s_0 \times \{x_0\}$

$\langle z_3 \rangle \hookrightarrow$ *Stat7*(*Stat10*) $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat12*: $\neg\langle \exists z \mid [u_0,z] \in d_0 \rangle$

Suppose $\Rightarrow$ $t_0 \neq u_0$

$\langle t_0, u_0\rangle \hookrightarrow Stat1a(Stat3\star) \Rightarrow$ $Stat13: \langle \exists z \mid ([t_0, z] \in d_0 \leftrightarrow [u_0, z] \in d_0) \to t_0 = u_0\rangle$
$\langle z_4\rangle \hookrightarrow Stat13 \Rightarrow$ AUTO
$\langle z_4\rangle \hookrightarrow Stat9 \Rightarrow$ AUTO
$\langle z_4\rangle \hookrightarrow Stat12(Stat12\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO
$\langle x_0\rangle \hookrightarrow Stat2(Stat10, Stat10\star) \Rightarrow$ $[u_0, x_0] \in d_0 \cup s_0 \times \{x_0\} \leftrightarrow [w_0, x_0] \in d_0 \cup s_0 \times \{x_0\}$
EQUAL$(Stat10) \Rightarrow$ $Stat14: [t_0, x_0] \in d_0 \cup s_0 \times \{x_0\} \leftrightarrow [x_0, x_0] \in d_0 \cup s_0 \times \{x_0\}$
$\langle [x_0, x_0], v_0, v_0\rangle \hookrightarrow T\mathrm{cartesian}_0(Stat3\star) \Rightarrow$ $[x_0, x_0] \in s_0 \times \{x_0\}$
$\langle [x_0, x_0], s_0, \{x_0\}\rangle \hookrightarrow T\mathrm{cartesian}_0(Stat3\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO
$(Stat2\star)$ELEM $\Rightarrow$ $w_0 \in v_0 \;\&\; u_0 = x_0$
Suppose $\Rightarrow$ $Stat15: \langle \exists z \mid [w_0, z] \in d_0\rangle$
$\langle z_5\rangle \hookrightarrow Stat15(Stat15\star) \Rightarrow$ $[w_0, z_5] \in d_0 \cup s_0 \times \{x_0\}$
$\langle z_5\rangle \hookrightarrow Stat2(Stat10\star) \Rightarrow$ $[u_0, z_5] \in d_0 \cup s_0 \times \{x_0\}$
$\langle z_5\rangle \hookrightarrow Stat7(Stat10) \Rightarrow$ false; Discharge $\Rightarrow$ $Stat16: \neg\langle \exists z \mid [w_0, z] \in d_0\rangle$
Suppose $\Rightarrow$ $t_0 \neq w_0$
$\langle t_0, w_0\rangle \hookrightarrow Stat1a(Stat3\star) \Rightarrow$ $Stat17: \langle \exists z \mid ([t_0, z] \in d_0 \leftrightarrow [w_0, z] \in d_0) \to t_0 = w_0\rangle$
$\langle z_6\rangle \hookrightarrow Stat17 \Rightarrow$ AUTO
$\langle z_6\rangle \hookrightarrow Stat9 \Rightarrow$ AUTO
$\langle z_6\rangle \hookrightarrow Stat16(Stat16\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO
$\langle x_0\rangle \hookrightarrow Stat2(Stat10, Stat10\star) \Rightarrow$ $[u_0, x_0] \in d_0 \cup s_0 \times \{x_0\} \leftrightarrow [w_0, x_0] \in d_0 \cup s_0 \times \{x_0\}$
EQUAL$(Stat10) \Rightarrow$ $Stat18: [x_0, x_0] \in d_0 \cup s_0 \times \{x_0\} \leftrightarrow [w_0, x_0] \in d_0 \cup s_0 \times \{x_0\}$
$(Stat3)$ELEM $\Rightarrow$ $[x_0, x_0] \in d_0 \cup s_0 \times \{x_0\}$
$\langle [x_0, x_0], v_0, v_0\rangle \hookrightarrow T\mathrm{cartesian}_0(Stat3\star) \Rightarrow$ $[x_0, x_0] \in s_0 \times \{x_0\}$
$\langle [x_0, x_0], s_0, \{x_0\}\rangle \hookrightarrow T\mathrm{cartesian}_0(Stat3\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

THM extensionality$_2$: [An extensional digraph has at most one sink] $\mathrm{Extensional}(V, D) \;\&\; X, Y \in V \;\&\; \langle \forall z \mid [X, z] \notin D\rangle \;\&\; \langle \forall z \mid [Y, z] \notin D\rangle \to X = Y$. PROOF:
Suppose_not$(v_0, d_0, x_0, y_0) \Rightarrow$ AUTO
Use_def(Extensional) $\Rightarrow$ $Stat1: \langle \forall x \in v_0, y \in v_0, \exists z \mid ([x, z] \in d_0 \leftrightarrow [y, z] \in d_0) \to x = y\rangle$
$\langle x_0, y_0\rangle \hookrightarrow Stat1(\star) \Rightarrow$ $Stat2: \langle \exists z \mid ([x_0, z] \in d_0 \leftrightarrow [y_0, z] \in d_0) \to x_0 = y_0\rangle$
$\langle z_0\rangle \hookrightarrow Stat2(\star) \Rightarrow$ $[x_0, z_0] \in d_0 \vee [y_0, z_0] \in d_0$
ELEM $\Rightarrow$ $Stat3: \langle \forall z \mid [x_0, z] \notin d_0\rangle \;\&\; \langle \forall z \mid [y_0, z] \notin d_0\rangle$
$\langle z_0, z_0\rangle \hookrightarrow Stat3(\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

2.4 Acyclicity

We can characterize acyclicity as well-foundedness, in view of the finiteness of the graphs on which we are focusing: a set of nodes which has no sinks, in fact, is not necessarily a cycle, but it contains one.

DEF acyclicity: [Acyclicity] $\mathrm{Acyclic}(V, D) \leftrightarrow_{\mathrm{Def}} \langle \forall w \subseteq V \mid w \neq \emptyset \to \langle \exists t \in w \mid \emptyset = \{y \in w \mid [t, y] \in D\}\rangle\rangle$

|| About acyclicity, in this section we will prove the collection of statement displayed here:

THM $\mathsf{acyclicity}_0$: [Adjunction of an outer vertex to a digraph cannot disrupt acyclicity]
$\mathsf{V} \times \mathsf{V} \supseteq \mathsf{D} \;\&\; \mathsf{X} \notin \mathsf{V} \;\&\; \mathsf{V} \supseteq \mathsf{S} \;\&\; \mathsf{Acyclic}(\mathsf{V}, \mathsf{D}) \rightarrow \mathsf{Acyclic}(\mathsf{V} \cup \{\mathsf{X}\}, \mathsf{D} \cup \{\mathsf{X}\} \times \mathsf{S})$

THM $\mathsf{acyclicity}_1$: [Reduction of the set of edges of a digraph preserves its acyclicity]
$\mathsf{Acyclic}(\mathsf{V}, \mathsf{D}) \;\&\; \mathsf{V}' \subseteq \mathsf{V} \;\&\; \mathsf{D}' \subseteq \mathsf{D} \rightarrow \mathsf{Acyclic}(\mathsf{V}', \mathsf{D}')$

THM $\mathsf{acyclicity}_2$: [Acyclic digraphs are devoid of self-loops and of symmetrical arcs]
$\mathsf{Acyclic}(\mathsf{V}, \mathsf{D}) \;\&\; \{\mathsf{Y}, \mathsf{X}\} \subseteq \mathsf{V} \;\&\; [\mathsf{X}, \mathsf{Y}] \in \mathsf{D} \rightarrow [\mathsf{Y}, \mathsf{X}] \notin \mathsf{D} \;\&\; \mathsf{X} \neq \mathsf{Y}$

THM $\mathsf{acyclicity}_3$: [Local sources exist in any acyclic graph]
$\mathsf{W} \neq \emptyset \;\&\; \mathsf{Acyclic}(\mathsf{V}, \mathsf{D}) \;\&\; \mathsf{Finite}(\mathsf{V}) \;\&\; \mathsf{V} \supseteq \mathsf{W} \rightarrow$
$\langle \exists \mathsf{t} \in \mathsf{W} \mid \emptyset = \{\mathsf{y} \in \mathsf{W} \mid [\mathsf{y}, \mathsf{t}] \in \mathsf{D}\} \rangle$

THM $\mathsf{acyclicity}_4$: [Every acyclic graph has sinks and sources]
$\mathsf{Acyclic}(\mathsf{V}, \mathsf{D}) \;\&\; \mathsf{Finite}(\mathsf{V}) \;\&\; \mathsf{V} \neq \emptyset \rightarrow$
$\langle \exists \mathsf{s} \in \mathsf{V}, \mathsf{t} \in \mathsf{V} \mid \emptyset = \{\mathsf{y} \in \mathsf{V} \mid [\mathsf{s}, \mathsf{y}] \in \mathsf{D} \vee [\mathsf{y}, \mathsf{t}] \in \mathsf{D}\} \rangle$

THM $\mathsf{acyclicity}_5$: [No triangle inside an acyclic digraph]
$\mathsf{Acyclic}(\mathsf{V}, \mathsf{D}) \;\&\; \{\mathsf{X}, \mathsf{Y}, \mathsf{Z}\} \subseteq \mathsf{V} \;\&\; \{[\mathsf{X}, \mathsf{Y}], [\mathsf{Y}, \mathsf{Z}]\} \subseteq \mathsf{D} \rightarrow [\mathsf{Z}, \mathsf{X}] \notin \mathsf{D}$

THM $\mathsf{acyclicity}_6$: [Adjunction of an inner vertex to a digraph cannot disrupt acyclicity]
$\mathsf{V} \times \mathsf{V} \supseteq \mathsf{D} \;\&\; \mathsf{X} \notin \mathsf{V} \;\&\; \mathsf{V} \supseteq \mathsf{S} \;\&\; \mathsf{Acyclic}(\mathsf{V}, \mathsf{D}) \rightarrow \mathsf{Acyclic}(\mathsf{V} \cup \{\mathsf{X}\}, \mathsf{D} \cup \mathsf{S} \times \{\mathsf{X}\})$

THM $\mathsf{acyclicity}_0$: [Adjunction of an outer vertex to a digraph cannot disrupt acyclicity] $\mathsf{V} \times \mathsf{V} \supseteq \mathsf{D} \;\&\; \mathsf{X} \notin \mathsf{V} \;\&\; \mathsf{V} \supseteq \mathsf{S} \;\&\; \mathsf{Acyclic}(\mathsf{V}, \mathsf{D}) \rightarrow \mathsf{Acyclic}(\mathsf{V} \cup \{\mathsf{X}\}, \mathsf{D} \cup \{\mathsf{X}\} \times \mathsf{S})$. PROOF:

$\mathsf{Suppose\_not}(\mathsf{v}_0, \mathsf{d}_0, \mathsf{x}_0, \mathsf{s}_0) \Rightarrow$ AUTO

$\mathsf{Use\_def}(\mathsf{Acyclic}) \Rightarrow$ $Stat1:\ \neg\langle \forall \mathsf{w} \subseteq \mathsf{v}_0 \cup \{\mathsf{x}_0\} \mid \mathsf{w} \neq \emptyset \rightarrow \langle \exists \mathsf{t} \in \mathsf{w} \mid \emptyset = \{\mathsf{y} \in \mathsf{w} \mid [\mathsf{t}, \mathsf{y}] \in \mathsf{d}_0 \cup \{\mathsf{x}_0\} \times \mathsf{s}_0\} \rangle\rangle \;\&$
$Stat2:\ \langle \forall \mathsf{w} \subseteq \mathsf{v}_0 \mid \mathsf{w} \neq \emptyset \rightarrow \langle \exists \mathsf{t} \in \mathsf{w} \mid \emptyset = \{\mathsf{y} \in \mathsf{w} \mid [\mathsf{t}, \mathsf{y}] \in \mathsf{d}_0\} \rangle\rangle$

$\langle \mathsf{w}_0 \rangle \hookrightarrow Stat1(\star) \Rightarrow$ $Stat3:\ \neg\langle \exists \mathsf{t} \in \mathsf{w}_0 \mid \emptyset = \{\mathsf{y} \in \mathsf{w}_0 \mid [\mathsf{t}, \mathsf{y}] \in \mathsf{d}_0 \cup \{\mathsf{x}_0\} \times \mathsf{s}_0\} \rangle \;\&\; \mathsf{w}_0 \subseteq \mathsf{v}_0 \cup \{\mathsf{x}_0\} \;\&\; \mathsf{w}_0 \neq \emptyset \;\&\; \mathsf{x}_0 \notin \mathsf{v}_0 \;\&\; \mathsf{v}_0 \times \mathsf{v}_0 \supseteq \mathsf{d}_0$

$\langle \mathsf{w}_0 \rangle \hookrightarrow Stat2 \Rightarrow$ $\mathsf{w}_0 \subseteq \mathsf{v}_0 \rightarrow \langle \exists \mathsf{t} \in \mathsf{w}_0 \mid \emptyset = \{\mathsf{y} \in \mathsf{w}_0 \mid [\mathsf{t}, \mathsf{y}] \in \mathsf{d}_0\} \rangle$

$\mathsf{Suppose} \Rightarrow$ $Stat4:\ \langle \exists \mathsf{t} \in \mathsf{w}_0 \mid \emptyset = \{\mathsf{y} \in \mathsf{w}_0 \mid [\mathsf{t}, \mathsf{y}] \in \mathsf{d}_0\} \rangle \;\&\; \mathsf{w}_0 \subseteq \mathsf{v}_0$

$\langle \mathsf{t}_0 \rangle \hookrightarrow Stat4 \Rightarrow$ $Stat5:\ \{\mathsf{y} \in \mathsf{w}_0 \mid [\mathsf{t}_0, \mathsf{y}] \in \mathsf{d}_0\} = \emptyset \;\&\; \mathsf{t}_0 \in \mathsf{w}_0$

$\langle \mathsf{t}_0 \rangle \hookrightarrow Stat3 \Rightarrow$ $Stat6:\ \{\mathsf{y} \in \mathsf{w}_0 \mid [\mathsf{t}_0, \mathsf{y}] \in \mathsf{d}_0 \cup \{\mathsf{x}_0\} \times \mathsf{s}_0\} \neq \emptyset$

$\langle \mathsf{y}_0 \rangle \hookrightarrow Stat6 \Rightarrow$ $\mathsf{y}_0 \in \mathsf{w}_0 \;\&\; [\mathsf{t}_0, \mathsf{y}_0] \in \mathsf{d}_0 \cup \{\mathsf{x}_0\} \times \mathsf{s}_0$

$\langle \mathsf{y}_0 \rangle \hookrightarrow Stat5 \Rightarrow$ $Stat7:\ [\mathsf{t}_0, \mathsf{y}_0] \in \{\mathsf{x}_0\} \times \mathsf{s}_0$

$\langle [\mathsf{t}_0, \mathsf{y}_0], \{\mathsf{x}_0\}, \mathsf{s}_0 \rangle \hookrightarrow T\mathsf{cartesian}_0(Stat7) \Rightarrow$ $\mathsf{t}_0 = \mathsf{x}_0$

$\mathsf{Discharge} \Rightarrow$ $\mathsf{x}_0 \in \mathsf{w}_0$

$\langle \mathsf{x}_0 \rangle \hookrightarrow Stat3(Stat3\star) \Rightarrow$ $Stat9:\ \{\mathsf{y} \in \mathsf{w}_0 \mid [\mathsf{x}_0, \mathsf{y}] \in \mathsf{d}_0 \cup \{\mathsf{x}_0\} \times \mathsf{s}_0\} \neq \emptyset$

$\langle \mathsf{y}_2 \rangle \hookrightarrow Stat9 \Rightarrow$ $Stat10:\ \mathsf{y}_2 \in \mathsf{w}_0 \;\&\; [\mathsf{x}_0, \mathsf{y}_2] \in \mathsf{d}_0 \cup \{\mathsf{x}_0\} \times \mathsf{s}_0$

$\langle [\mathsf{x}_0, \mathsf{y}_2], \mathsf{v}_0, \mathsf{v}_0 \rangle \hookrightarrow T\mathsf{cartesian}_0(Stat11, Stat3) \Rightarrow$ $Stat11:\ [\mathsf{x}_0, \mathsf{y}_2] \notin \mathsf{v}_0 \times \mathsf{v}_0$

$\langle [\mathsf{x}_0, \mathsf{y}_2], \{\mathsf{x}_0\}, \mathsf{s}_0 \rangle \hookrightarrow T\mathsf{cartesian}_0(Stat3, Stat10, Stat11) \Rightarrow$ $\mathsf{y}_2 \in \mathsf{s}_0$

ELEM $\Rightarrow$ $y_2 \in w_0 \setminus \{x_0\}$
$\langle w_0 \setminus \{x_0\} \rangle \hookrightarrow Stat2 \Rightarrow$ $Stat13: \langle \exists t \in w_0 \setminus \{x_0\} \mid \emptyset = \{y \in w_0 \setminus \{x_0\} \mid [t, y] \in d_0\} \rangle$
$\langle t_1 \rangle \hookrightarrow Stat13 \Rightarrow$ $Stat14: \{y \in w_0 \setminus \{x_0\} \mid [t_1, y] \in d_0\} = \emptyset \;\&\; t_1 \in w_0 \setminus \{x_0\}$
$\langle t_1 \rangle \hookrightarrow Stat3 \Rightarrow$ $Stat15: \{y \in w_0 \mid [t_1, y] \in d_0 \cup \{x_0\} \times s_0\} \neq \emptyset$
$\langle y_5 \rangle \hookrightarrow Stat15 \Rightarrow$ $y_5 \in w_0 \;\&\; [t_1, y_5] \in d_0 \cup \{x_0\} \times s_0$
Suppose $\Rightarrow$ $y_5 \neq x_0$
$\langle y_5 \rangle \hookrightarrow Stat14 \Rightarrow$ $Stat16: [t_1, y_5] \in \{x_0\} \times s_0$
$\langle [t_1, y_5], \{x_0\}, s_0 \rangle \hookrightarrow T\mathsf{cartesian}_0(Stat16) \Rightarrow$ $t_1 = x_0$
$(Stat14\star)$Discharge $\Rightarrow$ AUTO
EQUAL$(Stat15) \Rightarrow$ $Stat18: [t_1, x_0] \in d_0 \cup \{x_0\} \times s_0$
Suppose $\Rightarrow$ $Stat19: [t_1, x_0] \in d_0$
$\langle [t_1, x_0], v_0, v_0 \rangle \hookrightarrow T\mathsf{cartesian}_0(Stat19, Stat3) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO
$\langle [t_1, x_0], \{x_0\}, s_0 \rangle \hookrightarrow T\mathsf{cartesian}_0(Stat18) \Rightarrow$ $t_1 = x_0$
$(Stat14\star)$Discharge $\Rightarrow$ QED

THM acyclicity$_1$: [Reduction of the set of edges of a digraph preserves its acyclicity] $\mathsf{Acyclic}(V, D) \;\&\; V' \subseteq V \;\&\; D' \subseteq D \rightarrow \mathsf{Acyclic}(V', D')$. PROOF:
Suppose_not$(v_0, d_0, v_1, d_1) \Rightarrow$ AUTO
Use_def(Acyclic) $\Rightarrow$ $Stat1: \neg\langle \forall w \subseteq v_1 \mid w \neq \emptyset \rightarrow \langle \exists t \in w \mid \emptyset = \{y \in w \mid [t, y] \in d_1\} \rangle\rangle \;\&\; Stat2: \langle \forall w \subseteq v_0 \mid w \neq \emptyset \rightarrow \langle \exists t \in w \mid \emptyset = \{y \in w \mid [t, y] \in d_0\} \rangle\rangle$
$\langle w_0 \rangle \hookrightarrow Stat1 \Rightarrow$ AUTO
$\langle w_0 \rangle \hookrightarrow Stat2(\star) \Rightarrow$ $Stat3: \langle \exists t \in w_0 \mid \emptyset = \{y \in w_0 \mid [t, y] \in d_0\} \rangle \;\&\; \neg\langle \exists t \in w_0 \mid \emptyset = \{y \in w_0 \mid [t, y] \in d_1\} \rangle$
$\langle t_0, t_0 \rangle \hookrightarrow Stat3(\star) \Rightarrow$ $Stat4: \emptyset \neq \{y \in w_0 \mid [t_0, y] \in d_1\} \;\&\; \emptyset = \{y \in w_0 \mid [t_0, y] \in d_0\}$
$\langle y_0, y_0 \rangle \hookrightarrow Stat4(\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

THM acyclicity$_2$: [Acyclic digraphs are devoid of self-loops and of symmetrical arcs] $\mathsf{Acyclic}(V, D) \;\&\; \{Y, X\} \subseteq V \;\&\; [X, Y] \in D \rightarrow [Y, X] \notin D \;\&\; X \neq Y$. PROOF:
Suppose_not$(v_0, d_0, y_0, x_0) \Rightarrow$ AUTO
Use_def(Acyclic) $\Rightarrow$ $Stat1: \langle \forall w \subseteq v_0 \mid w \neq \emptyset \rightarrow \langle \exists t \in w \mid \emptyset = \{y \in w \mid [t, y] \in d_0\} \rangle\rangle$
$\langle \{y_0, x_0\} \rangle \hookrightarrow Stat1 \Rightarrow$ $Stat2: \langle \exists t \in \{y_0, x_0\} \mid \emptyset = \{y \in \{y_0, x_0\} \mid [t, y] \in d_0\} \rangle \;\&\; [x_0, y_0] \in d_0$
$\langle t_0 \rangle \hookrightarrow Stat2(Stat2\star) \Rightarrow$ $Stat3: \emptyset = \{y \in \{y_0, x_0\} \mid [t_0, y] \in d_0\} \;\&\; t_0 \in \{y_0, x_0\}$
$\langle y_0 \rangle \hookrightarrow Stat3(Stat2) \Rightarrow$ $t_0 \neq x_0$
$\langle x_0 \rangle \hookrightarrow Stat3(Stat2) \Rightarrow$ $[y_0, x_0] \notin d_0$
Discharge $\Rightarrow$ QED

> The following theorem states that every non-null set W of vertices in an acyclic digraph V, D has at least one source; that is, it has a vertex s whose in-neighborhood does not intersect W. Although it deserves some interest of its own, this theorem will not be exploited in the present scenario: we are including it only as a down-sized, easier-to-read version of the subsequent THM acyclicity$_4$.

THM acyclicity$_3$: [Local sources exist in any acyclic graph] $W \neq \emptyset$ & $\mathsf{Acyclic}(V, D)$ & $\mathsf{Finite}(V)$ & $V \supseteq W \rightarrow \langle \exists t \in W \mid \emptyset = \{y \in W \mid [y, t] \in D\} \rangle$. PROOF:
$\mathsf{Suppose\_not}(w_2, v_0, d_0) \Rightarrow$ AUTO

Arguing by contradiction, suppose that there is a counterexample w_2, v_0, d_0 to the claim. Then, using finite induction, we can pick a counterexample w_1, v_0, d_0 whose w_1 is minimal.

$\mathsf{Use\_def}(\mathsf{Acyclic}) \Rightarrow$ *Stat1*: $\langle \forall w \subseteq v_0 \mid w \neq \emptyset \rightarrow \langle \exists t \in w \mid \emptyset = \{y \in w \mid [t, y] \in d_0\} \rangle \rangle$ & $\mathsf{Finite}(w_2)$
APPLY $\langle \mathsf{fin}_\Theta : w_1 \rangle$ $\mathsf{finiteInduction}(s_0 \mapsto w_2, P(W) \mapsto (W \subseteq v_0$ & $W \neq \emptyset$ & $\neg \langle \exists t \in W \mid \emptyset = \{y \in W \mid [y, t] \in d_0\} \rangle)) \Rightarrow$
Stat7: $\langle \forall v \mid v \subseteq w_1 \rightarrow \mathsf{Finite}(v)$ & $(v \subseteq v_0$ & $v \neq \emptyset$ & $\neg \langle \exists t \in v \mid \emptyset = \{y \in v \mid [y, t] \in d_0\} \rangle \leftrightarrow v = w_1) \rangle$
$\langle w_1 \rangle \hookrightarrow$ *Stat7*(*Stat7*$\star$) $\Rightarrow$ *Stat8*: $\neg \langle \exists t \in w_1 \mid \emptyset = \{y \in w_1 \mid [y, t] \in d_0\} \rangle$ & $w_1 \neq \emptyset$ & $w_1 \subseteq v_0$

Now consider a sink a of w_1. It is easily seen that $w_1 \neq \{a\}$.

$\langle w_1 \rangle \hookrightarrow$ *Stat1*(*Stat8*$\star$) $\Rightarrow$ *Stat9*: $\langle \exists t \in w_1 \mid \emptyset = \{y \in w_1 \mid [t, y] \in d_0\} \rangle$
$\langle a \rangle \hookrightarrow$ *Stat9*(*Stat9*$\star$) $\Rightarrow$ *Stat10*: $\{y \in w_1 \mid [a, y] \in d_0\} = \emptyset$ & $a \in w_1$
Suppose $\Rightarrow$ $w_1 = \{a\}$
$\langle a \rangle \hookrightarrow$ *Stat8*(*Stat10*$\star$) $\Rightarrow$ *Stat11*: $\{y \in w_1 \mid [y, a] \in d_0\} \neq \emptyset$
$\langle c \rangle \hookrightarrow$ *Stat11*(*Stat10*) $\Rightarrow$ $[a, a] \in d_0$
$\langle a \rangle \hookrightarrow$ *Stat10* $\Rightarrow$ AUTO
(*Stat11*)Discharge $\Rightarrow$ AUTO
$\langle w_1 \setminus \{a\} \rangle \hookrightarrow$ *Stat7*(*Stat8*$\star$) $\Rightarrow$ *Stat14*: $\langle \exists t \in w_1 \setminus \{a\} \mid \emptyset = \{y \in w_1 \setminus \{a\} \mid [y, t] \in d_0\} \rangle$
$\langle t_0 \rangle \hookrightarrow$ *Stat14*(*Stat14*$\star$) $\Rightarrow$ *Stat15*: $\{y \in w_1 \setminus \{a\} \mid [y, t_0] \in d_0\} = \emptyset$ & $t_0 \in w_1 \setminus \{a\}$

If t_0, which is a source of $w_1 \setminus \{a\}$, is not a source of w_1, then $[a, t_0]$ must be an edge; but this conflicts with the way a has been chosen.

$\langle t_0 \rangle \hookrightarrow$ *Stat8*(*Stat15*$\star$) $\Rightarrow$ *Stat16*: $\{y \in w_1 \mid [y, t_0] \in d_0\} \neq \emptyset$
$\langle b \rangle \hookrightarrow$ *Stat16*(*Stat15*$\star$) $\Rightarrow$ *Stat17*: $b \notin \{y \in w_1 \setminus \{a\} \mid [y, t_0] \in d_0\}$ & $[b, t_0] \in d_0$ & $b \in w_1$
$\langle b \rangle \hookrightarrow$ *Stat17*(*Stat17*) $\Rightarrow$ *Stat18*: $[a, t_0] \in d_0$
$\langle t_0 \rangle \hookrightarrow$ *Stat10*(*Stat15*, *Stat18*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ QED

The following theorem states that every acyclic digraph V, D (with $V \neq \emptyset$) has at least one source and one sink, namely it has vertices s, t whose in-neighborhood and out-neighborhood, respectively, are empty.

THM acyclicity$_4$: [Every acyclic graph has sinks and sources] $\mathsf{Acyclic}(V, D)$ & $\mathsf{Finite}(V)$ & $V \neq \emptyset \rightarrow \langle \exists s \in V, t \in V \mid \emptyset = \{y \in V \mid [s, y] \in D \vee [y, t] \in D\} \rangle$. PROOF:
$\mathsf{Suppose\_not}(w_2, d_0) \Rightarrow$ AUTO

Arguing by contradiction, suppose that there is a counterexample w_2, d_0 to the claim. It readily follows from the definition of acyclicity that w_2, d_0 has sinks; therefore w_2, d_0 has no sources.

$\mathsf{Use\_def}(\mathsf{Acyclic}) \Rightarrow$ *Stat1*: $\langle \forall w \subseteq w_2 \mid w \neq \emptyset \rightarrow \langle \exists t \in w \mid \emptyset = \{y \in w \mid [t, y] \in d_0\} \rangle \rangle$
Suppose $\Rightarrow$ *Stat2*: $\langle \exists t \in w_2 \mid \emptyset = \{y \in w_2 \mid [y, t] \in d_0\} \rangle$

$\langle t_1 \rangle \hookrightarrow$ *Stat2*$(\star) \Rightarrow$ *Stat3*: $\neg\langle \exists s \in w_2, t \in w_2 \mid \emptyset = \{y \in w_2 \mid [s,y] \in d_0 \vee [y,t] \in d_0\} \rangle \,\&\, \emptyset = \{y \in w_2 \mid [y,t_1] \in d_0\} \,\&\, t_1 \in w_2$
$\langle w_2 \rangle \hookrightarrow$ *Stat1*$(\star) \Rightarrow$ *Stat4*: $\langle \exists s \in w_2 \mid \emptyset = \{y \in w_2 \mid [s,y] \in d_0\} \rangle$
$\langle s_0 \rangle \hookrightarrow$ *Stat4* $\Rightarrow$ AUTO
$\langle s_0, t_1 \rangle \hookrightarrow$ *Stat3*(*Stat3*$\star$) $\Rightarrow$ *Stat5*: $\emptyset \neq \{y \in w_2 \mid [s_0,y] \in d_0 \vee [y,t_1] \in d_0\} \,\&\, \emptyset = \{y \in w_2 \mid [s_0,y] \in d_0\} \,\&\, \emptyset = \{y \in w_2 \mid [y,t_1] \in d_0\} \,\&\, s_0 \in w_2 \,\&\, t_1 \in w_2$
$\langle y_0, y_0, y_0 \rangle \hookrightarrow$ *Stat5*(*Stat5*$\star$) $\Rightarrow$ false
Discharge $\Rightarrow$ AUTO

‖ Then, using finite induction, we can pick a counterexample w_1, d_0 whose w_1 is minimal.

APPLY $\langle fin_\Theta : w_1 \rangle$ finiteInduction$(s_0 \mapsto w_2, P(W) \mapsto (W \subseteq w_2 \,\&\, W \neq \emptyset \,\&\, \neg\langle \exists t \in W \mid \emptyset = \{y \in W \mid [y,t] \in d_0\} \rangle)) \Rightarrow$
Stat7: $\langle \forall v \mid v \subseteq w_1 \rightarrow \mathrm{Finite}(v) \,\&\, (v \subseteq w_2 \,\&\, v \neq \emptyset \,\&\, \neg\langle \exists t \in v \mid \emptyset = \{y \in v \mid [y,t] \in d_0\} \rangle \leftrightarrow v = w_1) \rangle$
$\langle w_1 \rangle \hookrightarrow$ *Stat7*(*Stat7*$\star$) $\Rightarrow$ *Stat8*: $\neg\langle \exists t \in w_1 \mid \emptyset = \{y \in w_1 \mid [y,t] \in d_0\} \rangle \,\&\, w_1 \neq \emptyset \,\&\, w_1 \subseteq w_2$

‖ Now consider a sink a of w_1. It is easily seen that $w_1 \neq \{a\}$.

$\langle w_1 \rangle \hookrightarrow$ *Stat1*(*Stat8*$\star$) $\Rightarrow$ *Stat9*: $\langle \exists t \in w_1 \mid \emptyset = \{y \in w_1 \mid [t,y] \in d_0\} \rangle$
$\langle a \rangle \hookrightarrow$ *Stat9*(*Stat9*$\star$) $\Rightarrow$ *Stat10*: $\{y \in w_1 \mid [a,y] \in d_0\} = \emptyset \,\&\, a \in w_1$
Suppose $\Rightarrow$ $w_1 = \{a\}$
$\langle a \rangle \hookrightarrow$ *Stat8*(*Stat10*$\star$) $\Rightarrow$ *Stat11*: $\{y \in w_1 \mid [y,a] \in d_0\} \neq \emptyset$
$\langle c \rangle \hookrightarrow$ *Stat11*(*Stat10*) $\Rightarrow$ $[a,a] \in d_0$
$\langle a \rangle \hookrightarrow$ *Stat10* $\Rightarrow$ AUTO
(*Stat11*)Discharge $\Rightarrow$ AUTO
$\langle w_1 \setminus \{a\} \rangle \hookrightarrow$ *Stat7*(*Stat8*$\star$) $\Rightarrow$ *Stat14*: $\langle \exists t \in w_1 \setminus \{a\} \mid \emptyset = \{y \in w_1 \setminus \{a\} \mid [y,t] \in d_0\} \rangle$
$\langle t_0 \rangle \hookrightarrow$ *Stat14*(*Stat14*$\star$) $\Rightarrow$ *Stat15*: $\{y \in w_1 \setminus \{a\} \mid [y,t_0] \in d_0\} = \emptyset \,\&\, t_0 \in w_1 \setminus \{a\}$

‖ If t_0, which is a source of $w_1 \setminus \{a\}$, is not a source of w_1, then $[a, t_0]$ must be an edge; but this conflicts with the way a has been chosen.

$\langle t_0 \rangle \hookrightarrow$ *Stat8*(*Stat15*$\star$) $\Rightarrow$ *Stat16*: $\{y \in w_1 \mid [y,t_0] \in d_0\} \neq \emptyset$
$\langle b \rangle \hookrightarrow$ *Stat16*(*Stat15*$\star$) $\Rightarrow$ *Stat17*: $b \notin \{y \in w_1 \setminus \{a\} \mid [y,t_0] \in d_0\} \,\&\, [b,t_0] \in d_0 \,\&\, b \in w_1$
$\langle b \rangle \hookrightarrow$ *Stat17*(*Stat17*) $\Rightarrow$ *Stat18*: $[a,t_0] \in d_0$
$\langle t_0 \rangle \hookrightarrow$ *Stat10*(*Stat15*, *Stat18*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ QED

THM acyclicity$_5$: [No triangle inside an acyclic digraph] $\mathrm{Acyclic}(V,D) \,\&\, \{X,Y,Z\} \subseteq V \,\&\, \{[X,Y],[Y,Z]\} \subseteq D \rightarrow [Z,X] \notin D$. PROOF:
Suppose_not$(v_0, d_0, x_0, y_0, z_0) \Rightarrow$ AUTO
Use_def(Acyclic) $\Rightarrow$ *Stat1*: $\langle \forall w \subseteq v_0 \mid w \neq \emptyset \rightarrow \langle \exists t \in w \mid \emptyset = \{y \in w \mid [t,y] \in d_0\} \rangle \rangle$
$\langle \{x_0,y_0,z_0\} \rangle \hookrightarrow$ *Stat1*$(\star) \Rightarrow$ *Stat2*: $\langle \exists t \in \{x_0,y_0,z_0\} \mid \emptyset = \{y \in \{x_0,y_0,z_0\} \mid [t,y] \in d_0\} \rangle$
$\langle t_0 \rangle \hookrightarrow$ *Stat2*(*Stat2*$\star$) $\Rightarrow$ *Stat3*: $t_0 \in \{x_0,y_0,z_0\} \,\&\, \{y \in \{x_0,y_0,z_0\} \mid [t_0,y] \in d_0\} = \emptyset$
Suppose $\Rightarrow$ $t_0 = x_0$

EQUAL($Stat3$) ⇒ $Stat4: y_0 \notin \{y \in \{x_0, y_0, z_0\} \mid [x_0, y] \in d_0\}$

$\langle y_0 \rangle \hookrightarrow Stat4(\star)$ ⇒ false; Discharge ⇒ AUTO

Suppose ⇒ $t_0 = y_0$

EQUAL($Stat3$) ⇒ $Stat5: z_0 \notin \{y \in \{x_0, y_0, z_0\} \mid [y_0, y] \in d_0\}$

$\langle z_0 \rangle \hookrightarrow Stat5(\star)$ ⇒ false; Discharge ⇒ $t_0 = z_0$

EQUAL($Stat3$) ⇒ $Stat6: x_0 \notin \{y \in \{x_0, y_0, z_0\} \mid [z_0, y] \in d_0\}$

$\langle x_0 \rangle \hookrightarrow Stat6(\star)$ ⇒ false; Discharge ⇒ QED

THM acyclicity$_6$: [Adjunction of an inner vertex to a digraph cannot disrupt acyclicity] $V \times V \supseteq D \,\&\, X \notin V \,\&\, V \supseteq S \,\&\, \text{Acyclic}(V, D) \rightarrow \text{Acyclic}(V \cup \{X\}, D \cup S \times \{X\})$. PROOF:

Suppose_not(v_0, d_0, x_0, s_0) ⇒ AUTO

Use_def(Acyclic) ⇒ $Stat1: \neg\langle \forall w \subseteq v_0 \cup \{x_0\} \mid w \neq \emptyset \rightarrow \langle \exists t \in w \mid \emptyset = \{y \in w \mid [t, y] \in d_0 \cup s_0 \times \{x_0\}\}\rangle\rangle \,\&\, \langle \forall w \subseteq v_0 \mid w \neq \emptyset \rightarrow \langle \exists t \in w \mid \emptyset = \{y \in w \mid [t, y] \in d_0\}\rangle\rangle$

$\langle w_0, w_0 \rangle \hookrightarrow Stat1(Stat1\star)$ ⇒ $Stat2: \neg\langle \exists t \in w_0 \mid \emptyset = \{y \in w_0 \mid [t, y] \in d_0 \cup s_0 \times \{x_0\}\}\rangle \,\&\, w_0 \subseteq v_0 \cup \{x_0\} \,\&\, w_0 \neq \emptyset \,\&\, (x_0 \notin w_0 \rightarrow \langle \exists t \in w_0 \mid \emptyset = \{y \in w_0 \mid [t, y] \in d_0\}\rangle)$

ELEM ⇒ $Stat3: v_0 \times v_0 \supseteq d_0 \,\&\, x_0 \notin v_0 \,\&\, v_0 \supseteq s_0$

Suppose ⇒ $Stat4: \langle \exists t \in w_0 \mid \emptyset = \{y \in w_0 \mid [t, y] \in d_0\}\rangle$

$\langle t_0 \rangle \hookrightarrow Stat4(Stat4\star)$ ⇒ $Stat5: \{y \in w_0 \mid [t_0, y] \in d_0\} = \emptyset \,\&\, t_0 \in w_0$

$\langle t_0 \rangle \hookrightarrow Stat2(Stat5\star)$ ⇒ $Stat6: \{y \in w_0 \mid [t_0, y] \in d_0 \cup s_0 \times \{x_0\}\} \neq \emptyset$

$\langle y_0 \rangle \hookrightarrow Stat6(Stat6\star)$ ⇒ $y_0 \in w_0 \,\&\, [t_0, y_0] \in d_0 \cup s_0 \times \{x_0\}$

$\langle y_0 \rangle \hookrightarrow Stat5(Stat6\star)$ ⇒ $Stat7: [t_0, y_0] \in s_0 \times \{x_0\}$

$\langle [t_0, y_0], s_0, \{x_0\} \rangle \hookrightarrow T\text{cartesian}_0(Stat7)$ ⇒ $t_0 \in s_0 \,\&\, y_0 = x_0$

EQUAL($Stat6$) ⇒ $Stat8: x_0 \in w_0$

$\langle x_0 \rangle \hookrightarrow Stat2(Stat8\star)$ ⇒ $Stat9: \{y \in w_0 \mid [x_0, y] \in d_0 \cup s_0 \times \{x_0\}\} \neq \emptyset$

$\langle y_2 \rangle \hookrightarrow Stat9(Stat9\star)$ ⇒ $[x_0, y_2] \in d_0 \cup s_0 \times \{x_0\}$

Suppose ⇒ $[x_0, y_2] \in v_0 \times v_0$

$\langle [x_0, y_2], v_0, v_0 \rangle \hookrightarrow T\text{cartesian}_0(Stat3)$ ⇒ false; Discharge ⇒ AUTO

($Stat3\star$)ELEM ⇒ $Stat10: [x_0, y_2] \in s_0 \times \{x_0\}$

$\langle [x_0, y_2], s_0, \{x_0\} \rangle \hookrightarrow T\text{cartesian}_0(Stat10)$ ⇒ $x_0 \in s_0$

($Stat3\star$)Discharge ⇒ $x_0 \in w_0$

$\langle x_0 \rangle \hookrightarrow Stat2(Stat3\star)$ ⇒ $Stat11: \{y \in w_0 \mid [x_0, y] \in d_0 \cup s_0 \times \{x_0\}\} \neq \emptyset$

$\langle y_3 \rangle \hookrightarrow Stat11(Stat11\star)$ ⇒ $[x_0, y_3] \in d_0 \cup s_0 \times \{x_0\}$

Suppose ⇒ $[x_0, y_3] \in v_0 \times v_0$

$\langle [x_0, y_3], v_0, v_0 \rangle \hookrightarrow T\text{cartesian}_0(Stat3)$ ⇒ false; Discharge ⇒ AUTO

($Stat3\star$)ELEM ⇒ $Stat12: [x_0, y_3] \in s_0 \times \{x_0\}$

$\langle [x_0, y_3], s_0, \{x_0\} \rangle \hookrightarrow T\text{cartesian}_0(Stat12)$ ⇒ $x_0 \in s_0$

($Stat3\star$)Discharge ⇒ QED

THM voidgraph$_1$: [The void has all virtues] $V \subseteq \{X\} \rightarrow \text{Extensional}(V, \emptyset) \,\&\, \text{Orientates}(\emptyset, V, E)$. PROOF:

Suppose_not(v_0, x_0, e_0) ⇒ AUTO

Suppose ⇒ $\neg\text{Extensional}(v_0, \emptyset)$

Use_def(Extensional) ⇒ *Stat1*: $\neg\langle\forall x \in v_0, y \in v_0, \exists z \mid ([x,z] \in \emptyset \leftrightarrow [y,z] \in \emptyset) \rightarrow x = y\rangle$

$\langle x_1, y_1\rangle \hookrightarrow Stat1(\star)$ ⇒ *Stat2*: $\neg\langle\exists z \mid ([x_1,z] \in \emptyset \leftrightarrow [y_1,z] \in \emptyset) \rightarrow x_1 = y_1\rangle \;\&\; x_1 = x_0 \;\&\; y_1 = x_0$

$\langle\emptyset\rangle \hookrightarrow Stat2(Stat2\star)$ ⇒ false

Discharge ⇒ Auto

Use_def(Orientates) ⇒ *Stat3*: $\{\{x,y\} : x \in v_0, y \in v_0 \setminus \{x\}\} \neq \emptyset \vee \{\{p^{[1]}, p^{[2]}\} : p \in \emptyset \mid p = [p^{[1]}, p^{[2]}]\} \neq \emptyset$

$\langle x_2, y_2, p\rangle \hookrightarrow Stat3(\star)$ ⇒ false; Discharge ⇒ Qed

Thm voidgraph$_2$: [The void has all virtues, 2] $\text{Acyclic}(V, \emptyset)$. Proof:

Suppose_not(v_0) ⇒ Auto

Use_def(Acyclic) ⇒ *Stat1*: $\neg\langle\forall w \subseteq v_0 \mid w \neq \emptyset \rightarrow \langle\exists t \in w \mid \emptyset = \{y \in w \mid [t,y] \in \emptyset\}\rangle\rangle$

$\langle w_0\rangle \hookrightarrow Stat1(\star)$ ⇒ *Stat2*: $w_0 \neq \emptyset \;\&\; \neg\langle\exists t \in w_0 \mid \emptyset = \{y \in w_0 \mid [t,y] \in \emptyset\}\rangle$

$\langle t_0, t_0\rangle \hookrightarrow Stat2(Stat2\star)$ ⇒ *Stat3*: $\emptyset \neq \{y \in w_0 \mid [t_0, y] \in \emptyset\}$

$\langle y_0\rangle \hookrightarrow Stat3(Stat3\star)$ ⇒ false; Discharge ⇒ Qed

Theory finAcycLabeling$(v_0, d_0, h(S,X))$

Is_map(d_0)

Acyclic(v_0, d_0)

Finite(v_0)

End finAcycLabeling

Enter_theory finAcycLabeling

Thm finAcycLabeling$_0$: [Recursive labeling of an acyclic graph] $\langle\exists f \mid \text{Svm}(f) \;\&\; \mathbf{dom}(f) = v_0 \;\&\; \langle\forall x \in v_0 \mid f\restriction x = h(\{f\restriction p^{[2]} : p \in d_{0|\{x\}} \mid p^{[2]} \in v_0\}, x)\rangle\rangle$. Proof:

Suppose_not() ⇒ Auto

Assump ⇒ *Stat0*: $\text{Finite}(v_0) \;\&\; \text{Acyclic}(v_0, d_0) \;\&\; \text{Is\_map}(d_0)$

Suppose that the contrary holds, so that v_0 has a subset v_1 which cannot be labeled in the way specified for v_0 in the claim.

Suppose ⇒ *Stat1*: $\neg\langle\exists v \subseteq v_0 \mid \text{Finite}(v) \;\&\; \neg\langle\exists f \mid \text{Svm}(f) \;\&\; \mathbf{dom}(f) = v \;\&\; \langle\forall x \in v \mid f\restriction x = h(\{f\restriction p^{[2]} : p \in d_{0|\{x\}} \mid p^{[2]} \in v\}, x)\rangle\rangle\rangle$

$\langle v_0\rangle \hookrightarrow Stat1(\star)$ ⇒ false; Discharge ⇒ *Stat2*: $\langle\exists v \subseteq v_0 \mid \text{Finite}(v) \;\&\; \neg\langle\exists f \mid \text{Svm}(f) \;\&\; \mathbf{dom}(f) = v \;\&\; \langle\forall x \in v \mid f\restriction x = h(\{f\restriction p^{[2]} : p \in d_{0|\{x\}} \mid p^{[2]} \in v\}, x)\rangle\rangle\rangle$

$\langle v_1\rangle \hookrightarrow Stat2(Stat2\star)$ ⇒ $\text{Finite}(v_1) \;\&\; v_1 \subseteq v_0 \;\&\; \neg\langle\exists f \mid \text{Svm}(f) \;\&\; \mathbf{dom}(f) = v_1 \;\&\; \langle\forall x \in v_1 \mid f\restriction x = h(\{f\restriction p^{[2]} : p \in d_{0|\{x\}} \mid p^{[2]} \in v_1\}, x)\rangle\rangle$

In view of the finiteness of v_0, we can consider an inclusion-minimal such subset, v_2. Plainly, $v_2 \neq \emptyset$ must hold.

APPLY $\langle \text{fin}_\ominus : v_2\rangle$ finiteInduction$\big(s_0 \mapsto v_1, P(V) \mapsto (V \subseteq v_0 \;\&\; \neg\langle\exists f \mid \text{Svm}(f) \;\&\; \mathbf{dom}(f) = V \;\&\; \langle\forall x \in V \mid f\restriction x = h(\{f\restriction p^{[2]} : p \in d_{0|\{x\}} \mid p^{[2]} \in V\}, x)\rangle\rangle)\big)$ ⇒

$Stat3:\ \langle\forall V\,|\,V\subseteq v_2\to \mathsf{Finite}(V)\ \&\ \big(V\subseteq v_0\ \&\ \neg\langle\exists f\,|\,\mathsf{Svm}(f)\ \&\ \mathbf{dom}(f)=V\ \&\ \langle\forall x\in V\,|\,f\upharpoonright x=h(\{f\upharpoonright p^{[2]}:\ p\in d_{0|\{x\}}\,|\,p^{[2]}\in V\},x)\rangle\rangle\leftrightarrow V=v_2\big)\rangle$

$\langle v_2\rangle\hookrightarrow Stat3(Stat3\star)\Rightarrow\quad Stat4:\ \neg\langle\exists f\,|\,\mathsf{Svm}(f)\ \&\ \mathbf{dom}(f)=v_2\ \&\ \langle\forall x\in v_2\,|\,f\upharpoonright x=h(\{f\upharpoonright p^{[2]}:\ p\in d_{0|\{x\}}\,|\,p^{[2]}\in v_2\},x)\rangle\rangle\ \&\ v_2\subseteq v_0\ \&\ \mathsf{Finite}(v_2)$

$\mathsf{Suppose}\Rightarrow\quad v_2=\emptyset$

$\langle\emptyset\rangle\hookrightarrow T\mathsf{domain}_1(Stat4\star)\Rightarrow\quad \mathbf{dom}(\emptyset)=v_2$

$\langle\emptyset\rangle\hookrightarrow T\mathsf{image}_0(Stat4\star)\Rightarrow\quad \mathsf{Svm}(\emptyset)$

$\langle\emptyset\rangle\hookrightarrow Stat4(Stat4\star)\Rightarrow\quad Stat5:\ \neg\langle\forall x\in v_2\,|\,\emptyset\upharpoonright x=h(\{\emptyset\upharpoonright p^{[2]}:\ p\in d_{0|\{x\}}\,|\,p^{[2]}\in v_2\},x)\rangle$

$\langle x_0\rangle\hookrightarrow Stat5(Stat4\star)\Rightarrow\quad \mathsf{false};\qquad \mathsf{Discharge}\Rightarrow\quad \textsc{Auto}$

The digraph v_2, d_0 inherits from v_0, d_0 the acyclicity property; therefore it has a source, s_0. The minimality of v_2 implies that $v_2\setminus\{s_0\}$ can be labeled in the way specified for v_0 in the claim. Let f_1 be such a labeling for $v_2\setminus\{s_0\}$.

$\langle v_0,d_0,v_2,d_0\rangle\hookrightarrow T\mathsf{acyclicity}_1(\star)\Rightarrow\quad \mathsf{Acyclic}(v_2,d_0)$

$\langle v_2,d_0\rangle\hookrightarrow T\mathsf{acyclicity}_4(Stat4\star)\Rightarrow\quad Stat6:\ \langle\exists t\in v_2, s\in v_2\,|\,\emptyset=\{y\in v_2\,|\,[t,y]\in d_0\vee[y,s]\in d_0\}\rangle$

$\langle t_0,s_0\rangle\hookrightarrow Stat6(Stat6\star)\Rightarrow\quad Stat7:\ \emptyset=\{y\in v_2\,|\,[t_0,y]\in d_0\vee[y,s_0]\in d_0\}\ \&\ s_0\in v_2$

$\langle v_2\setminus\{s_0\}\rangle\hookrightarrow Stat3(Stat4\star)\Rightarrow\quad Stat8:\ \langle\exists f\,|\,\mathsf{Svm}(f)\ \&\ \mathbf{dom}(f)=v_2\setminus\{s_0\}\ \&\ \langle\forall x\in v_2\setminus\{s_0\}\,|\,f\upharpoonright x=h(\{f\upharpoonright p^{[2]}:\ p\in d_{0|\{x\}}\,|\,p^{[2]}\in v_2\setminus\{s_0\}\},x)\rangle\rangle$

$\langle f_1\rangle\hookrightarrow Stat8(Stat8\star)\Rightarrow\quad Stat9:\ \langle\forall x\in v_2\setminus\{s_0\}\,|\,f_1\upharpoonright x=h(\{f_1\upharpoonright p^{[2]}:\ p\in d_{0|\{x\}}\,|\,p^{[2]}\in v_2\setminus\{s_0\}\},x)\rangle\ \&\ \mathsf{Svm}(f_1)\ \&\ \mathbf{dom}(f_1)=v_2\setminus\{s_0\}$

We will define a single-valued map f_2 of domain v_2 such that $f_2\upharpoonright x=h(\{f_2\upharpoonright p^{[2]}:\ p\in d_{0|\{x\}}\,|\,p^{[2]}\in v_2\},x)$ holds for all $x\in v_2$, contrary to what Stat4 claims. This contradiction will lead us to the desired conclusion.

$\mathsf{Loc\_def}\Rightarrow\quad Stat10:\ f_2=f_1\cup\{[s_0,h(\{f_1\upharpoonright p^{[2]}:\ p\in d_{0|\{s_0\}}\,|\,p^{[2]}\in v_2\setminus\{s_0\}\},s_0)]\}$

After defining f_2 in the way just shown, we readily check that f_2 is a single-valued map and that its domain is v_2. According to Stat4, f_2 fails to meet the labeling condition.

$\langle\{[s_0,h(\{f_1\upharpoonright p^{[2]}:\ p\in d_{0|\{s_0\}}\,|\,p^{[2]}\in v_2\setminus\{s_0\}\},s_0)]\},s_0,h(\{f_1\upharpoonright p^{[2]}:\ p\in d_{0|\{s_0\}}\,|\,p^{[2]}\in v_2\setminus\{s_0\}\},s_0)\rangle\hookrightarrow T\mathsf{singletonMap}_1(Stat10\star)\Rightarrow$
$\quad\mathsf{Svm}(\{[s_0,h(\{f_1\upharpoonright p^{[2]}:\ p\in d_{0|\{s_0\}}\,|\,p^{[2]}\in v_2\setminus\{s_0\}\},s_0)]\})\ \&\ \mathbf{dom}(\{[s_0,h(\{f_1\upharpoonright p^{[2]}:\ p\in d_{0|\{s_0\}}\,|\,p^{[2]}\in v_2\setminus\{s_0\}\},s_0)]\})=\{s_0\}$

$\langle f_1,\{[s_0,h(\{f_1\upharpoonright p^{[2]}:\ p\in d_{0|\{s_0\}}\,|\,p^{[2]}\in v_2\setminus\{s_0\}\},s_0)]\}\rangle\hookrightarrow T\mathsf{svm}_1(Stat8\star)\Rightarrow\quad \mathsf{Svm}(f_1\cup\{[s_0,h(\{f_1\upharpoonright p^{[2]}:\ p\in d_{0|\{s_0\}}\,|\,p^{[2]}\in v_2\setminus\{s_0\}\},s_0)]\})$

$\langle f_1\cup\{[s_0,h(\{f_1\upharpoonright p^{[2]}:\ p\in d_{0|\{s_0\}}\,|\,p^{[2]}\in v_2\setminus\{s_0\}\},s_0)]\},f_1,\{[s_0,h(\{f_1\upharpoonright p^{[2]}:\ p\in d_{0|\{s_0\}}\,|\,p^{[2]}\in v_2\setminus\{s_0\}\},s_0)]\}\rangle\hookrightarrow T\mathsf{domain}_1(Stat6\star)\Rightarrow$
$\quad\mathbf{dom}(f_1\cup\{[s_0,h(\{f_1\upharpoonright p^{[2]}:\ p\in d_{0|\{s_0\}}\,|\,p^{[2]}\in v_2\setminus\{s_0\}\},s_0)]\})=v_2$

$\mathsf{EQUAL}(Stat10)\Rightarrow\quad \mathsf{Svm}(f_2)\ \&\ \mathbf{dom}(f_2)=v_2$

Where is the map f_2 defective? Let x_2 be a vertex in v_2 where the labeling condition is violated.

$\langle f_2\rangle\hookrightarrow Stat4(Stat10\star)\Rightarrow\quad Stat11:\ \neg\langle\forall x\in v_2\,|\,f_2\upharpoonright x=h(\{f_2\upharpoonright p^{[2]}:\ p\in d_{0|\{x\}}\,|\,p^{[2]}\in v_2\},x)\rangle$

$\langle x_2\rangle\hookrightarrow Stat11(Stat11\star)\Rightarrow\quad x_2\in v_2\ \&\ f_2\upharpoonright x_2\neq h(\{f_2\upharpoonright p^{[2]}:\ p\in d_{0|\{x_2\}}\,|\,p^{[2]}\in v_2\},x_2)$

TELEM ⇒ $\{[s_0, h(\{f_1 \upharpoonright p^{[2]} : p \in d_{0|\{s_0\}} \mid p^{[2]} \in v_2 \setminus \{s_0\}\}, s_0)]\} \cup f_1 = f_1 \cup \{[s_0, h(\{f_1 \upharpoonright p^{[2]} : p \in d_{0|\{s_0\}} \mid p^{[2]} \in v_2 \setminus \{s_0\}\}, s_0)]\}$

Notice that $\{f_1 \upharpoonright p^{[2]} : p \in d_{0|\{x_2\}} \mid p^{[2]} \in v_2 \setminus \{s_0\}\} \neq \{f_2 \upharpoonright p^{[2]} : p \in d_{0|\{x_2\}} \mid p^{[2]} \in v_2\}$. This is proved in two slightly different ways depending on whether x_2 is the selected source of v_2.

Suppose ⇒ $\{f_1 \upharpoonright p^{[2]} : p \in d_{0|\{x_2\}} \mid p^{[2]} \in v_2 \setminus \{s_0\}\} = \{f_2 \upharpoonright p^{[2]} : p \in d_{0|\{x_2\}} \mid p^{[2]} \in v_2\}$

Discard first the case when $x_2 = s_0$:

Suppose ⇒ $x_2 = s_0$

$\langle x_2, f_1, \{[s_0, h(\{f_1 \upharpoonright p^{[2]} : p \in d_{0|\{s_0\}} \mid p^{[2]} \in v_2 \setminus \{s_0\}\}, s_0)]\}\rangle \hookrightarrow Timage_2(Stat8\star) \Rightarrow$

$(f_1 \cup \{[s_0, h(\{f_1 \upharpoonright p^{[2]} : p \in d_{0|\{s_0\}} \mid p^{[2]} \in v_2 \setminus \{s_0\}\}, s_0)]\}) \upharpoonright x_2 = \{[s_0, h(\{f_1 \upharpoonright p^{[2]} : p \in d_{0|\{s_0\}} \mid p^{[2]} \in v_2 \setminus \{s_0\}\}, s_0)]\} \upharpoonright x_2$

$\langle \{[s_0, h(\{f_1 \upharpoonright p^{[2]} : p \in d_{0|\{s_0\}} \mid p^{[2]} \in v_2 \setminus \{s_0\}\}, s_0)]\}, s_0, h(\{f_1 \upharpoonright p^{[2]} : p \in d_{0|\{s_0\}} \mid p^{[2]} \in v_2 \setminus \{s_0\}\}, s_0)\rangle \hookrightarrow TsingletonMap_2(Stat11\star) \Rightarrow$

$\{[s_0, h(\{f_1 \upharpoonright p^{[2]} : p \in d_{0|\{s_0\}} \mid p^{[2]} \in v_2 \setminus \{s_0\}\}, s_0)]\} \upharpoonright s_0 = h(\{f_1 \upharpoonright p^{[2]} : p \in d_{0|\{s_0\}} \mid p^{[2]} \in v_2 \setminus \{s_0\}\}, s_0)$

EQUAL($Stat10$) ⇒ false; Discharge ⇒ AUTO

But then x_2 must be an element of $v_2 \setminus \{s_0\}$, where f_2 behaves exactly like f_1 and hence meets the labeling condition. By reaching a contradiction also in this case, we obtain the sought intermediate conclusion.

$\langle x_2\rangle \hookrightarrow Stat9(Stat11\star) \Rightarrow$ $x_2 \in v_2 \setminus \{s_0\}$ & $f_1 \upharpoonright x_2 = h(\{f_1 \upharpoonright p^{[2]} : p \in d_{0|\{x_2\}} \mid p^{[2]} \in v_2 \setminus \{s_0\}\}, x_2)$

$\langle x_2, \{[s_0, h(\{f_1 \upharpoonright p^{[2]} : p \in d_{0|\{s_0\}} \mid p^{[2]} \in v_2 \setminus \{s_0\}\}, s_0)]\}, f_1\rangle \hookrightarrow Timage_2(Stat8\star) \Rightarrow$ $(\{[s_0, h(\{f_1 \upharpoonright p^{[2]} : p \in d_{0|\{s_0\}} \mid p^{[2]} \in v_2 \setminus \{s_0\}\}, s_0)]\} \cup f_1) \upharpoonright x_2 = f_1 \upharpoonright x_2$

EQUAL($Stat10$) ⇒ false; Discharge ⇒ $Stat15: \{f_1 \upharpoonright p^{[2]} : p \in d_{0|\{x_2\}} \mid p^{[2]} \in v_2 \setminus \{s_0\}\} \neq \{f_2 \upharpoonright p^{[2]} : p \in d_{0|\{x_2\}} \mid p^{[2]} \in v_2\}$

$\langle p_3\rangle \hookrightarrow Stat15(Stat15\star) \Rightarrow$ $Stat16: p_3 \in d_{0|\{x_2\}}$ & $(p_3{}^{[2]} \in v_2 \setminus \{s_0\} \neq p_3{}^{[2]} \in v_2) \vee f_1 \upharpoonright p_3{}^{[2]} \neq f_2 \upharpoonright p_3{}^{[2]}$

$\langle d_0, p_3, x_2\rangle \hookrightarrow Trestr_3(Stat0, Stat16\star) \Rightarrow$ $p_3 = [x_2, p_3{}^{[2]}]$ & $p_3 \in d_0$

Suppose ⇒ $p_3{}^{[2]} = s_0$

EQUAL($Stat16$) ⇒ $[x_2, s_0] \in d_0$

$\langle x_2\rangle \hookrightarrow Stat7(Stat11\star) \Rightarrow$ false; Discharge ⇒ AUTO

$\langle p_3{}^{[2]}, \{[s_0, h(\{f_1 \upharpoonright p^{[2]} : p \in d_{0|\{s_0\}} \mid p^{[2]} \in v_2 \setminus \{s_0\}\}, s_0)]\}, f_1\rangle \hookrightarrow Timage_2(Stat9\star) \Rightarrow$ $(\{[s_0, h(\{f_1 \upharpoonright p^{[2]} : p \in d_{0|\{s_0\}} \mid p^{[2]} \in v_2 \setminus \{s_0\}\}, s_0)]\} \cup f_1) \upharpoonright p_3{}^{[2]} = f_1 \upharpoonright p_3{}^{[2]}$

EQUAL($Stat10$) ⇒ $f_1 \upharpoonright p_3{}^{[2]} = f_2 \upharpoonright p_3{}^{[2]}$

($Stat16\star$)Discharge ⇒ QED

APPLY $\langle v1_\Theta : lab_\Theta\rangle$ Skolem⇒

THM finAcycLabeling$_1$. $Svm(lab_\Theta)$ & $\mathbf{dom}(lab_\Theta) = v_0$ & $\langle \forall x \in v_0 \mid lab_\Theta \upharpoonright x = h(\{lab_\Theta \upharpoonright p^{[2]} : p \in d_{0|\{x\}} \mid p^{[2]} \in v_0\}, x)\rangle$.

ENTER_THEORY Set_theory

DISPLAY finAcycLabeling

THEORY finAcycLabeling$(\mathsf{v}_0, \mathsf{d}_0, \mathsf{h}(\mathsf{s}, \mathsf{x}))$
  $\mathsf{Acyclic}(\mathsf{v}_0, \mathsf{d}_0)$
  $\mathsf{Finite}(\mathsf{v}_0)$
$\Rightarrow (\mathsf{lab}_\ominus)$
  $\mathsf{Svm}(\mathsf{lab}_\ominus)$
  $\mathbf{dom}(\mathsf{lab}_\ominus) = \mathsf{v}_0$
  $\langle \forall \mathsf{x} \in \mathsf{v}_0 \mid \mathsf{lab}_\ominus \upharpoonright \mathsf{x} = \mathsf{h}(\{\mathsf{lab}_\ominus \upharpoonright \mathsf{p}^{[2]} : \mathsf{p} \in \mathsf{d}_{0|\{\mathsf{x}\}} \mid \mathsf{p}^{[2]} \in \mathsf{v}_0\}, \mathsf{x})\rangle$
END finAcycLabeling

2.5 Connectivity

A handy introduction to connectivity is achieved along the path developed below. We will say that a (finite) graph is connected if and only if it has a spanning tree. A spanning tree must have the same vertices as the graph V, E which it spans, and, for edges, a subset of E. In the special case when V is singleton, we choose to represent the tree as a self-loop; in all other cases, a tree shall be devoid of hanks (i. e., it must be cycle-free) and will admit no non-trivial partition of its set of edges into vertex-disjoint blocks.

2.5.1 The unionset operation

DEF unionset: [Family of all members of members of a set] $\bigcup \mathsf{S} =_{\mathrm{Def}} \{\mathsf{u} : \mathsf{v} \in \mathsf{S}, \mathsf{u} \in \mathsf{v}\}$

About the sum-set operation, in this section we will prove the collection of statement displayed here:

THM un_0: [Unionset operation yielding null result] $\bigcup \mathsf{X} = \emptyset \leftrightarrow \mathsf{X} \subseteq \{\emptyset\}$

THM un_1: [Unionset operation yielding sigleton result] $\bigcup \mathsf{X} = \{\mathsf{Y}\} \leftrightarrow \mathsf{X} \subseteq \{\emptyset, \{\mathsf{Y}\}\} \;\&\; \{\mathsf{Y}\} \in \mathsf{X}$

THM un_2: [Unionset operation combined with set adjunction] $\mathsf{X} \cup \{\mathsf{Y}\} = \mathsf{Z} \rightarrow \bigcup \mathsf{Z} = \bigcup \mathsf{X} \cup \mathsf{Y}$

THM un_3: [Unionset operation combined with single removal and adjunction] $\mathsf{B} \in \mathsf{S} \rightarrow \bigcup(\mathsf{S} \setminus \{\mathsf{B}\} \cup \{\mathsf{B} \cup \{\mathsf{A}\}\}) = \bigcup \mathsf{S} \cup \{\mathsf{A}\}$

THM un_4: [Unionset operation applied to a singleton] $\{\mathsf{Y}\} = \mathsf{Z} \rightarrow \bigcup \mathsf{Z} = \mathsf{Y}$

THM un_5: [Unionset operation applied to a singleton, 2] $\mathsf{U} \neq \emptyset \;\&\; \bigcup \mathsf{S} = \mathsf{U} \;\&\; \mathsf{S} \subseteq \{\mathsf{X}\} \rightarrow \mathsf{S} = \{\mathsf{U}\}$

THM un_0: [Unionset operation yielding null result] $\bigcup \mathsf{X} = \emptyset \leftrightarrow \mathsf{X} \subseteq \{\emptyset\}$. PROOF:
Suppose_not$(\mathsf{x}_0) \Rightarrow$ AUTO
  Use_def$(\bigcup \mathsf{x}_0) \Rightarrow$ AUTO
  Suppose $\Rightarrow$ *Stat1* : $\mathsf{x}_0 \nsubseteq \{\emptyset\} \;\&\; \{\mathsf{y} : \mathsf{x} \in \mathsf{x}_0, \mathsf{y} \in \mathsf{x}\} = \emptyset$

$\langle x_1, x_1, \mathbf{arb}(x_1)\rangle \hookrightarrow Stat1(Stat1) \Rightarrow$ false; Discharge $\Rightarrow$ $Stat2:\ \{y:\ x \in x_0, y \in x\} \neq \emptyset \ \&\ x_0 \subseteq \{\emptyset\}$
$\langle x_2, y_2\rangle \hookrightarrow Stat2(Stat2\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

THM un_1: [Unionset operation yielding sigleton result] $\bigcup X = \{Y\} \leftrightarrow X \subseteq \{\emptyset, \{Y\}\} \ \&\ \{Y\} \in X$. PROOF:
Suppose_not$(x_0, y_0) \Rightarrow$ AUTO
Use_def$(\bigcup) \Rightarrow$ $Stat0:\ \bigcup x_0 = \{y:\ x \in x_0, y \in x\}$
Suppose $\Rightarrow$ $Stat1:\ x_0 \not\subseteq \{\emptyset, \{y_0\}\} \ \&\ \bigcup x_0 = \{y_0\}$
$\langle x_1\rangle \hookrightarrow Stat1(Stat1) \Rightarrow$ $Stat2:\ \mathbf{arb}(x_1 \setminus \{y_0\}) \in x_1 \setminus \{y_0\} \ \&\ x_1 \in x_0$
Loc_def $\Rightarrow$ $a = \mathbf{arb}(x_1 \setminus \{y_0\})$
EQUAL$(Stat0) \Rightarrow$ $Stat3:\ a \notin \{y:\ x \in x_0, y \in x\}$
$\langle x_1, a\rangle \hookrightarrow Stat3(Stat2\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO
Suppose $\Rightarrow$ $\{y_0\} \notin x_0 \ \&\ \bigcup x_0 = \{y_0\}$
$\langle x_0\rangle \hookrightarrow Tun_0(Stat0\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO
Suppose $\Rightarrow$ $Stat4:\ y_0 \notin \{y:\ x \in x_0, y \in x\} \ \&\ \{y_0\} \in x_0$
$\langle \{y_0\}, y_0\rangle \hookrightarrow Stat4(Stat4\star) \Rightarrow$ false; Discharge $\Rightarrow$ $Stat5:\ \bigcup x_0 \not\subseteq \{y_0\} \ \&\ x_0 \subseteq \{\emptyset, \{y_0\}\}$
$\langle e\rangle \hookrightarrow Stat5(Stat0\star) \Rightarrow$ $Stat6:\ e \in \{y:\ x \in x_0, y \in x\} \ \&\ e \neq y_0$
$\langle x_2, y_2\rangle \hookrightarrow Stat6(Stat5) \Rightarrow$ $x_2 = \{y_0\} \ \&\ e \in x_2$
$(Stat6\star)$Discharge $\Rightarrow$ QED

THM un_2: [Unionset operation combined with set adjunction] $X \cup \{Y\} = Z \rightarrow \bigcup Z = \bigcup X \cup Y$. PROOF:
Suppose_not$(x_0, y_0, z_0) \Rightarrow$ AUTO
EQUAL $\Rightarrow$ $Stat0:\ \bigcup(x_0 \cup \{y_0\}) \neq \bigcup x_0 \cup y_0$
Use_def$(\bigcup) \Rightarrow$ $Stat1:\ \{y:\ x \in x_0 \cup \{y_0\}, y \in x\} \neq \{y':\ x' \in x_0, y' \in x'\} \cup y_0$
$\langle c\rangle \hookrightarrow Stat1(Stat1\star) \Rightarrow$ $c \in \{y:\ x \in x_0 \cup \{y_0\}, y \in x\} \neq c \in \{y':\ x' \in x_0, y' \in x'\} \cup y_0$
Suppose $\Rightarrow$ $Stat2:\ c \in \{y:\ x \in x_0 \cup \{y_0\}, y \in x\} \ \&\ c \notin \{y':\ x' \in x_0, y' \in x'\} \ \&\ c \notin y_0$
$\langle x_1, y_1, x_1, y_1\rangle \hookrightarrow Stat2(Stat2\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO
Suppose $\Rightarrow$ $Stat3:\ c \notin \{y:\ x \in x_0 \cup \{y_0\}, y \in x\} \ \&\ c \in y_0$
$\langle y_0, c\rangle \hookrightarrow Stat3(Stat3\star) \Rightarrow$ false; Discharge $\Rightarrow$ $Stat4:\ c \in \{y':\ x' \in x_0, y' \in x'\} \ \&\ c \notin \{y:\ x \in x_0 \cup \{y_0\}, y \in x\}$
$\langle x_2, y_2, x_2, y_2\rangle \hookrightarrow Stat4(Stat4\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

THM un_3: [Unionset operation combined with single removal and adjunction] $B \in S \rightarrow \bigcup(S \setminus \{B\} \cup \{B \cup \{A\}\}) = \bigcup S \cup \{A\}$. PROOF:
Suppose_not$(b_0, s_0, a_0) \Rightarrow$ AUTO
$\langle s_0 \setminus \{b_0\}, b_0, s_0\rangle \hookrightarrow Tun_2(\star) \Rightarrow$ $\bigcup s_0 = \bigcup(s_0 \setminus \{b_0\}) \cup b_0$
$\langle s_0 \setminus \{b_0\}, b_0 \cup \{a_0\}, s_0 \setminus \{b_0\} \cup \{b_0 \cup \{a_0\}\}\rangle \hookrightarrow Tun_2(\star) \Rightarrow$ $\bigcup(s_0 \setminus \{b_0\} \cup \{b_0 \cup \{a_0\}\}) = \bigcup(s_0 \setminus \{b_0\}) \cup b_0 \cup \{a_0\}$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

THM un_4: [Unionset operation applied to a singleton] $\{Y\} = Z \rightarrow \bigcup Z = Y$. PROOF:
$\mathsf{Suppose\_not}(y_0, z_0) \Rightarrow$ AUTO
$\langle \emptyset \rangle \hookrightarrow T\mathsf{un}_0(\star) \Rightarrow \bigcup \emptyset = \emptyset$
$\langle \emptyset, y_0, z_0 \rangle \hookrightarrow T\mathsf{un}_2(\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

THM un_5: [Unionset operation applied to a singleton, 2] $U \neq \emptyset \;\&\; \bigcup S = U \;\&\; S \subseteq \{X\} \rightarrow S = \{U\}$. PROOF:
$\mathsf{Suppose\_not}(u_0, s_0, x_0) \Rightarrow$ AUTO
$\langle s_0 \rangle \hookrightarrow T\mathsf{un}_0(\star) \Rightarrow \{x_0\} = s_0$
$\langle x_0, s_0 \rangle \hookrightarrow T\mathsf{un}_4(\star) \Rightarrow u_0 = x_0$
Discharge $\Rightarrow$ QED

2.5.2 Trees

A first characterization of connectivity, which plays a limited role in the ongoing formal development but which we need in order to define trees, is that a set of edges is connected if it can nohow be partitioned into multiple vertex-disjoint blocks. We do not need to adopt officially the definition shown in this comment, because we will build it directly into the definition of tree:
DEF $\mathsf{connectivity}_0$: [A set of edges that cannot be split into multiple vertex-disjoint blocks]
$$\mathsf{PreTree}(E) \leftrightarrow_{\mathrm{Def}} \{b \subseteq E \mid \bigcup b \cap \bigcup (E \setminus b) = \emptyset\} \subseteq \{\emptyset, E\}$$

DEF hank_free: [A hank-free graph is one whose edges do not include a hank] $\mathsf{HankFree}(G) \leftrightarrow_{\mathrm{Def}} \langle \forall e \subseteq G \mid e = \emptyset \vee \langle \exists a \in e \mid a \not\subseteq \bigcup (e \setminus \{a\}) \rangle \rangle$

DEF tree_1: [A tree is a hank-free graph whose edges cannot be partitioned into multiple vertex-disjoint blocks]
$$\mathsf{Is\_tree}(T) \leftrightarrow_{\mathrm{Def}} T \neq \emptyset \;\&\; \{b \subseteq T \mid \bigcup b \cap \bigcup (T \setminus b) = \emptyset\} \subseteq \{\emptyset, T\} \;\&\; \mathsf{HankFree}(X)$$

About trees, in this section we will prove the collection of statement displayed here:

THM tree_0: [A tree cannot be null or have a null edge] $\mathsf{Is\_tree}(T) \rightarrow \emptyset \notin T \cup \{T\}$

THM tree_1: [Non-singleton trees can be pruned] $\mathsf{Is\_tree}(T) \;\&\; T \neq \{\mathbf{arb}(T)\} \;\&\; T \subseteq \{\{x, y\} : p \in T, x \in p, y \in p\} \rightarrow$
$\langle \exists e \in T, u \in e \mid \{a \in T \mid u \notin a\} = T \setminus \{e\} \;\&\; \mathsf{Is\_tree}(T \setminus \{e\}) \rangle$

THM tree_2: [Every singleton other than $\{\emptyset\}$ is a tree] $A \neq \emptyset \leftrightarrow \mathsf{Is\_tree}(\{A\})$

THM tree_3: [Irreducible singleton edges in trees] $\mathsf{Is\_tree}(T) \;\&\; \{X\} \in T \;\&\; \{e \in T \mid X \notin e\} = T \setminus \{\{X\}\} \rightarrow \neg \mathsf{Is\_tree}(T \setminus \{\{X\}\})$

THM tree_4: [In a tree resulting from removal of an edge from a tree, only one vertex gets lost]
$\mathsf{Is\_tree}(T) \;\&\; \{X, Y\} = A \;\&\; A \in T \;\&\; \mathsf{Is\_tree}(T \setminus \{A\}) \;\&\; \{e \in T \mid X \notin e\} = T \setminus \{A\} \rightarrow \bigcup (T \setminus \{A\}) = \bigcup T \setminus \{X\}$

THM tree_0: [A tree cannot be null or have a null edge] $\mathsf{Is\_tree}(T) \rightarrow \emptyset \notin T \cup \{T\}$. PROOF:
$\mathsf{Suppose\_not}(t_0) \Rightarrow$ AUTO

Use_def(HankFree(t_0)) ⇒ AUTO
Use_def(Is_tree) ⇒ *Stat0* : $\emptyset \in t_0$ & *Stat1* : $\langle \forall e \subseteq t_0 \mid e = \emptyset \vee \langle \exists a \in e \mid a \not\subseteq \bigcup(e \setminus \{a\})\rangle\rangle$
$\langle \{\emptyset\} \rangle \hookrightarrow$ *Stat1(Stat0⋆)* ⇒ *Stat2* : $\langle \exists a \in \{\emptyset\} \mid a \not\subseteq \bigcup(\{\emptyset\} \setminus \{a\})\rangle$
$\langle x_0 \rangle \hookrightarrow$ *Stat2(Stat0⋆)* ⇒ false; Discharge ⇒ QED

THM tree_1: [Non-singleton trees can be pruned] $\text{Is\_tree}(T)$ & $T \neq \{\mathbf{arb}(T)\}$ & $T \subseteq \{\{x,y\} : p \in T, x \in p, y \in p\} \rightarrow$
$\langle \exists e \in T, u \in e \mid \{a \in T \mid u \notin a\} = T \setminus \{e\}$ & $\text{Is\_tree}(T \setminus \{e\})\rangle$. PROOF:
Suppose_not(t_0) ⇒ AUTO

> Supposing that the contrary holds for t_0, consider an edge a_0 of t_0 one of whose endpoints, u_0, is not a vertex of $t_0 \setminus \{a_0\}$. Consequently, either $t_0 \setminus \{a_0\}$ differs from the set of edges not incident to u_0 or $t_0 \setminus \{a_0\}$ is not a tree.

Loc_def ⇒ $a' = \mathbf{arb}(t_0)$
Use_def(HankFree(t_0)) ⇒ AUTO
Use_def(Is_tree) ⇒ *Stat0* : $\langle \forall e \subseteq t_0 \mid e = \emptyset \vee \langle \exists a \in e \mid a \not\subseteq \bigcup(e \setminus \{a\})\rangle\rangle$ & *Stat0a* : $\{b \subseteq t_0 \mid \bigcup b \cap \bigcup(t_0 \setminus b) = \emptyset\} \subseteq \{\emptyset, t_0\}$
$\langle t_0 \rangle \hookrightarrow T\text{tree}_0$ ⇒ *Stat1* : $t_0 \supseteq \{a'\}$ & $t_0 \neq \{a'\}$
$\langle t_0 \rangle \hookrightarrow$ *Stat0(Stat0⋆)* ⇒ *Stat2* : $\langle \exists a \in t_0 \mid a \not\subseteq \bigcup(t_0 \setminus \{a\})\rangle$
$\langle a_0 \rangle \hookrightarrow$ *Stat2(⋆)* ⇒ *Stat3* : $a_0 \not\subseteq \bigcup(t_0 \setminus \{a_0\})$ & $\neg\langle \exists e \in t_0, u \in e \mid \{a \in t_0 \mid u \notin a\} = t_0 \setminus \{e\}$ & $\text{Is\_tree}(t_0 \setminus \{e\})\rangle$ & $a_0 \in t_0$ & $t_0 \subseteq \{\{x,y\} : p \in t_0, x \in p, y \in p\}$
$\langle u_0, a_0, u_0 \rangle \hookrightarrow$ *Stat3(Stat3⋆)* ⇒ *Stat4* : $u_0 \notin \bigcup(t_0 \setminus \{a_0\})$ & $\neg(\{a \in t_0 \mid u_0 \notin a\} = t_0 \setminus \{a_0\}$ & $\text{Is\_tree}(t_0 \setminus \{a_0\}))$ & $u_0 \in a_0$

> The first condition of the alternative is readily discharged; hence $t_0 \setminus \{a_0\}$ is not a tree and it must violate at least one of the two conditions for being a tree.

Suppose ⇒ *Stat7* : $\{a \in t_0 \mid u_0 \notin a\} \neq t_0 \setminus \{a_0\}$
  $\langle a_1 \rangle \hookrightarrow$ *Stat7* ⇒ AUTO
  Suppose ⇒ *Stat8* : $a_1 \notin \{a \in t_0 \mid u_0 \notin a\}$
    $\langle t_0 \setminus \{a_0\}, a_1, t_0 \setminus \{a_0\} \rangle \hookrightarrow T\text{un}_2$ *(Stat4⋆)* ⇒ $a_1 \in t_0$ & $u_0 \notin a_1$
    $\langle a_1 \rangle \hookrightarrow$ *Stat8(Stat8⋆)* ⇒ false
  Discharge ⇒ *Stat9* : $a_1 \in \{a \in t_0 \mid u_0 \notin a\}$ & $a_1 \notin t_0 \setminus \{a_0\}$
  $\langle\, \rangle \hookrightarrow$ *Stat9(Stat9, Stat4⋆)* ⇒ false
Discharge ⇒ $\{a \in t_0 \mid u_0 \notin a\} = t_0 \setminus \{a_0\}$

> One easily sees that the violated condition cannot be cycle-freeness; hence it must be the other condition. Consider, accordingly, a partition p_0 of $t_0 \setminus \{a_0\}$ into two equivalence classes, b_0 and $t_0 \setminus \{a_0\} \setminus b_0$, devoid of common vertices. Plainly, $\{a_0\}$ does not belong to p_0.

Set_monot ⇒ $\langle \forall e \subseteq t_0 \mid e = \emptyset \vee \langle \exists a \in e \mid a \not\subseteq \bigcup(e \setminus \{a\})\rangle\rangle \rightarrow \langle \forall e \subseteq t_0 \setminus \{a_0\} \mid e = \emptyset \vee \langle \exists a \in e \mid a \not\subseteq \bigcup(e \setminus \{a\})\rangle\rangle$
Use_def(HankFree($t_0 \setminus \{a_0\}$)) ⇒ AUTO

Use_def(Is_tree) $\Rightarrow$ $\quad Stat10: \{b \subseteq t_0 \setminus \{a_0\} \mid \bigcup b \cap \bigcup (t_0 \setminus \{a_0\} \setminus b) = \emptyset\} \not\subseteq \{\emptyset, t_0 \setminus \{a_0\}\}$
$\langle b_0 \rangle \hookrightarrow Stat10(Stat10, Stat1\star) \Rightarrow \quad Stat12: b_0 \in \{b \subseteq t_0 \setminus \{a_0\} \mid \bigcup b \cap \bigcup (t_0 \setminus \{a_0\} \setminus b) = \emptyset\}$ & $b_0 \neq \emptyset$ & $b_0 \neq t_0 \setminus \{a_0\}$
$\langle\, \rangle \hookrightarrow Stat12(Stat12\star) \Rightarrow \quad Stat13: b_0 \subseteq t_0 \setminus \{a_0\}$ & $\bigcup b_0 \cap \bigcup (t_0 \setminus \{a_0\} \setminus b_0) = \emptyset$
Use_def(Is_tree) $\Rightarrow$ $\quad \{b \subseteq t_0 \mid \bigcup b \cap \bigcup (t_0 \setminus b) = \emptyset\} \subseteq \{\emptyset, t_0\}$

We discard first the possibility that the block $t_0 \setminus \{a_0\} \setminus b_0$ of the partition p_0 has $\bigcup (t_0 \setminus \{a_0\} \setminus b_0)$ intersecting a_0, by showing that if this were the case, then p_0 could be extended into a multiple partition p_1 of t_0 in which no two blocks share vertices. Such a p_1 is simply obtained by adjoining a_0 to the block b_0 of p_0.

Suppose $\Rightarrow$ $\quad a_0 \cap \bigcup (t_0 \setminus \{a_0\} \setminus b_0) = \emptyset$
  Loc_def $\Rightarrow$ $\quad b_1 = b_0 \cup \{a_0\}$
  $\langle b_0, a_0, b_1 \rangle \hookrightarrow Tun_2\ (Stat12\star) \Rightarrow \quad Stat14: b_1 \notin \{b \subseteq t_0 \mid \bigcup b \cap \bigcup (t_0 \setminus b) = \emptyset\}$ & $\bigcup b_1 = \bigcup b_0 \cup a_0$
  $\langle b_1 \rangle \hookrightarrow Stat14(Stat3\star) \Rightarrow \quad (\bigcup b_0 \cup a_0) \cap \bigcup (t_0 \setminus b_1) \neq \emptyset$ & $t_0 \setminus \{a_0\} \setminus b_0 = t_0 \setminus b_1$
  EQUAL $\Rightarrow$ $\quad \bigcup b_0 \cap \bigcup (t_0 \setminus b_1) = \emptyset$ & $a_0 \cap \bigcup (t_0 \setminus b_1) = \emptyset$
$(Stat14\star)$Discharge $\Rightarrow$ $\quad$ Auto

In sight of showing that $a_0 \cap \bigcup b_0 = \emptyset$, we observe that a_0 can neither consist of u_0 alone nor have more than two elements; hence let w_0 be its endpoint distinct from u_0. This must be the only member of $a_0 \cap \bigcup (t_0 \setminus \{a_0\} \setminus b_0)$.

Set_monot $\Rightarrow$ $\quad \{v : u \in t_0 \setminus \{a_0\} \setminus b_0, v \in u\} \subseteq \{v : u \in t_0 \setminus \{a_0\}, v \in u\}$
Use_def($\bigcup$) $\Rightarrow$ $\quad u_0 \notin \bigcup (t_0 \setminus \{a_0\} \setminus b_0)$
$(Stat3\star)$ELEM $\Rightarrow$ $\quad Stat15: a_0 \in \{\{x, y\} : q \in t_0, x \in q, y \in q\}$ & $a_0 \not\subseteq \{u_0\}$
$\langle q_0, x_0, y_0, w_0 \rangle \hookrightarrow Stat15(Stat15, Stat4\star) \Rightarrow \quad \{u_0, w_0\} = a_0$ & $u_0 \neq w_0$
$(Stat12\star)$ELEM $\Rightarrow$ $\quad a_0 \cap \bigcup (t_0 \setminus \{a_0\} \setminus b_0) = \{w_0\}$

Now suppose that $\bigcup b_0$ also intersects a_0. If this were the case then $\bigcup b_0$ and $\bigcup (t_0 \setminus \{a_0\} \setminus b_0)$ would intersect; indeed, neither $\bigcup b_0$ nor $\bigcup (t_0 \setminus \{a_0\} \setminus b_0)$ can have u_0 for element, hence these two unions must share w_0. However, we know that in p_0 no two blocks share vertices.

Suppose $\Rightarrow$ $\quad a_0 \cap \bigcup b_0 \neq \emptyset$
  Set_monot $\Rightarrow$ $\quad \{v : u \in b_0, v \in u\} \subseteq \{v : u \in t_0 \setminus \{a_0\}, v \in u\}$
  Use_def($\bigcup$) $\Rightarrow$ $\quad u_0 \notin \bigcup b_0$
  $(Stat12\star)$ELEM $\Rightarrow$ $\quad$ false
Discharge $\Rightarrow$ $\quad Stat16: a_0 \cap \bigcup b_0 = \emptyset$

Now extend p_0 by adjoining the edge a_0 to its block $t_0 \setminus \{a_0\} \setminus b_0$. This extension will result into a multiple partition p_2 of t_0 in which no two blocks share vertices.

Loc_def $\Rightarrow$ $b_2 = t_0 \setminus \{a_0\} \setminus b_0 \cup \{a_0\}$
$\langle t_0 \setminus \{a_0\} \setminus b_0, a_0, b_2 \rangle \hookrightarrow Tun_2$ $(Stat12\star) \Rightarrow$ $\bigcup b_2 = a_0 \cup \bigcup(t_0 \setminus \{a_0\} \setminus b_0)$
$(Stat12\star)$ELEM $\Rightarrow$ $Stat17: \ b_2 \notin \{b \subseteq t_0 \mid \bigcup b \cap \bigcup(t_0 \setminus b) = \emptyset\}$
$\langle b_2 \rangle \hookrightarrow Stat17(Stat3\star) \Rightarrow$ $(a_0 \cup \bigcup(t_0 \setminus \{a_0\} \setminus b_0)) \cap \bigcup(t_0 \setminus b_2) \neq \emptyset \ \& \ b_0 = t_0 \setminus b_2$
EQUAL $\langle Stat17 \rangle \Rightarrow$ $Stat18: \ a_0 \cap \bigcup b_0 \cup \bigcup(t_0 \setminus \{a_0\} \setminus b_0) \cap \bigcup b_0 \neq \emptyset$
$(Stat13, Stat16, Stat18\star)$Discharge $\Rightarrow$ QED

THM tree$_2$: [Every singleton other than $\{\emptyset\}$ is a tree] $A \neq \emptyset \leftrightarrow \text{Is\_tree}(\{A\})$. PROOF:
Suppose_not$(a_0) \Rightarrow$ AUTO

> Recall that $\{\emptyset\}$ is not a tree and observe that no singleton a_0 cannot be partitioned into more than one equivalence class.

$\langle \{a_0\} \rangle \hookrightarrow T\text{tree}_0 \Rightarrow$ AUTO
Suppose $\Rightarrow$ $Stat1: \ \{b \subseteq \{a_0\} \mid \bigcup b \cap \bigcup(\{a_0\} \setminus b) = \emptyset\} \not\subseteq \{\emptyset, \{a_0\}\}$
$\langle b_0 \rangle \hookrightarrow Stat1(Stat1\star) \Rightarrow$ $Stat2: \ b_0 \in \{b \subseteq \{a_0\} \mid \bigcup b \cap \bigcup(\{a_0\} \setminus b) = \emptyset\} \ \& \ b_0 \neq \emptyset \ \& \ b_0 \neq \{a_0\}$
$\langle b_1 \rangle \hookrightarrow Stat2(Stat2\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO

> Consequently, the only reason why a singleton $a_0 \neq \{\emptyset\}$ might fail to be a tree could be its lack of cycle-freeness. But this is also untenable.

Use_def$(\text{HankFree}(\{a_0\})) \Rightarrow$ AUTO
Use_def(Is_tree) $\Rightarrow$ $Stat3: \ \neg\langle \forall e \subseteq \{a_0\} \mid e = \emptyset \vee \langle \exists a \in e \mid a \not\subseteq \bigcup(e \setminus \{a\}) \rangle \rangle \ \& \ a_0 \neq \emptyset$
$\langle e_1 \rangle \hookrightarrow Stat3(Stat3\star) \Rightarrow$ $Stat4: \ \neg\langle \exists a \in e_1 \mid a \not\subseteq \bigcup(e_1 \setminus \{a\}) \rangle \ \& \ e_1 = \{a_0\}$
$\langle a_0 \rangle \hookrightarrow Stat4(Stat4\star) \Rightarrow$ $a_0 \subseteq \bigcup(e_1 \setminus \{a_0\})$
$\langle e_1 \setminus \{a_0\} \rangle \hookrightarrow Tun_0$ $(Stat3\star) \Rightarrow$ $a_0 = \emptyset$
$(Stat3\star)$Discharge $\Rightarrow$ QED

THM tree$_3$: [Irreducible singleton edges in trees] $\text{Is\_tree}(T) \ \& \ \{X\} \in T \ \& \ \{e \in T \mid X \notin e\} = T \setminus \{\{X\}\} \to \neg\text{Is\_tree}(T \setminus \{\{X\}\})$. PROOF:
Suppose_not$(t_0, x_0) \Rightarrow$ $Stat0: \ \{e \in t_0 \mid x_0 \notin e\} = t_0 \setminus \{\{x_0\}\} \ \& \ \text{Is\_tree}(t_0) \ \& \ \{x_0\} \in t_0 \ \& \ \text{Is\_tree}(t_0 \setminus \{\{x_0\}\})$
Use_def(Is_tree) $\Rightarrow$ $\{b \subseteq t_0 \mid \bigcup b \cap \bigcup(t_0 \setminus b) = \emptyset\} \subseteq \{\emptyset, t_0\}$
$\langle t_0 \setminus \{\{x_0\}\} \rangle \hookrightarrow T\text{tree}_0$ $(\star) \Rightarrow$ $Stat1: \ t_0 \setminus \{\{x_0\}\} \notin \{b \subseteq t_0 \mid \bigcup b \cap \bigcup(t_0 \setminus b) = \emptyset\} \ \& \ t_0 \setminus (t_0 \setminus \{\{x_0\}\}) = \{\{x_0\}\}$
$\langle t_0 \setminus \{\{x_0\}\} \rangle \hookrightarrow Stat1(Stat1\star) \Rightarrow$ $\bigcup(t_0 \setminus \{\{x_0\}\}) \cap \bigcup(t_0 \setminus (t_0 \setminus \{\{x_0\}\})) \neq \emptyset$
$\langle \{x_0\}, \{\{x_0\}\} \rangle \hookrightarrow Tun_4$ $(Stat2\star) \Rightarrow$ $Stat2: \ \bigcup\{\{x_0\}\} = \{x_0\}$
Use_def$(\bigcup(t_0 \setminus \{\{x_0\}\})) \Rightarrow$ AUTO
EQUAL $\langle Stat1 \rangle \Rightarrow$ $Stat3: \ x_0 \in \{v : \ u \in t_0 \setminus \{\{x_0\}\}, v \in u\}$
$\langle u_0, v_0 \rangle \hookrightarrow Stat3(Stat3, Stat0\star) \Rightarrow$ $Stat4: \ u_0 \in \{e \in t_0 \mid x_0 \notin e\} \ \& \ x_0 \in u_0$
$\langle \, \rangle \hookrightarrow Stat4(Stat4\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

THM tree_4: [In a tree resulting from removal of an edge from a tree, only one vertex gets lost]

$\text{Is\_tree}(T) \,\&\, \{X, Y\} = A \,\&\, A \in T \,\&\, \text{Is\_tree}(T \setminus \{A\}) \,\&\, \{e \in T \,|\, X \notin e\} = T \setminus \{A\} \rightarrow \bigcup(T \setminus \{A\}) = \bigcup T \setminus \{X\}$. PROOF:

$\text{Suppose\_not}(t_0, u_0, w_0, a_0) \Rightarrow$ AUTO

Contrary to the claim, suppose that $\bigcup(t_0 \setminus \{a_0\}) \neq \bigcup t_0 \setminus \{u_0\} \,\&\, \text{Is\_tree}(t_0) \,\&\, \{u_0, w_0\} = a_0 \,\&\, a_0 \in t_0 \,\&\, \text{Is\_tree}(t_0 \setminus \{a_0\}) \,\&\, \{e \in t_0 \,|\, u_0 \notin e\} = t_0 \setminus \{a_0\}$ holds for t_0, u_0, w_0, a_0. Then we must have $u_0 \neq w_0$ as the only element differentiating $\bigcup(t_0 \setminus \{a_0\})$ from $\bigcup t_0 \setminus \{u_0\}$.

Suppose $\Rightarrow$ $\mathit{Stat0}: \bigcup t_0 \setminus \{u_0\} \neq \bigcup(t_0 \setminus \{a_0\}) \cup \{w_0\}$

Suppose $\Rightarrow$ $\bigcup(t_0 \setminus \{a_0\}) \not\subseteq \bigcup t_0 \setminus \{u_0\}$

$\text{Use\_def}(\bigcup) \Rightarrow$ $\mathit{Stat1}: \{x : y \in t_0 \setminus \{a_0\}, x \in y\} \not\subseteq \{x : y \in t_0, x \in y\} \setminus \{u_0\}$

$\text{Set\_monot}\ \langle \mathit{Stat1} \rangle \Rightarrow$ $\{x : y \in t_0 \setminus \{a_0\}, x \in y\} \subseteq \{x : y \in t_0, x \in y\}$

$(\mathit{Stat1}\star)\text{ELEM} \Rightarrow$ $\mathit{Stat2}: u_0 \in \{x : y \in t_0 \setminus \{a_0\}, x \in y\}$

$\langle y_1, x_1 \rangle \hookrightarrow \mathit{Stat2}(\star) \Rightarrow$ $\mathit{Stat3}: y_1 \in \{e \in t_0 \,|\, u_0 \notin e\} \,\&\, u_0 \in y_1$

$\langle\,\rangle \hookrightarrow \mathit{Stat3}(\mathit{Stat3}\star) \Rightarrow$ false

Discharge $\Rightarrow$ AUTO

$\langle t_0, a_0, t_0 \rangle \hookrightarrow \mathit{Tun}_2 \Rightarrow$ AUTO

Suppose $\Rightarrow$ $a_0 = \{u_0\}$

EQUAL $\Rightarrow$ $\mathit{Stat4}: \text{Is\_tree}(t_0) \,\&\, \{u_0\} \in t_0 \,\&\, \{e \in t_0 \,|\, u_0 \notin e\} = t_0 \setminus \{\{u_0\}\} \,\&\, \text{Is\_tree}(t_0 \setminus \{\{u_0\}\})$

$\langle t_0, u_0 \rangle \hookrightarrow \mathit{T}\text{tree}_3\ (\mathit{Stat4}\star) \Rightarrow$ false

Discharge $\Rightarrow$ $u_0 \neq w_0$

$\langle u_1 \rangle \hookrightarrow \mathit{Stat0}(\star) \Rightarrow$ $u_1 \in \bigcup t_0 \setminus \{u_0\} \,\&\, u_1 \notin \bigcup(t_0 \setminus \{a_0\}) \cup \{w_0\}$

$\text{Use\_def}(\bigcup) \Rightarrow$ $\mathit{Stat5}: u_1 \in \{x : y \in t_0, x \in y\} \,\&\, u_1 \notin \{x : y \in t_0 \setminus \{a_0\}, x \in y\} \,\&\, u_1 \neq u_0 \,\&\, u_1 \neq w_0$

$\langle y_2, x_2, y_2, x_2 \rangle \hookrightarrow \mathit{Stat5}(\mathit{Stat5}\star) \Rightarrow$ $u_1 \in a_0 \,\&\, u_1 \neq u_0 \,\&\, u_1 \neq w_0$

Discharge $\Rightarrow$ AUTO

Consider now the partition $\{t_0 \setminus \{a_0\}, \{a_0\}\}$ of $\bigcup t_0$. The two blocks of the said partition must share a vertex, because t_0 is a tree. The shared vertex must be either u_0 or w_0, because u_0, w_0 are the only vertices in a_0; however, as we already know, neither of u_0, w_0 belongs to $\bigcup(t_0 \setminus \{a_0\})$ (see the statement Stat0 discharged above and bear in mind that $\bigcup(t_0 \setminus \{a_0\}) \neq \bigcup t_0 \setminus \{u_0\}$). This contradiction completes our argument and proves the claim.

$\text{Use\_def}(\text{Is\_tree}) \Rightarrow$ $\{b \subseteq t_0 \,|\, \bigcup b \cap \bigcup(t_0 \setminus b) = \emptyset\} \subseteq \{\emptyset, t_0\}$

$\langle t_0 \setminus \{a_0\} \rangle \hookrightarrow \mathit{T}\text{tree}_0\ (\star) \Rightarrow$ $\mathit{Stat7}: t_0 \setminus \{a_0\} \notin \{b \subseteq t_0 \,|\, \bigcup b \cap \bigcup(t_0 \setminus b) = \emptyset\} \,\&\, t_0 \setminus (t_0 \setminus \{a_0\}) = \{a_0\}$

$\langle t_0 \setminus \{a_0\} \rangle \hookrightarrow \mathit{Stat7}(\mathit{Stat7}\star) \Rightarrow$ $\bigcup(t_0 \setminus \{a_0\}) \cap \bigcup(t_0 \setminus (t_0 \setminus \{a_0\})) \neq \emptyset$

$\langle a_0, \{a_0\} \rangle \hookrightarrow \mathit{Tun}_4\ (\mathit{Stat8}\star) \Rightarrow$ $\mathit{Stat8}: \bigcup \{a_0\} = a_0$

EQUAL $\Rightarrow$ $\mathit{Stat9}: \bigcup(t_0 \setminus \{a_0\}) \cap \bigcup \{a_0\} \neq \emptyset$

Discharge $\Rightarrow$ QED

2.5.3 Spanning trees and cut vertices

Next we bring into play an alternative notion of connectivity, based on a spanning tree, more directly exploitable than the connectivity notion which helped us in characterizing trees in what precedes.

DEF connectivity$_1$: [A graph endowed with a spanning tree] $\mathsf{HasSpanningTree}(V,E) \leftrightarrow_{\text{Def}} \langle \exists t \mid \mathsf{Is\_tree}(t) \ \& \ \bigcup t = V \ \& \ (V = \{\mathbf{arb}(V)\} \lor t \subseteq E)\rangle$

About graphs which are endowed with spanning trees, in this section we will prove the collection of statement displayed here:

THM connectivity$_0$: [Non-triviality of connectivity] $\mathsf{HasSpanningTree}(V,E) \rightarrow V \neq \emptyset$

THM connectivity$_1$: [No vertex is isolated in a graph endowed with a spanning tree]
$\mathsf{HasSpanningTree}(V,E) \ \& \ E \subseteq \{\{x,y\} : x \in Z, y \in Z \setminus \{x\}\} \ \& \ U \in V \ \& \ V \setminus \{U\} \neq \emptyset \rightarrow \langle \exists w \in V \setminus \{U\} \mid \{U,w\} \in E\rangle$

THM connectivity$_2$: [Every graph endowed with a spanning tree has a non-cut vertex]
$\mathsf{HasSpanningTree}(V,E) \ \& \ E \subseteq \{\{x,y\} : x \in V, y \in V \setminus \{x\}\} \ \& \ V \neq \{\mathbf{arb}(V)\} \rightarrow \langle \exists u \in V \mid \mathsf{HasSpanningTree}(V \setminus \{u\}, \{a \in E \mid u \notin a\})\rangle$

THM connectivity$_0$: [Non-triviality of connectivity] $\mathsf{HasSpanningTree}(V,E) \rightarrow V \neq \emptyset$. PROOF:

Suppose_not$(v_0, e_0) \Rightarrow$ AUTO

Use_def(HasSpanningTree) $\Rightarrow$ *Stat1*: $\langle \exists t \mid \mathsf{Is\_tree}(t) \ \& \ \bigcup t = v_0 \ \& \ v_0 = \{\mathbf{arb}(v_0)\} \lor t \subseteq e_0\rangle$

$\langle t_0\rangle \hookrightarrow Stat1(\star) \Rightarrow$ $\mathsf{Is\_tree}(t_0) \ \& \ \bigcup t_0 = \emptyset$

$\langle t_0\rangle \hookrightarrow T\mathsf{un}_0(Stat1\star) \Rightarrow$ $t_0 \subseteq \{\emptyset\}$

$\langle t_0\rangle \hookrightarrow T\mathsf{tree}_0(Stat1\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

THM connectivity$_1$: [No vertex is isolated in a graph endowed with a spanning tree]
$\mathsf{HasSpanningTree}(V,E) \ \& \ E \subseteq \{\{x,y\} : x \in Z, y \in Z \setminus \{x\}\} \ \& \ U \in V \ \& \ V \setminus \{U\} \neq \emptyset \rightarrow \langle \exists w \in V \setminus \{U\} \mid \{U,w\} \in E\rangle$. PROOF:

Suppose_not$(v_0, e_0, v_1, u_0) \Rightarrow$ AUTO

Suppose that v_0, e_0 is a graph endowed with at least two vertices u_0, w_0 and also endowed with a spanning tree t_0.

Use_def(HasSpanningTree) $\Rightarrow$ *Stat1*: $\langle \exists t \mid \mathsf{Is\_tree}(t) \ \& \ \bigcup t = v_0 \ \& \ v_0 = \{\mathbf{arb}(v_0)\} \lor t \subseteq e_0\rangle$

Use_def$(\bigcup t_0) \Rightarrow$ AUTO

$\langle t_0\rangle \hookrightarrow Stat1(\star) \Rightarrow$ *Stat2*: $u_0 \in \{y : x \in t_0, y \in x\} \ \& \ v_0 = \{y : x \in t_0, y \in x\} \ \& \ t_0 \subseteq e_0 \ \&$ *Stat2a*: $\neg\langle \exists w \in v_0 \setminus \{u_0\} \mid \{u_0, w\} \in e_0\rangle$

$\langle a_0, y_1\rangle \hookrightarrow Stat2(\star) \Rightarrow$ $Stat3:\ a_0 \in \{\{x,y\}:\ x \in v_1, y \in v_1 \setminus \{x\}\}\ \&\ a_0 \in t_0\ \&\ u_0 \in a_0$
$\langle x_2, y_2\rangle \hookrightarrow Stat3(Stat3\star) \Rightarrow$ $a_0 = \{x_2, y_2\}\ \&\ x_2 \neq y_2$
Suppose $\Rightarrow$ $Stat4:\ x_2 \notin \{y:\ x \in t_0, y \in x\} \vee y_2 \notin \{y:\ x \in t_0, y \in x\}$
$\langle a_0, x_2, a_0, y_2\rangle \hookrightarrow Stat4(Stat3\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO
Suppose $\Rightarrow$ $u_0 = x_2$
$\langle y_2\rangle \hookrightarrow Stat2a(Stat2\star) \Rightarrow$ false; Discharge $\Rightarrow$ $u_0 = y_2$
$\langle x_2\rangle \hookrightarrow Stat2a(Stat2\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

Any vertex whose degree is less than 2 in the spanning tree of a graph will be a non-cut vertex: namely, removing such a vertex along with the edges incident to it does not disrupt connectivity. Such a vertex must exist, as proved above, in view of the cycle-freeness of trees.

THM connectivity$_2$: [Every graph endowed with a spanning tree has a non-cut vertex]
$\text{HasSpanningTree}(V, E)\ \&\ E \subseteq \{\{x,y\}:\ x \in V, y \in V \setminus \{x\}\}\ \&\ V \neq \{\mathbf{arb}(V)\} \rightarrow \langle \exists u \in V \mid \text{HasSpanningTree}(V \setminus \{u\}, \{a \in E \mid u \notin a\})\rangle$. PROOF:
Suppose_not$(v_0, e_0) \Rightarrow$ AUTO

Suppose that v_0, e_0 is a graph endowed with more than one vertex, also endowed with a spanning tree t_0, and such that no vertex can be removed from it without disrupting its connectivity.

Use_def(HasSpanningTree) $\Rightarrow$ $Stat1:\ \langle \exists t \mid \text{Is\_tree}(t)\ \&\ \bigcup t = v_0\ \&\ v_0 = \{\mathbf{arb}(v_0)\} \vee t \subseteq e_0\rangle\ \&$
$Stat1a:\ \neg\langle \exists u \in v_0 \mid \text{HasSpanningTree}(v_0 \setminus \{u\}, \{a \in e_0 \mid u \notin a\})\rangle\ \&\ v_0 \neq \{\mathbf{arb}(\mathbf{arb}(t_0))\}$
$\langle t_0\rangle \hookrightarrow Stat1(\star) \Rightarrow$ $\text{Is\_tree}(t_0)\ \&\ \bigcup t_0 = v_0\ \&\ t_0 \subseteq e_0\ \&\ t_0 \subseteq \{\{x,y\}:\ x \in v_0, y \in v_0 \setminus \{x\}\}$
Suppose $\Rightarrow$ $Stat2:\ t_0 \not\subseteq \{\{x,y\}:\ p \in t_0, x \in p, y \in p\}$
$\langle p_0\rangle \hookrightarrow Stat2(Stat1\star) \Rightarrow$ $Stat3:\ p_0 \in \{\{x,y\}:\ x \in v_0, y \in v_0 \setminus \{x\}\}\ \&\ p_0 \notin \{\{x,y\}:\ p \in t_0, x \in p, y \in p\}\ \&\ p_0 \in t_0$
$\langle x_0, y_0, p_0, x_0, y_0\rangle \hookrightarrow Stat3(Stat3\star) \Rightarrow$ false
Discharge $\Rightarrow$ AUTO

Suppose first that t_0 consists of a sole edge. After observing that this edge must have two endpoints x_1, y_1, we see that removal of one endpoint — say y_1 for specificity — would lead to a spanning tree for the original graph deprived of the vertex in question. Since the reduced graph would retain connectivity, we must discard this case.

Suppose $\Rightarrow$ $Stat4:\ t_0 = \{\mathbf{arb}(t_0)\}$
$(Stat4)$ELEM $\Rightarrow$ $\mathbf{arb}(t_0) \in t_0$
$\langle \mathbf{arb}(t_0), t_0\rangle \hookrightarrow Tun_4(\star) \Rightarrow$ $Stat5:\ v_0 \in \{\{x,y\}:\ x \in v_0, y \in v_0 \setminus \{x\}\}\ \&\ v_0 \neq \{\mathbf{arb}(v_0)\}$
$\langle x_1, y_1\rangle \hookrightarrow Stat5(Stat5) \Rightarrow$ $Stat6:\ v_0 = \{x_1, y_1\}\ \&\ x_1 \neq y_1$
$\langle y_1\rangle \hookrightarrow Stat1a(Stat5\star) \Rightarrow$ $\neg\text{HasSpanningTree}(v_0 \setminus \{y_1\}, \{a \in e_0 \mid y_1 \notin a\})\ \&\ v_0 \setminus \{y_1\} = \{x_1\}$
Suppose $\Rightarrow$ $Stat7:\ \{a \in e_0 \mid y_1 \notin a\} \neq \emptyset$
$\langle a_1\rangle \hookrightarrow Stat7(\star) \Rightarrow$ $Stat8:\ a_1 \in \{\{x,y\}:\ x \in v_0, y \in v_0 \setminus \{x\}\}\ \&\ y_1 \notin a_1$
$\langle x_2, y_2\rangle \hookrightarrow Stat8(Stat8\star) \Rightarrow$ $Stat9:\ x_2, y_2 \in v_0\ \&\ x_2 \neq y_2\ \&\ y_1 \notin \{x_2, y_2\}$

$(\mathit{Stat6}, \mathit{Stat9}\star)\mathsf{Discharge} \Rightarrow \quad \{a \in e_0 \mid y_1 \notin a\} = \emptyset$

$\mathsf{Use\_def}\big(\mathsf{HasSpanningTree}(\{x_1\}, \emptyset)\big) \Rightarrow \quad \textsc{Auto}$

$\mathsf{EQUAL}(\mathit{Stat5}) \Rightarrow \quad \mathit{Stat10}: \neg\langle \exists t \mid \mathsf{Is\_tree}(t) \,\&\, \bigcup t = \{x_1\} \,\&\, \{x_1\} = \{\mathbf{arb}(\{x_1\})\} \vee t \subseteq \emptyset \rangle$

$\big\langle \{\{x_1\}\} \big\rangle \hookrightarrow \mathit{Stat10}(\mathit{Stat10}) \Rightarrow \quad \neg\big(\mathsf{Is\_tree}(\{\{x_1\}\}) \,\&\, \bigcup \{\{x_1\}\} = \{x_1\}\big)$

$\big\langle \{x_1\} \big\rangle \hookrightarrow T\mathsf{tree}_2(\mathit{Stat10}\star) \Rightarrow \quad \bigcup \{\{x_1\}\} \neq \{x_1\}$

$\big\langle \{x_1\}, \{\{x_1\}\} \big\rangle \hookrightarrow T\mathsf{un}_4(\mathit{Stat10}\star) \Rightarrow \quad \mathsf{false}$

$\mathsf{Discharge} \Rightarrow \quad \textsc{Auto}$

Knowing, now, that t_0 consists of more than one edge, we are ensured by THM tree_1 that t_0 has a vertex u_0 whose removal from t_0 leads to a sub-tree t_1. We will show that t_1 is a spanning tree for the original graph deprived of the vertex in question.

$\langle t_0 \rangle \hookrightarrow T\mathsf{tree}_1(\star) \Rightarrow \quad \mathit{Stat11}: \langle \exists e \in t_0, u \in e \mid \{a \in t_0 \mid u \notin a\} = t_0 \setminus \{e\} \,\&\, \mathsf{Is\_tree}(t_0 \setminus \{e\}) \rangle$

$\langle a_0, u_0 \rangle \hookrightarrow \mathit{Stat11}(\mathit{Stat11}\star) \Rightarrow \quad a_0 \in t_0 \,\&\, u_0 \in a_0 \,\&\, \{a \in t_0 \mid u_0 \notin a\} = t_0 \setminus \{a_0\} \,\&\, \mathsf{Is\_tree}(t_0 \setminus \{a_0\})$

$\langle t_0, a_0, t_0 \rangle \hookrightarrow T\mathsf{un}_2(\mathit{Stat11}\star) \Rightarrow \quad u_0 \in \bigcup t_0$

$\langle u_0 \rangle \hookrightarrow \mathit{Stat1a}(\mathit{Stat1}\star) \Rightarrow \quad \neg\mathsf{HasSpanningTree}(v_0 \setminus \{u_0\}, \{a \in e_0 \mid u_0 \notin a\})$

$\mathsf{Use\_def}(\mathsf{HasSpanningTree}) \Rightarrow \quad \mathit{Stat12}: \neg\langle \exists t \mid \mathsf{Is\_tree}(t) \,\&\, \bigcup t = v_0 \setminus \{u_0\} \,\&\, v_0 \setminus \{u_0\} = \{\mathbf{arb}(v_0 \setminus \{u_0\})\} \vee t \subseteq \{a \in e_0 \mid u_0 \notin a\} \rangle$

$\mathsf{Set\_monot}(\mathit{Stat1}) \Rightarrow \quad \{a \in t_0 \mid u_0 \notin a\} \subseteq \{a \in e_0 \mid u_0 \notin a\}$

$\langle t_0 \setminus \{a_0\} \rangle \hookrightarrow \mathit{Stat12}(\mathit{Stat11}\star) \Rightarrow \quad \bigcup(t_0 \setminus \{a_0\}) \neq v_0 \setminus \{u_0\}$

$(\mathit{Stat1}\star)\mathsf{ELEM} \Rightarrow \quad \mathit{Stat13}: a_0 \in \{\{x, y\} : x \in v_0, y \in v_0 \setminus \{x\}\}$

$\langle x_3, y_3 \rangle \hookrightarrow \mathit{Stat13}(\mathit{Stat13}\star) \Rightarrow \quad \mathit{Stat14}: a_0 \not\subseteq \{u_0\} \,\&\, a_0 = \{x_3, y_3\}$

$\langle w_0 \rangle \hookrightarrow \mathit{Stat14}(\mathit{Stat11}\star) \Rightarrow \quad a_0 = \{u_0, w_0\}$

$\langle t_0, u_0, w_0, a_0 \rangle \hookrightarrow T\mathsf{tree}_4 \Rightarrow \quad \textsc{Auto}$

$\mathsf{EQUAL}(\mathit{Stat1}) \Rightarrow \quad \bigcup(t_0 \setminus \{a_0\}) = v_0 \setminus \{u_0\}$

$(\mathit{Stat12}\star)\mathsf{Discharge} \Rightarrow \quad \textsc{Qed}$

3 Representation of a graph by way of a membership digraph

Before developing two important THEORYs about the representation of graphs via membership, we will prove the statements about orienting graphs which are displayed here:

THM xtensionalization$_0$: [Weakly extensional acyclic orientability of a finite nonvoid graph]

$$\mathsf{Finite}(V) \mathbin{\&} S \in V \rightarrow \langle \exists d \mid \mathsf{Orientates}(d, V, E) \mathbin{\&} \mathsf{Acyclic}(V, d) \mathbin{\&} \mathsf{WExtensional}(V, d) \mathbin{\&} S \notin \mathbf{range}(d) \rangle$$

THM xtensionalization$_1$.

$$W = V \cup \{U\} \mathbin{\&} U \notin V \mathbin{\&} \mathsf{Extensional}(V, D) \mathbin{\&} E \subseteq \{\{x, y\} : x \in Z, y \in Z \setminus \{x\}\} \mathbin{\&} D \subseteq V \times V \mathbin{\&} D' = D \cup \{U\} \times \{t \in V \mid \{U, t\} \in E\} \rightarrow$$
$$D' \subseteq W \times W \mathbin{\&} \left(\neg \mathsf{Extensional}(W, D') \rightarrow \langle \exists x \in V, \forall z \mid [U, z] \in D' \leftrightarrow [x, z] \in D \rangle\right)$$

THM xtensionalization$_2$: [Preservation of acyclicity and extensionality under adjunction of an inner vertex]

$$W = V \cup \{U\} \mathbin{\&} U \notin V \mathbin{\&} S \in V \mathbin{\&} \{y \in V \mid [S, y] \in D\} = \emptyset \mathbin{\&}$$
$$S \in \{t \in W \mid \{U, t\} \in E\} \mathbin{\&} E \subseteq \{\{x, y\} : x \in Z, y \in Z \setminus \{x\}\} \mathbin{\&} \mathsf{Orientates}(D, V, E) \mathbin{\&} \mathsf{Acyclic}(V, D) \mathbin{\&} \mathsf{Extensional}(V, D) \mathbin{\&} D \subseteq V \times V \rightarrow$$
$$\langle \exists d \subseteq W \times W \mid \mathsf{Orientates}(d, W, E) \mathbin{\&} \mathsf{Acyclic}(W, d) \mathbin{\&} \mathsf{Extensional}(W, d) \rangle$$

Our initial plan was to develop with Referee the following representation THEORYs, in order to explain why one can work with membership as a convenient surrogate for the edge relationship of general graphs. Initially, we meant to insist on a weak form of extensionality.

THEORY weaklyExtGraphRepr(v_1, e_0)
$e_0 \subseteq \{\{x,y\} : x \in v_1, y \in v_1 \mid x \neq y\}$ & $\mathsf{Finite}(v_1)$
$\Rightarrow$ (we_Θ)
$v_1 \times v_1 \supseteq we_\Theta$ & $\mathsf{Orientates}(we_\Theta, v_1, e_0)$
$\mathsf{Acyclic}(v_1, we_\Theta)$
$\mathsf{WExtensional}(v_1, we_\Theta)$
END weaklyExtGraphRepr

THEORY finMostowskiDecoration(v_0, d_0)
$v_0 \times v_0 \supseteq d_0$ & $v_0 \neq \emptyset$ & $\mathsf{Finite}(v_0)$
$\mathsf{Acyclic}(v_0, d_0)$
$\mathsf{WExtensional}(v_0, d_0)$
$\Rightarrow$ $(mski_\Theta)$
$\mathsf{One\_1\_map}(mski_\Theta)$ & $\mathbf{dom}(mski_\Theta) = v_0$
$\emptyset \in \mathbf{range}(mski_\Theta)$ & $\langle \forall m \in \mathbf{range}(mski_\Theta) \mid \mathsf{Finite}(m) \rangle$
$\langle \forall w \in \mathbf{dom}(d_0) \mid mski_\Theta \upharpoonright w = \{ mski_\Theta \upharpoonright p^{[2]} : p \in d_{0|\{w\}} \} \rangle$
$\langle \forall y, x \mid y \in v_0 \rightarrow (mski_\Theta \upharpoonright y \in mski_\Theta \upharpoonright x \leftrightarrow [x,y] \in d_0) \rangle$
END finMostowskiDecoration

THEORY finiteGraphRepr(v_0, e_0)
$e_0 \subseteq \{\{x,y\} : x \in v_0, y \in v_0 \mid x \neq y\}$ & $\mathsf{Finite}(v_0)$
$\Rightarrow$ (rho_Θ, nu_Θ)
$\mathsf{One\_1\_map}(rho_\Theta)$ & $\mathbf{dom}(rho_\Theta) = v_0$ & $\mathbf{range}(rho_\Theta) = nu_\Theta$
$\langle \forall x \in v_0, y \in v_0 \mid \{x,y\} \in e_0 \leftrightarrow rho_\Theta \upharpoonright x \in rho_\Theta \upharpoonright y \vee rho_\Theta \upharpoonright y \in rho_\Theta \upharpoonright x \rangle$
$\{x \in nu_\Theta \mid x \cap nu_\Theta \neq \emptyset\} \subseteq \mathcal{P} nu_\Theta$
END finiteGraphRepr

3.1 Weakly extensional, acyclic orientation of a graph whatsoever

It turned out to be doable, instead of developing the first of the three THEORYs just outlined, to prove via an inductive proof a single theorem of equipollent content, namely:

THM xtensionalization$_0$: [Weakly extensional acyclic orientability of a finite nonvoid graph] $\mathsf{Finite}(V)$ & $S \in V \rightarrow$
$\langle \exists d \mid \mathsf{Orientates}(d, V, E)$ & $\mathsf{Acyclic}(V, d)$ & $\mathsf{WExtensional}(V, d)$ & $S \notin \mathbf{range}(d) \rangle$. PROOF:
Suppose_not$(v_2, s_2, e_0) \Rightarrow$ AUTO

Arguing by contradiction, assume that there is a counterexample v_2, s_2, e_0 to the claim. Then, thanks to the finiteness hypothesis, we can take a minimal counterexample v_1, s_1, e_0.

Suppose ⇒ *Stat0* : $\neg\langle \exists s \in v_2 \mid \neg\langle \exists d \mid \mathsf{Orientates}(d, v_2, e_0) \;\&\; \mathsf{Acyclic}(v_2, d) \;\&\; \mathsf{WExtensional}(v_2, d) \;\&\; s \notin \mathbf{range}(d)\rangle\rangle$

$\langle s_2\rangle \hookrightarrow$ *Stat0*(⋆) ⇒ false; Discharge ⇒ AUTO

APPLY $\langle \mathsf{fin}_\Theta : v_1\rangle$ $\mathsf{finiteInduction}(s_0 \mapsto v_2, P(V) \mapsto \langle \exists t \in V \mid \neg\langle \exists d \mid \mathsf{Orientates}(d, V, e_0) \;\&\; \mathsf{Acyclic}(V, d) \;\&\; \mathsf{WExtensional}(V, d) \;\&\; t \notin \mathbf{range}(d)\rangle\rangle)$ ⇒

Stat1 : $\langle \forall V \mid V \subseteq v_1 \rightarrow \mathsf{Finite}(V) \;\&\; (\langle \exists t \in V \mid \neg\langle \exists d \mid \mathsf{Orientates}(d, V, e_0) \;\&\; \mathsf{Acyclic}(V, d) \;\&\; \mathsf{WExtensional}(V, d) \;\&\; t \notin \mathbf{range}(d)\rangle\rangle \leftrightarrow V = v_1)\rangle$

$\langle v_1\rangle \hookrightarrow$ *Stat1*(*Stat1*⋆) ⇒ *Stat2* : $\langle \exists t \in v_1 \mid \neg\langle \exists d \mid \mathsf{Orientates}(d, v_1, e_0) \;\&\; \mathsf{Acyclic}(v_1, d) \;\&\; \mathsf{WExtensional}(v_1, d) \;\&\; t \notin \mathbf{range}(d)\rangle\rangle$

$\langle s_1\rangle \hookrightarrow$ *Stat2*(*Stat2*⋆) ⇒ *Stat3* : $\neg\langle \exists d \mid \mathsf{Orientates}(d, v_1, e_0) \;\&\; \mathsf{Acyclic}(v_1, d) \;\&\; \mathsf{WExtensional}(v_1, d) \;\&\; s_1 \notin \mathbf{range}(d)\rangle \;\&\; s_1 \in v_1$

Now consider the strict subset $v_0 = v_1 \setminus \{s_1\}$ of v_1. Due to the minimality assumption, v_0, e_0 can be oriented in an acyclic and weakly extensional way, for any vertex $t \in v_0$, so that t plays the role of a source.

Loc_def ⇒ *Stat4* : $v_0 = v_1 \setminus \{s_1\}$

$\langle v_0\rangle \hookrightarrow$ *Stat1*(*Stat3*⋆) ⇒ *Stat5* : $\neg\langle \exists t \in v_0 \mid \neg\langle \exists d \mid \mathsf{Orientates}(d, v_0, e_0) \;\&\; \mathsf{Acyclic}(v_0, d) \;\&\; \mathsf{WExtensional}(v_0, d) \;\&\; t \notin \mathbf{range}(d)\rangle\rangle \;\&\; v_1 = v_0 \cup \{s_1\}$

Unless s_1 is an isolated vertex, an acyclic and weakly extensional orientation of $v_1 \setminus \{s_1\}$ having as source a neighbor t_1 of s_1 exists, by the assumed minimality of v_1. However, that orientation (conveniently restricted to the Cartesian square of the set of vertices) could trivially be extended to the overall graph v_1, e_0 so that s_1 becomes the new source. This contradiction shows that s_1 must be devoid of neighbors.

Suppose ⇒ *Stat6* : $\{x \in v_0 \mid \{s_1, x\} \in e_0\} \neq \emptyset$

$\langle t_1\rangle \hookrightarrow$ *Stat6*(*Stat6*⋆) ⇒ *Stat7* : $t_1 \in v_0 \;\&\; \{t_1, s_1\} \in e_0$

$\langle t_1\rangle \hookrightarrow$ *Stat5*(*Stat5*⋆) ⇒ *Stat8* : $\langle \exists d \mid \mathsf{Orientates}(d, v_0, e_0) \;\&\; \mathsf{Acyclic}(v_0, d) \;\&\; \mathsf{WExtensional}(v_0, d) \;\&\; t_1 \notin \mathbf{range}(d)\rangle$

$\langle d_2\rangle \hookrightarrow$ *Stat8*(*Stat8*⋆) ⇒ $\mathsf{Orientates}(d_2, v_0, e_0) \;\&\; \mathsf{Acyclic}(v_0, d_2) \;\&\; \mathsf{WExtensional}(v_0, d_2) \;\&\; t_1 \notin \mathbf{range}(d_2)$

Loc_def ⇒ $d_0 = d_2 \cap (v_0 \times v_0) \;\&\; d_1 = d_0 \cup \{s_1\} \times \{x \in v_0 \mid \{s_1, x\} \in e_0\}$

Suppose ⇒ $\neg(\mathsf{Orientates}(d_0, v_0, e_0) \;\&\; \mathsf{Acyclic}(v_0, d_0) \;\&\; \mathsf{WExtensional}(v_0, d_0) \;\&\; t_1 \notin \mathbf{range}(d_0))$

$\langle d_2, v_0, e_0\rangle \hookrightarrow T\mathsf{orientation}_2$(*Stat8*⋆) ⇒ $\mathsf{Orientates}(d_2 \cap (v_0 \times v_0), v_0, e_0)$

$\langle v_0, d_2\rangle \hookrightarrow T\mathsf{weaXtensionality}_0$(*Stat8*⋆) ⇒ $\mathsf{WExtensional}(v_0, d_2 \cap (v_0 \times v_0))$

EQUAL(*Stat8*) ⇒ $\mathsf{Orientates}(d_0, v_0, e_0) \;\&\; \mathsf{WExtensional}(v_0, d_0)$

$\langle v_0, d_2, v_0, d_0\rangle \hookrightarrow T\mathsf{acyclicity}_1$ ⇒ AUTO

$\langle d_2, d_0, d_2\rangle \hookrightarrow T\mathsf{range}_1$(*Stat8*⋆) ⇒ $\mathbf{range}(d_0) \subseteq \mathbf{range}(d_2)$

(*Stat8*⋆)Discharge ⇒ AUTO

$\langle d_0, v_0, e_0, v_1, s_1, d_1\rangle \hookrightarrow T\mathsf{orientation}_3$(*Stat3*⋆) ⇒ *Stat9* : $\mathsf{Orientates}(d_1, v_1, e_0)$

$\langle d_1\rangle \hookrightarrow$ *Stat3*(*Stat9*⋆) ⇒ $\neg(\mathsf{Acyclic}(v_1, d_1) \;\&\; \mathsf{WExtensional}(v_1, d_1) \;\&\; s_1 \notin \mathbf{range}(d_1))$

Suppose ⇒ $\neg\mathsf{Acyclic}(v_1, d_1)$

Suppose ⇒ *Stat10* : $\{x \in v_0 \mid \{s_1, x\} \in e_0\} \not\subseteq v_0$

$\langle x_1\rangle \hookrightarrow$ *Stat10*(*Stat10*⋆) ⇒ *Stat11* : $x_1 \in \{x \in v_0 \mid \{s_1, x\} \in e_0\} \;\&\; x_1 \notin v_0$

$\langle\,\rangle \hookrightarrow$ *Stat11*(*Stat11*⋆) ⇒ false

Discharge ⇒ AUTO

$\langle v_0, d_0, s_1, \{x \in v_0 \mid \{s_1, x\} \in e_0\} \rangle \hookrightarrow T\text{acyclicity}_0(\textit{Stat4}\star) \Rightarrow \quad \text{Acyclic}(v_0 \cup \{s_1\}, d_0 \cup (\{s_1\} \times \{x \in v_0 \mid \{s_1, x\} \in e_0\}))$

EQUAL(*Stat5*) $\Rightarrow$ false

Discharge $\Rightarrow$ AUTO

Suppose $\Rightarrow$ $\neg\text{WExtensional}(v_1, d_1)$

Suppose $\Rightarrow$ *Stat12* : $t_1 \notin \{x \in v_0 \mid \{s_1, x\} \in e_0\}$

$\langle t_1 \rangle \hookrightarrow \textit{Stat12}(\textit{Stat7}, \textit{Stat7}\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO

Set_monot $\Rightarrow$ $\{x \in v_0 \mid \{s_1, x\} \in e_0\} \subseteq \{x \in v_0 \mid \text{true}\}$

$\langle v_0, d_0, d_1, s_1, \{x \in v_0 \mid \{s_1, x\} \in e_0\}, v_1 \rangle \hookrightarrow T\text{weaXtensionality}_2(\textit{Stat4}\star) \Rightarrow$ false

Discharge $\Rightarrow$ AUTO

$\langle \{s_1\}, \{x \in v_0 \mid \{s_1, x\} \in e_0\} \rangle \hookrightarrow T\text{cartesian}_1 \Rightarrow$ AUTO

$\langle d_1, d_0, \{s_1\} \times \{x \in v_0 \mid \{s_1, x\} \in e_0\} \rangle \hookrightarrow T\text{range}_1(\textit{Stat4}\star) \Rightarrow \quad s_1 \in \mathbf{range}(d_0) \vee s_1 \in \{x \in v_0 \mid \{s_1, x\} \in e_0\}$

Suppose $\Rightarrow$ *Stat13* : $s_1 \in \{x \in v_0 \mid \{s_1, x\} \in e_0\}$

$\langle\,\rangle \hookrightarrow \textit{Stat13}(\textit{Stat4}, \textit{Stat4}\star) \Rightarrow$ false

Discharge $\Rightarrow$ $s_1 \in \mathbf{range}(d_0)$

$\langle v_0, v_0 \rangle \hookrightarrow T\text{cartesian}_1 \Rightarrow$ AUTO

$\langle v_0 \times v_0, d_0, v_0 \times v_0 \rangle \hookrightarrow T\text{range}_1(\textit{Stat4}\star) \Rightarrow$ false

Discharge $\Rightarrow$ AUTO

At this point we know that s_1 is an isolated vertex. However, we will find a contradiction also in this case: any orientation for v_0, in fact, works also as an orientation for v_1 and, as such, has each isolated vertex of v_1 — in particular s_1 — as a source. Hence we will conclude that the claim of this theorem is true.

Suppose $\Rightarrow$ *Stat14* : $v_0 \neq \emptyset$

The argument is split into two subcases depending on whether v_1 has vertices different from s_1 or s_1 is the sole vertex.

$\langle t_2 \rangle \hookrightarrow \textit{Stat14}(\textit{Stat3}\star) \Rightarrow$ $t_2 \in v_0$

$\langle t_2 \rangle \hookrightarrow \textit{Stat5}(\textit{Stat5}\star) \Rightarrow$ *Stat15* : $\langle \exists d \mid \text{Orientates}(d, v_0, e_0) \,\&\, \text{Acyclic}(v_0, d) \,\&\, \text{WExtensional}(v_0, d) \,\&\, t_2 \notin \mathbf{range}(d) \rangle$

$\langle d_3 \rangle \hookrightarrow \textit{Stat15}(\textit{Stat15}\star) \Rightarrow$ $\text{Orientates}(d_3, v_0, e_0) \,\&\, \text{Acyclic}(v_0, d_3) \,\&\, \text{WExtensional}(v_0, d_3)$

$\langle v_0 \times v_0, d_3 \cap (v_0 \times v_0), v_0 \times v_0 \rangle \hookrightarrow T\text{range}_1 \Rightarrow$ AUTO

$\langle v_0, v_0 \rangle \hookrightarrow T\text{cartesian}_1 \Rightarrow$ AUTO

$\langle d_3, v_0, e_0 \rangle \hookrightarrow T\text{orientation}_2(\textit{Stat5}\star) \Rightarrow$ $\text{Orientates}(d_3 \cap (v_0 \times v_0), v_0, e_0)$

$\langle d_3 \cap (v_0 \times v_0), v_0, e_0, v_1, s_1 \rangle \hookrightarrow T\text{orientation}_4(\textit{Stat3}\star) \Rightarrow$ $\text{Orientates}(d_3 \cap (v_0 \times v_0), v_1, e_0)$

$\langle d_3 \cap (v_0 \times v_0) \rangle \hookrightarrow \textit{Stat3}(\textit{Stat4}\star) \Rightarrow$ *Stat16* : $\neg\big(\text{Acyclic}(v_1, d_3 \cap (v_0 \times v_0)) \,\&\, \text{WExtensional}(v_1, d_3 \cap (v_0 \times v_0))\big)$

Suppose $\Rightarrow$ $\neg\text{Acyclic}(v_1, d_3 \cap (v_0 \times v_0))$

$\langle v_0, d_3, v_0, d_3 \cap (v_0 \times v_0) \rangle \hookrightarrow T\text{acyclicity}_1(\textit{Stat15}\star) \Rightarrow$ $\text{Acyclic}(v_0, d_3 \cap (v_0 \times v_0))$

$\langle v_0, d_3 \cap (v_0 \times v_0), s_1, \emptyset \rangle \hookrightarrow T\text{acyclicity}_0(\textit{Stat4}\star) \Rightarrow$ $\text{Acyclic}(v_0 \cup \{s_1\}, d_3 \cap (v_0 \times v_0) \cup \{s_1\} \times \emptyset)$

$\langle \{s_1\} \rangle \hookrightarrow T\text{cartesian}_3(\textit{Stat16}\star) \Rightarrow$ $d_3 \cap (v_0 \times v_0) \cup \{s_1\} \times \emptyset = d_3 \cap (v_0 \times v_0)$

EQUAL(*Stat5*) $\Rightarrow$ false

Discharge $\Rightarrow$ AUTO

$\langle v_0, d_3\rangle \hookrightarrow T$weaXtensionality$_0$(*Stat15*$\star$) $\Rightarrow$ $\neg$WExtensional$(v_1, d_3 \cap (v_0 \times v_0))$ & WExtensional$(v_0, d_3 \cap (v_0 \times v_0))$

$\langle d_3 \cap (v_0 \times v_0), v_1, e_0, v_0, s_1\rangle \hookrightarrow T$orientation$_5$(*Stat3*$\star$) $\Rightarrow$ $s_1 \notin \mathbf{dom}(d_3 \cap (v_0 \times v_0))$

$\langle v_0, d_3 \cap (v_0 \times v_0), s_1, v_1\rangle \hookrightarrow T$weaXtensionality$_1$(*Stat3*$\star$) $\Rightarrow$ false

Discharge $\Rightarrow$ AUTO

$\langle \emptyset\rangle \hookrightarrow$ *Stat3*(*Stat3*$\star$) $\Rightarrow$ *Stat17*: $v_1 = \{s_1\}$ & $\neg$(Orientates$(\emptyset, v_1, e_0)$ & Acyclic$(v_1, \emptyset)$ & WExtensional$(v_1, \emptyset)$ & $s_1 \notin \mathbf{range}(\emptyset)$)

$\langle v_1, \emptyset\rangle \hookrightarrow T$extensionality$_0$ $\Rightarrow$ AUTO

$\langle v_1, s_1\rangle \hookrightarrow T$voidgraph$_1$ $\Rightarrow$ AUTO

$\langle \{s_1\}\rangle \hookrightarrow T$voidgraph$_2$ $\Rightarrow$ AUTO

$\langle \emptyset\rangle \hookrightarrow T$range$_1$(*Stat17*$\star$) $\Rightarrow$ $s_1 \notin \mathbf{range}(\emptyset)$

EQUAL(*Stat17*) $\Rightarrow$ $\neg$(Orientates$(\emptyset, \{s_1\}, e_0)$ & $s_1 \notin \mathbf{range}(\emptyset)$)

$\langle \{s_1\}, s_1, e_0\rangle \hookrightarrow T$orientation$_1$(*Stat17*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ QED

THM xtensionalization$_1$.
$W = V \cup \{U\}$ & $U \notin V$ & Extensional(V, D) & $E \subseteq \{\{x, y\} : x \in Z, y \in Z \setminus \{x\}\}$ & $D \subseteq V \times V$ & $D' = D \cup \{U\} \times \{t \in V \mid \{U, t\} \in E\}$ $\rightarrow$
$D' \subseteq W \times W$ & $(\neg$Extensional$(W, D') \rightarrow \langle \exists x \in V, \forall z \mid [U, z] \in D' \leftrightarrow [x, z] \in D\rangle)$. PROOF:

Suppose_not$(v_1, v_0, x_0, d_0, e_2, v_2, d_1)$ $\Rightarrow$ *Stat0*: $d_1 = d_0 \cup \{x_0\} \times \{t \in v_0 \mid \{x_0, t\} \in e_2\}$ &
$\neg\Big(d_1 \subseteq v_1 \times v_1$ & $(\neg$Extensional$(v_1, d_1) \rightarrow \langle \exists x \in v_0, \forall z \mid [x_0, z] \in d_1 \leftrightarrow [x, z] \in d_0\rangle)\Big)$ & $x_0 \in v_1$ & $v_0 = v_1 \setminus \{x_0\}$ &
$e_2 \subseteq \{\{x, y\} : x \in v_2, y \in v_2 \setminus \{x\}\}$ & Extensional(v_0, d_0) & $d_0 \subseteq v_0 \times v_0$

Set_monot $\Rightarrow$ $\{t \in v_0 \mid \text{true}\} \supseteq \{t \in v_0 \mid \{x_0, t\} \in e_2\}$

(*Stat0*$\star$)ELEM $\Rightarrow$ *Stat2*: $v_1 = v_0 \cup \{x_0\}$ & $v_0 \supseteq \{t \in v_0 \mid \{x_0, t\} \in e_2\}$

Suppose $\Rightarrow$ $d_1 \not\subseteq v_1 \times v_1$

$\langle v_0, v_1, v_0, v_1\rangle \hookrightarrow T$cartesian$_5$(*Stat0*$\star$) $\Rightarrow$ $\{x_0\} \times \{t \in v_0 \mid \{x_0, t\} \in e_2\} \not\subseteq v_1 \times v_1$

$\langle \{x_0\}, v_1, \{t \in v_0 \mid \{x_0, t\} \in e_2\}, v_1\rangle \hookrightarrow T$cartesian$_5$(*Stat0*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO

(*Stat0*$\star$)ELEM $\Rightarrow$ *Stat99*: $\neg\langle \exists x \in v_0, \forall z \mid [x_0, z] \in d_1 \leftrightarrow [x, z] \in d_0\rangle$ & $\neg$Extensional(v_1, d_1)

Use_def(Extensional) $\Rightarrow$ *Stat15*: $\neg\langle \forall x \in v_1, y \in v_1, \exists z \mid ([x, z] \in d_1 \leftrightarrow [y, z] \in d_1) \rightarrow x = y\rangle$ &
Stat15a: $\langle \forall x \in v_0, y \in v_0, \exists z \mid ([x, z] \in d_0 \leftrightarrow [y, z] \in d_0) \rightarrow x = y\rangle$

$\langle x_2, y_2, x_2, y_2\rangle \hookrightarrow$ *Stat15*(*Stat15*$\star$) $\Rightarrow$ *Stat16*: $\neg\langle \exists z \mid ([x_2, z] \in d_1 \leftrightarrow [y_2, z] \in d_1) \rightarrow x_2 = y_2\rangle$ & $x_2, y_2 \in v_1$

Suppose $\Rightarrow$ *Stat17*: $x_0 \notin \{x_2, y_2\}$

$\langle x_2, y_2\rangle \hookrightarrow$ *Stat15a*(*Stat16*, *Stat17*, *Stat2*$\star$) $\Rightarrow$ *Stat18*: $\langle \exists z \mid ([x_2, z] \in d_0 \leftrightarrow [y_2, z] \in d_0) \rightarrow x_2 = y_2\rangle$ & $\neg\langle \exists z \mid ([x_2, z] \in d_1 \leftrightarrow [y_2, z] \in d_1) \rightarrow x_2 = y_2\rangle$

$\langle z_2, z_2\rangle \hookrightarrow$ *Stat18*(*Stat18*$\star$) $\Rightarrow$ $x_2 \neq y_2$ & $([x_2, z_2] \in d_0 \neq [y_2, z_2] \in d_0)$ & $[x_2, z_2] \in d_1 \leftrightarrow [y_2, z_2] \in d_1$

Suppose $\Rightarrow$ $([x_2, z_2] \in d_0 \leftrightarrow [x_2, z_2] \in d_1)$ & $([y_2, z_2] \in d_0 \leftrightarrow [y_2, z_2] \in d_1)$

EQUAL(*Stat18*) $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO

$\langle [x_2, z_2], \{x_0\}, \{t \in v_0 \mid \{x_0, t\} \in e_2\}\rangle \hookrightarrow T$cartesian$_0$(*Stat17*$\star$) $\Rightarrow$ $[x_2, z_2] \notin \{x_0\} \times \{t \in v_0 \mid \{x_0, t\} \in e_2\}$

$\langle [y_2, z_2], \{x_0\}, \{t \in v_0 \mid \{x_0, t\} \in e_2\}\rangle \hookrightarrow T$cartesian$_0$(*Stat17*$\star$) $\Rightarrow$ $[y_2, z_2] \notin \{x_0\} \times \{t \in v_0 \mid \{x_0, t\} \in e_2\}$

(*Stat0*$\star$)Discharge $\Rightarrow$ AUTO

Loc_def $\Rightarrow$ $x_1 = \textbf{if } x_0 = x_2 \textbf{ then } y_2 \textbf{ else } x_2 \textbf{ fi}$

$\langle \emptyset\rangle \hookrightarrow$ *Stat16*(*Stat16*$\star$) $\Rightarrow$ *Stat19*: $x_1 \in v_1 \setminus \{x_0\}$

Suppose $\Rightarrow$ *Stat20*: $\neg\langle \forall z \mid [x_0, z] \in d_1 \leftrightarrow [x_1, z] \in d_0\rangle$

$\langle z_1 \rangle \hookrightarrow$ *Stat20(Stat20⋆)* $\Rightarrow$ $[x_0, z_1] \in d_1 \neq [x_1, z_1] \in d_0$
(*Stat16⋆*)ELEM $\Rightarrow$ $x_0 \neq x_1 \;\&\; \{x_0, x_1\} = \{x_2, y_2\}$
$\langle z_1 \rangle \hookrightarrow$ *Stat16(Stat20⋆)* $\Rightarrow$ $[x_2, z_1] \in d_1 \leftrightarrow [y_2, z_1] \in d_1$
Suppose $\Rightarrow$ *Stat21*: $\neg([x_1, z_1] \in d_1 \leftrightarrow [x_1, z_1] \in d_0)$
(*Stat0, Stat21⋆*)ELEM $\Rightarrow$ $[x_1, z_1] \in \{x_0\} \times \{t \in v_0 \mid \{x_0, t\} \in e_2\}$
$\langle [x_1, z_1], \{x_0\}, \{t \in v_0 \mid \{x_0, t\} \in e_2\} \rangle \hookrightarrow$ *T*cartesian$_0$(*Stat20⋆*) $\Rightarrow$ false
Discharge $\Rightarrow$ AUTO
Suppose $\Rightarrow$ $x_0 = x_2 \;\&\; x_1 = y_2$
EQUAL(*Stat20*) $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO
(*Stat20⋆*)ELEM $\Rightarrow$ $x_0 = y_2 \;\&\; x_1 = x_2$
EQUAL(*Stat20*) $\Rightarrow$ false
Discharge $\Rightarrow$ AUTO
$\langle x_1 \rangle \hookrightarrow$ *Stat99(Stat19, Stat2⋆)* $\Rightarrow$ $\neg\langle \forall z \mid [x_0, z] \in d_1 \leftrightarrow [x_1, z] \in d_0 \rangle$
(*Stat19⋆*)Discharge $\Rightarrow$ QED

THM xtensionalization$_2$: [Preservation of acyclicity and extensionality under adjunction of an inner vertex] $W = V \cup \{U\} \;\&\; U \notin V \;\&\; S \in V \;\&\; \{y \in V \mid [S, y] \in D\} = \emptyset \;\&$
$S \in \{t \in W \mid \{U, t\} \in E\} \;\&\; E \subseteq \{\{x, y\} : x \in Z, y \in Z \setminus \{x\}\} \;\&\; \mathrm{Orientates}(D, V, E) \;\&\; \mathrm{Acyclic}(V, D) \;\&\; \mathrm{Extensional}(V, D) \;\&\; D \subseteq V \times V \rightarrow$
$\langle \exists d \subseteq W \times W \mid \mathrm{Orientates}(d, W, E) \;\&\; \mathrm{Acyclic}(W, d) \;\&\; \mathrm{Extensional}(W, d) \rangle$. PROOF:
Suppose_not$(v_1, v_0, u_0, s_0, d_0, e_2, v_2)$ $\Rightarrow$ *Stat0*: $\neg\langle \exists d \subseteq v_1 \times v_1 \mid \mathrm{Orientates}(d, v_1, e_2) \;\&\; \mathrm{Acyclic}(v_1, d) \;\&$
$\mathrm{Extensional}(v_1, d) \rangle \;\&\; u_0 \in v_1 \;\&\; v_0 = v_1 \setminus \{u_0\} \;\&\; s_0 \in v_0 \;\&\; \{y \in v_0 \mid [s_0, y] \in d_0\} = \emptyset \;\&\; s_0 \in \{t \in v_1 \mid \{u_0, t\} \in e_2\} \;\&$
$e_2 \subseteq \{\{x, y\} : x \in v_2, y \in v_2 \setminus \{x\}\} \;\&\; \mathrm{Orientates}(d_0, v_0, e_2) \;\&\; \mathrm{Acyclic}(v_0, d_0) \;\&\; \mathrm{Extensional}(v_0, d_0) \;\&\; d_0 \subseteq v_0 \times v_0$
Suppose $\Rightarrow$ $\{u_0, u_0\} \in e_2$
(*Stat0⋆*)ELEM $\Rightarrow$ *Stat1*: $\{u_0, u_0\} \in \{\{x, y\} : x \in v_2, y \in v_2 \setminus \{x\}\}$
$\langle x_1, y_1 \rangle \hookrightarrow$ *Stat1(Stat1⋆)* $\Rightarrow$ false
Discharge $\Rightarrow$ *Stat2*: $\{u_0, u_0\} \notin e_2$

Orient the edges incident to u_0 as in-coming to u_0. The neighbors of u_0 through e_2 are $\{t \in v_1 \mid \{u_0, t\} \in e_2\}$.

Suppose $\Rightarrow$ *Stat3*: $v_0 \not\supseteq \{t \in v_1 \mid \{u_0, t\} \in e_2\}$
$\langle x_2 \rangle \hookrightarrow$ *Stat3(Stat3⋆)* $\Rightarrow$ *Stat4*: $x_2 \in \{t \in v_1 \mid \{u_0, t\} \in e_2\} \;\&\; x_2 \notin v_0$
$\langle \rangle \hookrightarrow$ *Stat4(Stat4, Stat0, Stat2⋆)* $\Rightarrow$ false
Discharge $\Rightarrow$ AUTO
Suppose $\Rightarrow$ *Stat5*: $\neg\langle \exists t \in \{t \in v_1 \mid \{u_0, t\} \in e_2\} \mid d_{0|\{t\}} = \emptyset \rangle$
$\langle s_0 \rangle \hookrightarrow$ *Stat5(Stat0⋆)* $\Rightarrow$ *Stat6*: $d_{0|\{s_0\}} \neq \emptyset \;\&\; \{y \in v_0 \mid [s_0, y] \in d_0\} = \emptyset \;\&\; d_0 \subseteq v_0 \times v_0$
Use_def(|) $\Rightarrow$ $d_{0|\{s_0\}} = \{p \in d_0 \mid p^{[1]} \in \{s_0\}\}$
$\langle p_0 \rangle \hookrightarrow$ *Stat6(Stat6⋆)* $\Rightarrow$ *Stat7*: $p_0 \in \{p \in d_0 \mid p^{[1]} \in \{s_0\}\}$
$\langle p_0, v_0, v_0 \rangle \hookrightarrow$ *T*cartesian$_0$ $\Rightarrow$ AUTO
$\langle \rangle \hookrightarrow$ *Stat7(Stat6⋆)* $\Rightarrow$ *Stat8*: $p_0^{[2]} \notin \{y \in v_0 \mid [s_0, y] \in d_0\} \;\&\; p_0 \in d_0 \;\&\; p_0 = [p_0^{[1]}, p_0^{[2]}] \;\&\; p_0^{[1]} = s_0 \;\&\; p_0^{[2]} \in v_0$

$\langle p_0{}^{[2]} \rangle \hookrightarrow$ *Stat8(Stat8)* $\Rightarrow$ false

Discharge $\Rightarrow$ AUTO

$\langle v_0, d_0, u_0, \{t \in v_1 \mid \{u_0, t\} \in e_2\} \rangle \hookrightarrow$ *Tacyclicity*$_6$(*Stat0*$\star$) $\Rightarrow$ *Stat9* : $\mathsf{Acyclic}(v_0 \cup \{u_0\}, d_0 \cup \{t \in v_1 \mid \{u_0, t\} \in e_2\} \times \{u_0\})$ & $v_1 = v_0 \cup \{u_0\}$

$\langle v_0, d_0, u_0, \{t \in v_1 \mid \{u_0, t\} \in e_2\} \rangle \hookrightarrow$ *Textensionality*$_1$(*Stat0*$\star$) $\Rightarrow$ $\mathsf{Extensional}(v_0 \cup \{u_0\}, d_0 \cup \{t \in v_1 \mid \{u_0, t\} \in e_2\} \times \{u_0\})$

Suppose $\Rightarrow$ $\neg\mathsf{Orientates}(d_0 \cup \{t \in v_1 \mid \{u_0, t\} \in e_2\} \times \{u_0\}, v_1, e_2)$

Use_def(Orientates) $\Rightarrow$ *Stat10* : $e_2 \cap \{\{x, y\} : x \in v_1, y \in v_1 \setminus \{x\}\} \neq \{\{p^{[1]}, p^{[2]}\} : p \in d_0 \cup \{t \in v_1 \mid \{u_0, t\} \in e_2\} \times \{u_0\} \mid p = [p^{[1]}, p^{[2]}]\}$ &
$e_2 \cap \{\{x, y\} : x \in v_0, y \in v_0 \setminus \{x\}\} = \{\{p^{[1]}, p^{[2]}\} : p \in d_0 \mid p = [p^{[1]}, p^{[2]}]\}$

Set_monot(*Stat10*) $\Rightarrow$ $\{\{p^{[1]}, p^{[2]}\} : p \in d_0 \mid p = [p^{[1]}, p^{[2]}]\} \subseteq \{\{p^{[1]}, p^{[2]}\} : p \in d_0 \cup \{t \in v_1 \mid \{u_0, t\} \in e_2\} \times \{u_0\} \mid p = [p^{[1]}, p^{[2]}]\}$

Suppose $\Rightarrow$ $\{\{x, y\} : x \in v_1, y \in v_1 \setminus \{x\}\} \not\supseteq \{\{x, y\} : x \in v_0, y \in v_0 \setminus \{x\}\}$

EQUAL $\Rightarrow$ *Stat11* : $\{\{x, y\} : x \in v_1, y \in v_1 \setminus \{x\}\} \not\supseteq \{\{x, y\} : x \in v_1 \setminus \{u_0\}, y \in v_1 \setminus \{u_0\} \setminus \{x\}\}$

$\langle q_2 \rangle \hookrightarrow$ *Stat11(Stat11*$\star$) $\Rightarrow$ *Stat12* : $q_2 \in \{\{x, y\} : x \in v_1 \setminus \{u_0\}, y \in v_1 \setminus \{u_0\} \setminus \{x\}\}$ & $q_2 \notin \{\{x, y\} : x \in v_1, y \in v_1 \setminus \{x\}\}$

$\langle x_3, y_3, x_3, y_3 \rangle \hookrightarrow$ *Stat12(Stat12*$\star$) $\Rightarrow$ false

Discharge $\Rightarrow$ AUTO

$\langle q_1 \rangle \hookrightarrow$ *Stat10(Stat10*$\star$) $\Rightarrow$ *Stat13* : $q_1 \notin \{\{p^{[1]}, p^{[2]}\} : p \in d_0 \mid p = [p^{[1]}, p^{[2]}]\}$ & $q_1 \notin e_2 \cap \{\{x, y\} : x \in v_0, y \in v_0 \setminus \{x\}\}$ &
$q_1 \in e_2 \cap \{\{x, y\} : x \in v_1, y \in v_1 \setminus \{x\}\} \neq q_1 \in \{\{p^{[1]}, p^{[2]}\} : p \in d_0 \cup \{t \in v_1 \mid \{u_0, t\} \in e_2\} \times \{u_0\} \mid p = [p^{[1]}, p^{[2]}]\}$

Suppose $\Rightarrow$ *Stat14* : $q_1 \in \{\{p^{[1]}, p^{[2]}\} : p \in d_0 \cup \{t \in v_1 \mid \{u_0, t\} \in e_2\} \times \{u_0\} \mid p = [p^{[1]}, p^{[2]}]\}$

$\langle p_1 \rangle \hookrightarrow$ *Stat14(Stat13*$\star$) $\Rightarrow$ *Stat15* : $\{p_1{}^{[1]}, p_1{}^{[2]}\} \notin \{\{p^{[1]}, p^{[2]}\} : p \in d_0 \mid p = [p^{[1]}, p^{[2]}]\}$ & $p_1 \in d_0 \cup \{t \in v_1 \mid \{u_0, t\} \in e_2\} \times \{u_0\}$ &
$p_1 = [p_1{}^{[1]}, p_1{}^{[2]}]$ & $q_1 = \{p_1{}^{[1]}, p_1{}^{[2]}\}$

$\langle p_1 \rangle \hookrightarrow$ *Stat15(Stat15*$\star$) $\Rightarrow$ $p_1 \in \{t \in v_1 \mid \{u_0, t\} \in e_2\} \times \{u_0\}$

$\langle p_1, \{t \in v_1 \mid \{u_0, t\} \in e_2\}, \{u_0\} \rangle \hookrightarrow$ *Tcartesian*$_0$(*Stat15*$\star$) $\Rightarrow$ *Stat16* : $p_1{}^{[1]} \in \{t \in v_1 \mid \{u_0, t\} \in e_2\}$ & $p_1{}^{[2]} = u_0$

$\langle\ \rangle \hookrightarrow$ *Stat16(Stat13*$\star$) $\Rightarrow$ *Stat17* : $\{p_1{}^{[1]}, u_0\} \notin \{\{x, y\} : x \in v_1, y \in v_1 \setminus \{x\}\}$ & $p_1{}^{[1]} \in v_1$ & $\{p_1{}^{[1]}, u_0\} \in e_2$

$\langle p_1{}^{[1]}, u_0 \rangle \hookrightarrow$ *Stat17(Stat17, Stat9, Stat2*$\star$) $\Rightarrow$ false

(*Stat17*$\star$)Discharge $\Rightarrow$ *Stat18* : $q_1 \in \{\{x, y\} : x \in v_1, y \in v_1 \setminus \{x\}\}$ & $q_1 \notin \{\{x, y\} : x \in v_0, y \in v_0 \setminus \{x\}\}$ &
Stat18a : $q_1 \notin \{\{p^{[1]}, p^{[2]}\} : p \in d_0 \cup \{t \in v_1 \mid \{u_0, t\} \in e_2\} \times \{u_0\} \mid p = [p^{[1]}, p^{[2]}]\}$ & $q_1 \in e_2$

$\langle x_4, y_4, x_4, y_4, [x_4, y_4] \rangle \hookrightarrow$ *Stat18(Stat18, Stat9*$\star$) $\Rightarrow$ *Stat19* : $[x_4, y_4] \notin \{t \in v_1 \mid \{u_0, t\} \in e_2\} \times \{u_0\}$ & $q_1 = \{x_4, y_4\}$ & $x_4 \neq y_4$ & $u_0 \in q_1$ & $x_4, y_4 \in v_1$

Suppose $\Rightarrow$ $y_4 = u_0$

$\langle [x_4, y_4], \{t \in v_1 \mid \{u_0, t\} \in e_2\}, \{u_0\} \rangle \hookrightarrow$ *Tcartesian*$_0$(*Stat19*$\star$) $\Rightarrow$ *Stat20* : $x_4 \notin \{t \in v_1 \mid \{u_0, t\} \in e_2\}$

$\langle x_4 \rangle \hookrightarrow$ *Stat20(Stat18*$\star$) $\Rightarrow$ false

(*Stat20*$\star$)Discharge $\Rightarrow$ AUTO

$\langle [y_4, x_4] \rangle \hookrightarrow$ *Stat18a(Stat18*$\star$) $\Rightarrow$ $[y_4, x_4] \notin \{t \in v_1 \mid \{u_0, t\} \in e_2\} \times \{u_0\}$

$\langle [y_4, x_4], \{t \in v_1 \mid \{u_0, t\} \in e_2\}, \{u_0\} \rangle \hookrightarrow$ *Tcartesian*$_0$(*Stat19*$\star$) $\Rightarrow$ *Stat21* : $y_4 \notin \{t \in v_1 \mid \{u_0, t\} \in e_2\}$

$\langle y_4 \rangle \hookrightarrow$ *Stat21(Stat18*$\star$) $\Rightarrow$ false

Discharge $\Rightarrow$ AUTO

Loc_def $\Rightarrow$ *Stat22* : $d_1 = d_0 \cup \{t \in v_1 \mid \{u_0, t\} \in e_2\} \times \{u_0\}$

EQUAL(*Stat9*) $\Rightarrow$ $\mathsf{Orientates}(d_1, v_1, e_2)$ & $\mathsf{Acyclic}(v_1, d_1)$ & $\mathsf{Extensional}(v_1, d_1)$

$\langle d_1 \rangle \hookrightarrow$ *Stat0(Stat22*$\star$) $\Rightarrow$ $d_0 \cup \{t \in v_1 \mid \{u_0, t\} \in e_2\} \times \{u_0\} \not\subseteq v_1 \times v_1$

$\langle v_0, v_1, v_0, v_1 \rangle \hookrightarrow$ *Tcartesian*$_5$(*Stat0*$\star$) $\Rightarrow$ $\{t \in v_1 \mid \{u_0, t\} \in e_2\} \times \{u_0\} \not\subseteq v_1 \times v_1$

Set_monot(*Stat22*) $\Rightarrow$ *Stat23* : $\{t \in v_1 \mid \{u_0, t\} \in e_2\} \subseteq \{t \in v_1 \mid \mathsf{true}\}$

$\langle \{t \in v_1 \mid \{u_0, t\} \in e_2\}, v_1, \{u_0\}, v_1 \rangle \hookrightarrow$ *Tcartesian*$_5$(*Stat0*$\star$) $\Rightarrow$ $\{t \in v_1 \mid \{u_0, t\} \in e_2\} \not\subseteq v_1$

$(\mathit{Stat23\star})$Discharge $\Rightarrow$ QED

3.2 Injective decoration of a weakly extensional and acyclic digraph

An outline of the construction needed for a decoration *à la* Mostowski is as follows:

- Send every sink w but one (because we must send one sink to $\emptyset$) to the set, $\{\{\mathsf{v}_0\} \cup (\mathsf{v}_0 \setminus \{\mathsf{w}\})\}$, whose rank is 2 bigger than the rank of v_0 and whose cardinality equals the cardinality of v_0.
- Extend mski_Θ consistently with the desideratum.
- Injectivity will follow from the fact that no two vertices have the same immediate successors, save for the sinks, which can neither collide with one another (because their images are, plainly, pairwise distinct) nor with internal nodes (because they differ from them for rank — as well as for cardinality).

THEORY finMostowskiDecoration$(\mathsf{v}_0, \mathsf{d}_0)$
$\mathsf{v}_0 \times \mathsf{v}_0 \supseteq \mathsf{d}_0 \;\&\; \mathsf{v}_0 \neq \emptyset \;\&\; \mathsf{Finite}(\mathsf{v}_0)$
$\mathsf{Acyclic}(\mathsf{v}_0, \mathsf{d}_0)$
$\mathsf{WExtensional}(\mathsf{v}_0, \mathsf{d}_0)$
END finMostowskiDecoration

ENTER_THEORY finMostowskiDecoration

THM finMostowskiDecoration$_0$: [Trivial inclusions between edge endpoints and vertices]
$\mathsf{Is\_map}(\mathsf{d}_0) \;\&\; \mathbf{dom}(\mathsf{d}_0) \subseteq \mathsf{v}_0 \;\&\; \mathbf{range}(\mathsf{d}_0) \subseteq \mathsf{v}_0 \;\&\; \langle \forall \mathsf{x} \in \mathbf{dom}(\mathsf{d}_0), \mathsf{y} \in \mathbf{dom}(\mathsf{d}_0), \exists \mathsf{z} \mid ([\mathsf{x},\mathsf{z}] \in \mathsf{d}_0 \leftrightarrow [\mathsf{y},\mathsf{z}] \in \mathsf{d}_0) \rightarrow \mathsf{x} = \mathsf{y} \rangle$. PROOF:
Suppose_not() $\Rightarrow$ AUTO
Assump $\Rightarrow$ $\mathsf{d}_0 \cap (\mathsf{v}_0 \times \mathsf{v}_0) = \mathsf{d}_0 \;\&\; \mathsf{v}_0 \neq \emptyset \;\&\; \mathsf{WExtensional}(\mathsf{v}_0, \mathsf{d}_0)$
$\langle \mathsf{v}_0 \times \mathsf{v}_0, \mathsf{d}_0, \mathsf{v}_0 \times \mathsf{v}_0 \rangle \hookrightarrow T\mathsf{range}_1$ $\Rightarrow$ AUTO
$\langle \mathsf{v}_0, \mathsf{v}_0 \rangle \hookrightarrow T\mathsf{cartesian}_1$ $\Rightarrow$ AUTO
$\langle \mathsf{v}_0 \times \mathsf{v}_0, \mathsf{d}_0, \mathsf{v}_0 \times \mathsf{v}_0 \rangle \hookrightarrow T\mathsf{domain}_1$ $\Rightarrow$ AUTO
$\langle \mathsf{v}_0, \mathsf{v}_0 \rangle \hookrightarrow T\mathsf{cartesian}_2$ $\Rightarrow$ AUTO
Use_def$(\mathsf{WExtensional}(\mathsf{v}_0, \mathsf{d}_0))$ $\Rightarrow$ AUTO
Use_def$(\mathsf{Extensional}(\mathbf{dom}(\mathsf{d}_0), \mathsf{d}_0))$ $\Rightarrow$ AUTO
ELEM $\Rightarrow$ $\mathsf{v}_0 \cap \mathbf{dom}(\mathsf{d}_0) = \mathbf{dom}(\mathsf{d}_0)$
EQUAL $\Rightarrow$ $\langle \forall \mathsf{x} \in \mathbf{dom}(\mathsf{d}_0), \mathsf{y} \in \mathbf{dom}(\mathsf{d}_0), \exists \mathsf{z} \mid ([\mathsf{x},\mathsf{z}] \in \mathsf{d}_0 \leftrightarrow [\mathsf{y},\mathsf{z}] \in \mathsf{d}_0) \rightarrow \mathsf{x} = \mathsf{y} \rangle$
$\langle \mathsf{d}_0, \mathsf{v}_0, \mathsf{v}_0 \rangle \hookrightarrow T\mathsf{cartesian}_6$ $\Rightarrow$ AUTO
Discharge $\Rightarrow$ QED

THM finMostowskiDecoration$_1$: [No self-loops in an acyclic digraph] $W \notin \mathbf{range}(d_{0|\{W\}})$ & $\mathbf{range}(d_{0|\{W\}}) \subseteq v_0$. PROOF:

Suppose_not$(w_0) \Rightarrow$ AUTO

Use_def$(|) \Rightarrow$ *Stat1* : $d_{0|\{w_0\}} = \{p \in d_0 \mid p^{[1]} \in \{w_0\}\}$

Suppose $\Rightarrow$ $w_0 \in \mathbf{range}(d_{0|\{w_0\}})$

Use_def$(\mathbf{range}(d_{0|\{w_0\}})) \Rightarrow$ AUTO

EQUAL $\Rightarrow$ $w_0 \in \{q^{[2]} : q \in \{p \in d_0 \mid p^{[1]} \in \{w_0\}\}\}$

SIMPLF $\Rightarrow$ *Stat2* : $w_0 \in \{p^{[2]} : p \in d_0 \mid p^{[1]} \in \{w_0\}\}$

$\langle p_0 \rangle \hookrightarrow$ *Stat2*(*Stat2*$\star$) $\Rightarrow$ $p_0 \in d_0$ & $p_0{}^{[1]} = p_0{}^{[2]}$

Assump $\Rightarrow$ Acyclic(v_0, d_0) & $v_0 \times v_0 \supseteq d_0$

$\langle v_0, d_0, p_0{}^{[2]}, p_0{}^{[1]} \rangle \hookrightarrow T$acyclicity$_2$(*Stat2*$\star$) $\Rightarrow$ *Stat3* : $\neg(p_0{}^{[1]} \in v_0$ & $[p_0{}^{[1]}, p_0{}^{[2]}] \in d_0)$

Suppose $\Rightarrow$ $p_0{}^{[1]} \notin v_0$

Use_def$(\mathbf{dom}(d_0)) \Rightarrow$ AUTO

$\langle\,\rangle \hookrightarrow T$finMostowskiDecoration$_0$(*Stat3*$\star$) $\Rightarrow$ *Stat4* : $p_0{}^{[1]} \notin \{p^{[1]} : p \in d_0\}$

$\langle p_0 \rangle \hookrightarrow$ *Stat4*(*Stat2*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO

(*Stat2*$\star$)ELEM $\Rightarrow$ $p_0 \neq [p_0{}^{[1]}, p_0{}^{[2]}]$

$\langle p_0, v_0, v_0 \rangle \hookrightarrow T$cartesian$_0$(*Stat2*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO

Set_monot $\Rightarrow$ $\{p \in d_0 \mid p^{[1]} \in \{w_0\}\} \subseteq \{p : p \in d_0\}$

$\langle d_0, d_{0|\{w_0\}}, d_0 \rangle \hookrightarrow T$range$_1$(*Stat1*$\star$) $\Rightarrow$ $\mathbf{range}(d_{0|\{w_0\}}) \subseteq \mathbf{range}(d_0)$

$\langle\,\rangle \hookrightarrow T$finMostowskiDecoration$_0(\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

> The following function assigns $\emptyset$ to every set which is not a sink of the current graph; also a designated sink is assigned the image $\emptyset$. Special non-zero sets are assigned to the sinks other than the designated one.

DEF finMostowskiDecoration$_1$: [Non-recursive part of labeling, used to enforce one-oneness]

$$\mathsf{lbl}(W) =_{\text{Def}} \textbf{if } W \in \mathbf{dom}(d_0) \cup \{\mathbf{arb}(v_0 \setminus \mathbf{dom}(d_0))\} \textbf{ then } \emptyset \textbf{ else } \{\{v_0\} \cup (v_0 \setminus \{W\})\} \textbf{ fi}$$

APPLY $\langle \mathsf{lab}_\Theta : \mathsf{mski}_\Theta \rangle$ finAcycLabeling$(v_0 \mapsto v_0, d_0 \mapsto d_0, h(s,x) \mapsto s \cup \mathsf{lbl}(x)) \Rightarrow$

THM finMostowskiDecoration$_{2a}$. $\mathsf{Svm}(\mathsf{mski}_\Theta)$ & $\mathbf{dom}(\mathsf{mski}_\Theta) = v_0$ & $\langle \forall x \in v_0 \mid \mathsf{mski}_\Theta \restriction x = \{\mathsf{mski}_\Theta \restriction p^{[2]} : p \in d_{0|\{x\}} \mid p^{[2]} \in v_0\} \cup \mathsf{lbl}(x) \rangle$.

THM finMostowskiDecoration$_2$. $\mathsf{Svm}(\mathsf{mski}_\Theta)$ & $\mathbf{dom}(\mathsf{mski}_\Theta) = v_0$. PROOF:

Suppose_not$() \Rightarrow$ AUTO

$\langle\,\rangle \hookrightarrow T$finMostowskiDecoration$_{2a}$ $\Rightarrow$ false; Discharge $\Rightarrow$ QED

THM finMostowskiDecoration$_3$: [Images of internal vertices under Mostowski's decoration]

$W \in \mathbf{dom}(d_0) \rightarrow \mathsf{mski}_\Theta \restriction W = \{\mathsf{mski}_\Theta \restriction p^{[2]} : p \in d_{0|\{W\}}\}$ & $\mathsf{mski}_\Theta \restriction W \neq \emptyset$. PROOF:

Suppose_not(w_0) $\Rightarrow$ AUTO

Suppose $\Rightarrow$ $\text{mski}_\Theta \upharpoonright w_0 \neq \{\text{mski}_\Theta \upharpoonright p^{[2]} : p \in d_{0|\{w_0\}}\} \cup \text{lbl}(w_0)$

$\langle\,\rangle \hookrightarrow T$finMostowskiDecoration$_{2a}(\star)$ $\Rightarrow$ *Stat1* : $\langle \forall x \in v_0 \mid \text{mski}_\Theta \upharpoonright x = \{\text{mski}_\Theta \upharpoonright p^{[2]} : p \in d_{0|\{x\}} \mid p^{[2]} \in v_0\} \cup \text{lbl}(x)\rangle$

$\langle\,\rangle \hookrightarrow T$finMostowskiDecoration$_0(\star)$ $\Rightarrow$ $\mathbf{dom}(d_0) \subseteq v_0$

$\langle w_0\rangle \hookrightarrow$ *Stat1*$(\star)$ $\Rightarrow$ *Stat2* : $\{\text{mski}_\Theta \upharpoonright p^{[2]} : p \in d_{0|\{w_0\}} \mid p^{[2]} \in v_0\} \neq \{\text{mski}_\Theta \upharpoonright p^{[2]} : p \in d_{0|\{w_0\}}\}$

$\langle p_0\rangle \hookrightarrow$ *Stat2*(*Stat2*$\star$) $\Rightarrow$ $p_0 \in d_{0|\{w_0\}}$ & $p_0{}^{[2]} \notin v_0$

Use_def(|) $\Rightarrow$ *Stat3* : $p_0 \in \{p \in d_0 \mid p^{[1]} \in \{w_0\}\}$

Assump $\Rightarrow$ $v_0 \times v_0 \supseteq d_0$

$\langle\,\rangle \hookrightarrow$ *Stat3*(*Stat3*$\star$) $\Rightarrow$ *Stat4* : $p_0 \in v_0 \times v_0$

$\langle p_0, v_0, v_0\rangle \hookrightarrow T$cartesian$_0$(*Stat4*$\star$) $\Rightarrow$ $p_0{}^{[2]} \in v_0$

(*Stat2*$\star$)Discharge $\Rightarrow$ AUTO

Use_def(lbl) $\Rightarrow$ $\text{mski}_\Theta \upharpoonright w_0 = \{\text{mski}_\Theta \upharpoonright p^{[2]} : p \in d_{0|\{w_0\}}\}$

Use_def(**dom**) $\Rightarrow$ *Stat6* : $w_0 \in \{p^{[1]} : p \in d_0\}$ & $\{\text{mski}_\Theta \upharpoonright p^{[2]} : p \in d_{0|\{w_0\}}\} = \emptyset$

Use_def(|) $\Rightarrow$ $d_{0|\{w_0\}} = \{p \in d_0 \mid p^{[1]} \in \{w_0\}\}$

$\langle p_1, p_1\rangle \hookrightarrow$ *Stat6*(*Stat6*$\star$) $\Rightarrow$ *Stat7* : $p_1 \notin \{p \in d_0 \mid p^{[1]} \in \{w_0\}\}$ & $w_0 = p_1{}^{[1]}$ & $p_1 \in d_0$

$\langle p_1\rangle \hookrightarrow$ *Stat7*(*Stat7*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ QED

THM finMostowskiDecoration$_4$: [Images of sinks under Mostowski's decoration] $W \in v_0 \backslash \mathbf{dom}(d_0) \rightarrow \text{mski}_\Theta \upharpoonright W = \text{lbl}(W)$ & $\text{mski}_\Theta \upharpoonright W \notin \{\{v_0\} \cup (v_0 \backslash \{\emptyset\})\}$. PROOF:

Suppose_not(w_0) $\Rightarrow$ AUTO

Suppose $\Rightarrow$ $\text{mski}_\Theta \upharpoonright w_0 \neq \text{lbl}(w_0)$

$\langle\,\rangle \hookrightarrow T$finMostowskiDecoration$_{2a}$ $\Rightarrow$ *Stat0* : $\langle \forall x \in v_0 \mid \text{mski}_\Theta \upharpoonright x = \{\text{mski}_\Theta \upharpoonright p^{[2]} : p \in d_{0|\{x\}} \mid p^{[2]} \in v_0\} \cup \text{lbl}(x)\rangle$

$\langle w_0\rangle \hookrightarrow$ *Stat0*$(\star)$ $\Rightarrow$ *Stat1* : $\{\text{mski}_\Theta \upharpoonright p^{[2]} : p \in d_{0|\{w_0\}} \mid p^{[2]} \in v_0\} \neq \emptyset$

Use_def(|) $\Rightarrow$ $d_{0|\{w_0\}} = \{p \in d_0 \mid p^{[1]} \in \{w_0\}\}$

Use_def($\mathbf{dom}(d_0)$) $\Rightarrow$ AUTO

$\langle p_0\rangle \hookrightarrow$ *Stat1*$(\star)$ $\Rightarrow$ *Stat2* : $p_0 \in \{p : p \in d_0 \mid p^{[1]} \in \{w_0\}\}$ & $w_0 \notin \{q^{[1]} : q \in d_0\}$

$\langle p_1, p_1\rangle \hookrightarrow$ *Stat2*(*Stat2*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO

‖ Recall: $\text{lbl}(W) =$ **if** $W \in \mathbf{dom}(d_0) \cup \{\mathbf{arb}(v_0 \backslash \mathbf{dom}(d_0))\}$ **then** $\emptyset$ **else** $\{\{v_0\} \cup (v_0 \backslash \{W\})\}$ **fi**

Assump $\Rightarrow$ $v_0 \neq \emptyset$

Use_def(lbl) $\Rightarrow$ false; Discharge $\Rightarrow$ QED

THM finMostowskiDecoration$_5$: [One image under Mostowski's decoration is null] $\emptyset \in \mathbf{range}(\text{mski}_\Theta)$. PROOF:

‖ In fact from $v_0 \neq \emptyset$, after putting $a_0 = \mathbf{arb}(v_0 \backslash \mathbf{dom}(d_0))$, we will get $a_0 \in v_0$, $\text{lbl}(a_0) = \emptyset$,
‖ $\text{mski}_\Theta \upharpoonright a_0 = \emptyset$.

Suppose_not() ⇒ AUTO

Use_def(lbl) ⇒ $\mathsf{lbl}(\mathbf{arb}(v_0 \setminus \mathbf{dom}(d_0))) = \emptyset$

Loc_def ⇒ $a_0 = \mathbf{arb}(v_0 \setminus \mathbf{dom}(d_0))$

EQUAL ⇒ $\mathsf{lbl}(a_0) = \emptyset$

Use_def($\mathbf{dom}(d_0)$) ⇒ AUTO

Suppose ⇒ $v_0 \subseteq \mathbf{dom}(d_0)$

Assump ⇒ $\mathsf{Acyclic}(v_0, d_0)\ \&\ \mathsf{Finite}(v_0)\ \&\ v_0 \neq \emptyset\ \&\ v_0 \times v_0 \supseteq d_0$

$\langle v_0, d_0 \rangle \hookrightarrow T\mathsf{acyclicity}_4(\star)$ ⇒ $Stat1:\ \langle \exists w \in v_0, s \in v_0 \mid \emptyset = \{y \in v_0 \mid [w,y] \in d_0 \vee [y,s] \in d_0\} \rangle$

$\langle w_0, s_0 \rangle \hookrightarrow Stat1(\star)$ ⇒ $Stat2:\ w_0 \in \{p^{[1]} : p \in d_0\}\ \&\ \{y \in v_0 \mid [w_0,y] \in d_0 \vee [y,s_0] \in d_0\} = \emptyset$

$\langle p_0, p_0{}^{[2]} \rangle \hookrightarrow Stat2(Stat2\star)$ ⇒ $w_0 = p_0{}^{[1]}\ \&\ p_0 \in d_0\ \&\ p_0{}^{[2]} \notin v_0 \vee [w_0, p_0{}^{[2]}] \notin d_0$

EQUAL($Stat2$) ⇒ $p_0 \in d_0\ \&\ p_0{}^{[2]} \notin v_0 \vee [p_0{}^{[1]}, p_0{}^{[2]}] \notin d_0$

$\langle\ \rangle \hookrightarrow T\mathsf{finMostowskiDecoration}_0(\star)$ ⇒ $\mathbf{range}(d_0) \subseteq v_0$

Use_def($\mathbf{range}(d_0)$) ⇒ AUTO

$\langle p_0, v_0, v_0 \rangle \hookrightarrow T\mathsf{cartesian}_0(\star)$ ⇒ $Stat3:\ p_0{}^{[2]} \notin \{p^{[2]} : p \in d_0\}$

$\langle p_0 \rangle \hookrightarrow Stat3(Stat2\star)$ ⇒ false; Discharge ⇒ $Stat4:\ a_0 \in v_0\ \&\ a_0 \notin \mathbf{dom}(d_0)$

$\langle a_0 \rangle \hookrightarrow T\mathsf{finMostowskiDecoration}_4(\star)$ ⇒ $\mathsf{mski}_\Theta \restriction a_0 = \emptyset$

Use_def($\mathbf{range}$) ⇒ $Stat7:\ \mathsf{mski}_\Theta \restriction a_0 \notin \{p^{[2]} : p \in \mathsf{mski}_\Theta\}$

$\langle [a_0, \mathsf{mski}_\Theta \restriction a_0] \rangle \hookrightarrow Stat7(Stat7)$ ⇒ $[a_0, \mathsf{mski}_\Theta \restriction a_0] \notin \mathsf{mski}_\Theta$

$\langle \mathsf{mski}_\Theta \rangle \hookrightarrow T\mathsf{image}_5$ ⇒ AUTO

$\langle\ \rangle \hookrightarrow T\mathsf{finMostowskiDecoration}_2(Stat4\star)$ ⇒ $Stat8:\ [a_0, \mathsf{mski}_\Theta \restriction a_0] \notin \{[x, \mathsf{mski}_\Theta \restriction x] : x \in \mathbf{dom}(\mathsf{mski}_\Theta)\}\ \&\ a_0 \in \mathbf{dom}(\mathsf{mski}_\Theta)$

$\langle a_0 \rangle \hookrightarrow Stat8(Stat8\star)$ ⇒ false; Discharge ⇒ QED

THM finMostowskiDecoration$_6$: [Finiteness of each image of Mostowski's decoration] $Y \in \mathbf{range}(\mathsf{mski}_\Theta) \rightarrow \mathsf{Finite}(Y)$. PROOF:

Suppose_not(y_0) ⇒ AUTO

> In fact, each $\mathsf{lbl}(w)$ is either $\emptyset$ or has as many elements as v_0, which we have supposed to be finite; moreover, each $\{\mathsf{mski}_\Theta \restriction p^{[2]} : p \in d_{0|\{x\}} \mid p^{[2]} \in v_0\}$ with $x \in v_0$ has fewer elements than v_0; consequently each $\mathsf{mski}_\Theta \restriction x$ with $x \in v_0$ must be finite, because it results from the union of two finite sets.

Use_def($\mathbf{range}(\mathsf{mski}_\Theta)$) ⇒ AUTO

EQUAL ⇒ $Stat1:\ y_0 \in \{p^{[2]} : p \in \mathsf{mski}_\Theta\}\ \&\ \neg\mathsf{Finite}(y_0)$

$\langle p_0 \rangle \hookrightarrow Stat1(Stat1\star)$ ⇒ $p_0 \in \mathsf{mski}_\Theta\ \&\ y_0 = p_0{}^{[2]}$

$\langle\ \rangle \hookrightarrow T\mathsf{finMostowskiDecoration}_{2a}(\star)$ ⇒ $Stat2:\ \langle \forall x \in v_0 \mid \mathsf{mski}_\Theta \restriction x = \{\mathsf{mski}_\Theta \restriction p^{[2]} : p \in d_{0|\{x\}} \mid p^{[2]} \in v_0\} \cup \mathsf{lbl}(x) \rangle\ \&\ \mathsf{Svm}(\mathsf{mski}_\Theta)\ \&\ \mathbf{dom}(\mathsf{mski}_\Theta) = v_0$

$\langle \mathsf{mski}_\Theta \rangle \hookrightarrow T\mathsf{image}_5(\star)$ ⇒ $Stat3:\ p_0 \in \{[x, \mathsf{mski}_\Theta \restriction x] : x \in \mathbf{dom}(\mathsf{mski}_\Theta)\}$

$\langle x_0 \rangle \hookrightarrow Stat3(Stat2\star)$ ⇒ $x_0 \in v_0\ \&\ p_0 = [x_0, \mathsf{mski}_\Theta \restriction x_0]$

$\langle x_0 \rangle \hookrightarrow Stat2(Stat3\star)$ ⇒ $\mathsf{mski}_\Theta \restriction x_0 = \{\mathsf{mski}_\Theta \restriction p^{[2]} : p \in d_{0|\{x_0\}} \mid p^{[2]} \in v_0\} \cup \mathsf{lbl}(x_0)$

TELEM ⇒ $[x_0, \mathsf{mski}_\Theta \restriction x_0]^{[1]} = x_0\ \&\ [x_0, \mathsf{mski}_\Theta \restriction x_0]^{[2]} = \mathsf{mski}_\Theta \restriction x_0$

EQUAL($Stat1$) ⇒ $Stat4:\ \neg\mathsf{Finite}(\{\mathsf{mski}_\Theta \restriction p^{[2]} : p \in d_{0|\{x_0\}} \mid p^{[2]} \in v_0\} \cup \mathsf{lbl}(x_0))$

Suppose ⇒ $\text{Finite}(\{\text{mski}_\Theta\upharpoonright p^{[2]} : p \in d_{0|\{x_0\}} \mid p^{[2]} \in v_0\})$
Use_def(lbl) ⇒ $Stat5: \text{lbl}(x_0) = \textbf{if } x_0 \in \textbf{dom}(d_0) \cup \{\textbf{arb}(v_0 \setminus \textbf{dom}(d_0))\} \textbf{ then } \emptyset \textbf{ else } \{\{v_0\} \cup (v_0 \setminus \{x_0\})\} \textbf{ fi}$
Suppose ⇒ $x_0 \in \textbf{dom}(d_0) \cup \{\textbf{arb}(v_0 \setminus \textbf{dom}(d_0))\}$
$(Stat5\star)$ELEM ⇒ $\{\text{mski}_\Theta\upharpoonright p^{[2]} : p \in d_{0|\{x_0\}} \mid p^{[2]} \in v_0\} \cup \text{lbl}(x_0) = \{\text{mski}_\Theta\upharpoonright p^{[2]} : p \in d_{0|\{x_0\}} \mid p^{[2]} \in v_0\}$
EQUAL$(Stat4)$ ⇒ false; Discharge ⇒ AUTO
$(Stat5\star)$ELEM ⇒ $\text{lbl}(x_0) = \{\{v_0\} \cup (v_0 \setminus \{x_0\})\}$
$\langle \{\text{mski}_\Theta\upharpoonright p^{[2]} : p \in d_{0|\{x_0\}} \mid p^{[2]} \in v_0\}, \{v_0\} \cup (v_0 \setminus \{x_0\})\rangle \hookrightarrow T\text{fin}_1(Stat4\star)$ ⇒ $\text{Finite}(\{\text{mski}_\Theta\upharpoonright p^{[2]} : p \in d_{0|\{x_0\}} \mid p^{[2]} \in v_0\} \cup \{\{v_0\} \cup (v_0 \setminus \{x_0\})\})$
EQUAL$(Stat4)$ ⇒ false; Discharge ⇒ AUTO
Assump ⇒ $v_0 \times v_0 \supseteq d_0 \;\&\; \text{Finite}(v_0)$
Suppose ⇒ $d_{0|\{x_0\}} \not\subseteq \{[x_0, y] : y \in v_0\}$
Use_def(|) ⇒ $Stat6: \{p \in d_0 \mid p^{[1]} \in \{x_0\}\} \not\subseteq \{[x_0, y] : y \in v_0\}$
$\langle p_1\rangle \hookrightarrow Stat6(Stat6\star)$ ⇒ $Stat7: p_1 \in \{p \in d_0 \mid p^{[1]} \in \{x_0\}\} \;\&\; Stat7a: p_1 \notin \{[x_0, y] : y \in v_0\}$
$\langle\,\rangle \hookrightarrow Stat7(Stat4\star)$ ⇒ $Stat8: p_1 \in v_0 \times v_0 \;\&\; p_1^{[1]} = x_0$
$\langle p_1, v_0, v_0\rangle \hookrightarrow T\text{cartesian}_0(Stat8)$ ⇒ $p_1 = [x_0, p_1^{[2]}] \;\&\; p_1^{[2]} \in v_0$
$\langle p_1^{[2]}\rangle \hookrightarrow Stat7a(Stat8\star)$ ⇒ false; Discharge ⇒ AUTO
APPLY $\langle\,\rangle$ finiteImage$(s_0 \mapsto v_0, f(X) \mapsto [x_0, X])$ ⇒ $\text{Finite}(\{[x_0, y] : y \in v_0\})$
$\langle \{[x_0, y] : y \in v_0\}, d_{0|\{x_0\}}\rangle \hookrightarrow T\text{fin}_0(Stat4\star)$ ⇒ $\text{Finite}(d_{0|\{x_0\}})$
APPLY $\langle\,\rangle$ finiteImage$(s_0 \mapsto d_{0|\{x_0\}}, f(X) \mapsto \text{mski}_\Theta\upharpoonright X^{[2]})$ ⇒ $\text{Finite}(\{\text{mski}_\Theta\upharpoonright p^{[2]} : p \in d_{0|\{x_0\}}\})$
Set_monot ⇒ $\{\text{mski}_\Theta\upharpoonright p^{[2]} : p \in d_{0|\{x_0\}} \mid p^{[2]} \in v_0\} \subseteq \{\text{mski}_\Theta\upharpoonright p^{[2]} : p \in d_{0|\{x_0\}}\}$
$\langle \{\text{mski}_\Theta\upharpoonright p^{[2]} : p \in d_{0|\{x_0\}}\}, \{\text{mski}_\Theta\upharpoonright p^{[2]} : p \in d_{0|\{x_0\}} \mid p^{[2]} \in v_0\}\rangle \hookrightarrow T\text{fin}_0(Stat4\star)$ ⇒ false; Discharge ⇒ QED

THM finMostowskiDecoration$_7$: [Under Mostowski's decoration, no image belongs to the image of a sink] $\{X, Y\} \subseteq v_0 \;\&\; \text{mski}_\Theta\upharpoonright Y \in \text{mski}_\Theta\upharpoonright X \to X \in \textbf{dom}(d_0)$. PROOF:
Suppose_not(w_0, w_1) ⇒ AUTO

> Arguing by contradiction, assume that w_0, w_1 is a counterexample; then w_0 must be a sink, but w_1 must not, else we would readily get a contradiction.

Use_def$(\text{lbl}(w_0))$ ⇒ AUTO
$\langle w_0\rangle \hookrightarrow T\text{finMostowskiDecoration}_4(\star)$ ⇒ $Stat1: w_1 \in v_0 \;\&\; \text{mski}_\Theta\upharpoonright w_1 = \{v_0\} \cup (v_0 \setminus \{w_0\})$
Suppose ⇒ $w_1 \notin \textbf{dom}(d_0)$
$\langle w_1\rangle \hookrightarrow T\text{finMostowskiDecoration}_4(Stat1\star)$ ⇒ $\text{lbl}(w_1) = \{v_0\} \cup (v_0 \setminus \{w_0\})$
Use_def$(\text{lbl}(w_1))$ ⇒ AUTO
$(Stat1\star)$Discharge ⇒ AUTO

Hence suppose that w_1, unlike w_0, is not a sink. Can then $lbl(w_0) = mski_\Theta \upharpoonright w_1$ hold? If this were the case, then

$$\left\{mski_\Theta \upharpoonright p^{[2]} : p \in d_{0|\{w_1\}}\right\} = \{\{v_0\} \cup (v_0 \setminus \{w_0\})\} .$$

$\langle w_1 \rangle \hookrightarrow T$finMostowskiDecoration$_3(\star) \Rightarrow$ $\quad Stat2: \left\{mski_\Theta \upharpoonright p^{[2]} : p \in d_{0|\{w_1\}}\right\} = \{v_0\} \cup (v_0 \setminus \{w_0\})$ & $w_1 \in \mathbf{dom}(d_0)$

Loc_def $\Rightarrow$ $\quad h_0 = \{[y, mski_\Theta \upharpoonright y] : y \in \mathbf{range}(d_{0|\{w_1\}})\}$

The following TELEM step makes implicit use of the THEORY isSvm seen at the beginning of this scenario.

TELEM $\Rightarrow$ $\quad Svm(\{[y, mski_\Theta \upharpoonright y] : y \in \mathbf{range}(d_{0|\{w_1\}})\})$ &
$\mathbf{range}(\{[y, mski_\Theta \upharpoonright y] : y \in \mathbf{range}(d_{0|\{w_1\}})\}) = \{mski_\Theta \upharpoonright y : y \in \mathbf{range}(d_{0|\{w_1\}})\}$ & $\mathbf{dom}(\{[y, mski_\Theta \upharpoonright y] : y \in \mathbf{range}(d_{0|\{w_1\}})\}) = \mathbf{range}(d_{0|\{w_1\}})$

The intuitive idea is that since $\mathbf{range}(d_{0|\{w_1\}})$ is a proper subset of v_0 (in fact $w_1 \notin \mathbf{range}(d_{0|\{w_1\}})$) and $\{v_0\} \cup (v_0 \setminus \{w_0\})$ has the same cardinality as v_0, the surjectivity of h_0 is untenable in view of the finiteness of v_0.

Use_def($\mathbf{range}(d_{0|\{w_1\}})$) $\Rightarrow$ $\quad$ AUTO

SIMPLF($Stat2$) $\Rightarrow$ $\quad \left\{mski_\Theta \upharpoonright y : y \in \left\{p^{[2]} : p \in d_{0|\{w_1\}}\right\}\right\} = \left\{mski_\Theta \upharpoonright p^{[2]} : p \in d_{0|\{w_1\}}\right\}$

$\langle w_1 \rangle \hookrightarrow T$finMostowskiDecoration$_1(Stat3\star) \Rightarrow$ $\quad Stat3: \mathbf{range}(d_{0|\{w_1\}}) \subseteq v_0$

EQUAL($Stat2$) $\Rightarrow$ $\quad Svm(h_0)$ & $\mathbf{range}(h_0) = \{v_0\} \cup (v_0 \setminus \{w_0\})$ & $\mathbf{dom}(h_0) \subseteq v_0$

Assump $\Rightarrow$ $\quad Finite(v_0)$ & $Acyclic(v_0, d_0)$ & $v_0 \neq \emptyset$ & $v_0 \times v_0 \supseteq d_0$

$\langle v_0, \mathbf{dom}(h_0) \rangle \hookrightarrow Tfin_0(Stat3\star) \Rightarrow$ $\quad Finite(\mathbf{dom}(h_0))$

However, since we have not developed a rich enough theory of cardinals within this scenario, we prefer to formalize the argument-by-contradiction outlined above in the following slicker (albeit slightly less intuitive) terms. We easily discard the case when $w_0 \notin \mathbf{dom}(h_0)$: in this case $\mathbf{dom}(h_0) \subseteq \mathbf{range}(h_0)$ and we can resort to THM part_whole$_1$ to get $\mathbf{range}(h_0) = \mathbf{dom}(h_0)$; but this conflicts with $v_0 \in \mathbf{range}(h_0)$ and $\mathbf{dom}(h_0) \subseteq v_0$.

$\langle h_0 \rangle \hookrightarrow T$part_whole$_0(Stat3\star) \Rightarrow$ $\quad Stat5: Finite(h_0)$

Use_def($\mathbf{dom}(h_0)$) $\Rightarrow$ $\quad$ AUTO

$\langle h_0 \rangle \hookrightarrow T$part_whole$_1(Stat3\star) \Rightarrow$ $\quad Stat6: w_0 \in \left\{p^{[1]} : p \in h_0\right\}$

On the other hand, if $w_0 \in \mathbf{dom}(h_0)$ then we can retouch h_0 by replacing its pair $[w_0, h_0 \upharpoonright w_0]$ by $[w_1, h_0 \upharpoonright w_0]$. Since $w_1 \notin \mathbf{range}(d_{0|\{w_1\}})$, the resulting single-valued map h_1 will have the same range as the original h_0 and will be finite, much like h_0. It will satisfy $\mathbf{dom}(h_1) \subseteq \mathbf{range}(h_1)$, making us able to derive $\mathbf{dom}(h_1) \subseteq v_0$ via THM part_whole$_1$, and leading us to a contradiction again.

$\langle p_0 \rangle \hookrightarrow Stat6(Stat6\star) \Rightarrow \quad p_0 \in h_0 \;\&\; w_0 = p_0^{[1]}$
$\langle h_0, p_0 \rangle \hookrightarrow Timage_4(Stat3\star) \Rightarrow \quad p_0 = [p_0^{[1]}, h_0 \upharpoonright p_0^{[1]}]$
Loc_def $\Rightarrow$ $\quad Stat7: \; y_0 = h_0 \upharpoonright p_0^{[1]} \;\&\; h_1 = h_0 \setminus \{[w_0, y_0]\} \cup \{[w_1, y_0]\}$
$\langle h_0 \setminus \{[w_0, y_0]\}, [w_1, y_0] \rangle \hookrightarrow Tfin_1(Stat5, Stat5) \Rightarrow \quad \mathsf{Finite}(h_0 \setminus \{[w_0, y_0]\} \cup \{[w_1, y_0]\})$
EQUAL$(Stat2) \Rightarrow \quad [w_0, y_0] \in h_0 \;\&\; \mathsf{Finite}(h_1) \;\&\; \mathbf{dom}(h_0) = \mathbf{range}(d_{0|\{w_1\}})$
$\langle w_1 \rangle \hookrightarrow T\mathsf{finMostowskiDecoration}_1(Stat1\star) \Rightarrow \quad w_1 \in v_0 \setminus \{w_0\} \setminus \mathbf{dom}(h_0)$
$\langle h_0, w_0, y_0, w_1, h_1 \rangle \hookrightarrow T\mathsf{singletonMap}_3(Stat3\star) \Rightarrow \quad \mathsf{Svm}(h_1) \;\&\; \mathbf{dom}(h_1) \subseteq v_0 \setminus \{w_0\} \;\&\; \mathbf{range}(h_1) = \{v_0\} \cup (v_0 \setminus \{w_0\})$
$\langle h_1 \rangle \hookrightarrow T\mathsf{part\_whole}_1(Stat7\star) \Rightarrow \quad \mathsf{false}; \qquad$ Discharge $\Rightarrow$ QED

THM finMostowskiDecoration$_8$: [Injectivity of Mostowski's decoration over the sinks] $\{X, Y\} \subseteq v_0 \;\&\; X \notin \mathbf{dom}(d_0) \;\&\; \mathsf{mski}_\Theta \upharpoonright X = \mathsf{mski}_\Theta \upharpoonright Y \to X = Y$. PROOF:
Suppose_not$(x_0, x_1) \Rightarrow$ AUTO

> Recalling that $\mathsf{lbl}(W) = \textbf{if } W \in \mathbf{dom}(d_0) \cup \{\mathbf{arb}(v_0 \setminus \mathbf{dom}(d_0))\} \textbf{ then } \emptyset \textbf{ else } \{\{v_0\} \cup (v_0 \setminus \{W\})\} \textbf{ fi}$,
> we readily get that distinct sinks x_0, x_1 cannot be sent to the same value by lbl.

Suppose $\Rightarrow \quad x_1 \notin \mathbf{dom}(d_0)$
$\langle x_0 \rangle \hookrightarrow T\mathsf{finMostowskiDecoration}_4(\star) \Rightarrow \quad \mathsf{mski}_\Theta \upharpoonright x_1 = \mathsf{lbl}(x_0)$
$\langle x_1 \rangle \hookrightarrow T\mathsf{finMostowskiDecoration}_4(\star) \Rightarrow \quad Stat1: \; \mathsf{lbl}(x_0) = \mathsf{lbl}(x_1) \;\&\; x_0 \in v_0 \;\&\; \{x_0, x_1\} \cap \mathbf{dom}(d_0) = \emptyset \;\&\; x_0 \neq x_1$
Use_def(lbl) $\Rightarrow \quad \mathsf{false}; \qquad$ Discharge $\Rightarrow$ AUTO
Use_def$(\mathsf{lbl}(x_0)) \Rightarrow$ AUTO
$\langle x_1 \rangle \hookrightarrow T\mathsf{finMostowskiDecoration}_3(\star) \Rightarrow \quad Stat2: \; \{\mathsf{mski}_\Theta \upharpoonright p^{[2]} : p \in d_{0|\{x_1\}}\} \neq \emptyset \;\&\; \mathsf{mski}_\Theta \upharpoonright x_0 = \{\mathsf{mski}_\Theta \upharpoonright p^{[2]} : p \in d_{0|\{x_1\}}\}$
$\langle p_0 \rangle \hookrightarrow Stat2(Stat2\star) \Rightarrow \quad p_0 \in d_{0|\{x_1\}}$
Suppose $\Rightarrow \quad Stat3: \; \mathsf{mski}_\Theta \upharpoonright p_0^{[2]} \notin \{\mathsf{mski}_\Theta \upharpoonright p^{[2]} : p \in d_{0|\{x_1\}}\}$
$\langle p_0 \rangle \hookrightarrow Stat3(Stat2\star) \Rightarrow \quad \mathsf{false}; \qquad$ Discharge $\Rightarrow$ AUTO
$\langle x_0, p_0^{[2]} \rangle \hookrightarrow T\mathsf{finMostowskiDecoration}_7(\star) \Rightarrow \quad Stat4: \; p_0^{[2]} \notin v_0$
Assump $\Rightarrow \quad d_0 \subseteq v_0 \times v_0$
$\langle d_0, \{x_1\} \rangle \hookrightarrow T\mathsf{restr}_0(Stat2\star) \Rightarrow \quad p_0 \in v_0 \times v_0$
$\langle p_0, v_0, v_0 \rangle \hookrightarrow T\mathsf{cartesian}_0(Stat4\star) \Rightarrow \quad p_0^{[2]} \in v_0$
Discharge $\Rightarrow$ QED

THM finMostowskiDecoration$_9$: [Injectivity of Mostowski's decoration] $\{X, Y\} \subseteq v_0 \;\&\; \mathsf{mski}_\Theta \upharpoonright X = \mathsf{mski}_\Theta \upharpoonright Y \to X = Y$. PROOF:
Suppose_not$(x_1, x_2) \Rightarrow$ AUTO

> Arguing by contradiction, suppose that
> $x_1 \in v_0 \;\&\; x_2 \in v_0 \;\&\; \mathsf{mski}_\Theta \upharpoonright x_1 = \mathsf{mski}_\Theta \upharpoonright x_2 \;\&\; x_1 \neq x_2$.

Loc_def $\Rightarrow \quad Stat0: \; a = \mathbf{arb}(\{\mathsf{mski}_\Theta \upharpoonright x : x \in v_0 \mid \langle \exists x' \in v_0 \setminus \{x\} \mid \mathsf{mski}_\Theta \upharpoonright x = \mathsf{mski}_\Theta \upharpoonright x' \rangle\})$

The a just defined (since the argument of **arb** in its definition is a nonnull set) will satisfy $\mathsf{a} = \mathsf{mski}_\Theta \upharpoonright \mathsf{x}_0$ and $\mathsf{a} = \mathsf{mski}_\Theta \upharpoonright \mathsf{x}_3$ for some $\mathsf{x}_0 \in \mathsf{v}_0$ and some $\mathsf{x}_3 \in \mathsf{v}_0$ such that $\mathsf{x}_0 \neq \mathsf{x}_3$.

Suppose $\Rightarrow$ *Stat1* : $\{\mathsf{mski}_\Theta \upharpoonright \mathsf{x} : \mathsf{x} \in \mathsf{v}_0 \mid \langle \exists \mathsf{x}' \in \mathsf{v}_0 \setminus \{\mathsf{x}\} \mid \mathsf{mski}_\Theta \upharpoonright \mathsf{x} = \mathsf{mski}_\Theta \upharpoonright \mathsf{x}' \rangle\} = \emptyset$
$\langle \mathsf{x}_1 \rangle \hookrightarrow$ *Stat1*$(\star) \Rightarrow$ *Stat2* : $\neg \langle \exists \mathsf{x}' \in \mathsf{v}_0 \setminus \{\mathsf{x}_1\} \mid \mathsf{mski}_\Theta \upharpoonright \mathsf{x}_1 = \mathsf{mski}_\Theta \upharpoonright \mathsf{x}' \rangle$
$\langle \mathsf{x}_2 \rangle \hookrightarrow$ *Stat2*$(\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO
(*Stat0*)ELEM $\Rightarrow$ *Stat3* : $\emptyset = \mathsf{a} \cap \{\mathsf{mski}_\Theta \upharpoonright \mathsf{x} : \mathsf{x} \in \mathsf{v}_0 \mid \langle \exists \mathsf{x}' \in \mathsf{v}_0 \setminus \{\mathsf{x}\} \mid \mathsf{mski}_\Theta \upharpoonright \mathsf{x} = \mathsf{mski}_\Theta \upharpoonright \mathsf{x}' \rangle\}$ & *Stat4* : $\mathsf{a} \in \{\mathsf{mski}_\Theta \upharpoonright \mathsf{x} : \mathsf{x} \in \mathsf{v}_0 \mid \langle \exists \mathsf{x}' \in \mathsf{v}_0 \setminus \{\mathsf{x}\} \mid \mathsf{mski}_\Theta \upharpoonright \mathsf{x} = \mathsf{mski}_\Theta \upharpoonright \mathsf{x}' \rangle\}$
$\langle \mathsf{x}_0 \rangle \hookrightarrow$ *Stat4*(*Stat3*$\star$) $\Rightarrow$ *Stat5* : $\langle \exists \mathsf{x}' \in \mathsf{v}_0 \setminus \{\mathsf{x}_0\} \mid \mathsf{mski}_\Theta \upharpoonright \mathsf{x}_0 = \mathsf{mski}_\Theta \upharpoonright \mathsf{x}' \rangle$ & $\mathsf{a} = \mathsf{mski}_\Theta \upharpoonright \mathsf{x}_0$ & $\mathsf{x}_0 \in \mathsf{v}_0$
$\langle \mathsf{x}_3 \rangle \hookrightarrow$ *Stat5*(*Stat5*$\star$) $\Rightarrow$ *Stat6* : $\mathsf{x}_3 \neq \mathsf{x}_0$ & $\mathsf{x}_3 \in \mathsf{v}_0$ & $\mathsf{mski}_\Theta \upharpoonright \mathsf{x}_0 = \mathsf{mski}_\Theta \upharpoonright \mathsf{x}_3$

On the basis of THM $\mathsf{finMostowskiDecoration}_8$, $\{\mathsf{x}_0, \mathsf{x}_3\} \subseteq \mathbf{dom}(\mathsf{d}_0)$ and therefore, by THM $\mathsf{finMostowskiDecoration}_3$, $\mathsf{a} \neq \emptyset$ and $\mathsf{a} = \{\mathsf{mski}_\Theta \upharpoonright \mathsf{p}^{[2]} : \mathsf{p} \in \mathsf{d}_{0|\{\mathsf{x}_0\}}\}$ and $\mathsf{a} = \{\mathsf{mski}_\Theta \upharpoonright \mathsf{p}^{[2]} : \mathsf{p} \in \mathsf{d}_{0|\{\mathsf{x}_3\}}\}$.

$\langle \mathsf{x}_0, \mathsf{x}_3 \rangle \hookrightarrow T\mathsf{finMostowskiDecoration}_8$(*Stat5*$\star$) $\Rightarrow$ $\mathsf{x}_0 \in \mathbf{dom}(\mathsf{d}_0)$
$\langle \mathsf{x}_3, \mathsf{x}_0 \rangle \hookrightarrow T\mathsf{finMostowskiDecoration}_8$(*Stat5*$\star$) $\Rightarrow$ $\mathsf{x}_3 \in \mathbf{dom}(\mathsf{d}_0)$
$\langle\ \rangle \hookrightarrow T\mathsf{finMostowskiDecoration}_0$(*Stat6*$\star$) $\Rightarrow$ *Stat7* : $\langle \forall \mathsf{x} \in \mathbf{dom}(\mathsf{d}_0), \mathsf{y} \in \mathbf{dom}(\mathsf{d}_0), \exists \mathsf{z} \mid ([\mathsf{x}, \mathsf{z}] \in \mathsf{d}_0 \leftrightarrow [\mathsf{y}, \mathsf{z}] \in \mathsf{d}_0) \rightarrow \mathsf{x} = \mathsf{y} \rangle$
$\langle \mathsf{x}_0 \rangle \hookrightarrow T\mathsf{finMostowskiDecoration}_3$(*Stat5*$\star$) $\Rightarrow$ $\mathsf{a} = \{\mathsf{mski}_\Theta \upharpoonright \mathsf{p}^{[2]} : \mathsf{p} \in \mathsf{d}_{0|\{\mathsf{x}_0\}}\}$
$\langle \mathsf{x}_3 \rangle \hookrightarrow T\mathsf{finMostowskiDecoration}_3$(*Stat5*$\star$) $\Rightarrow$ $\mathsf{a} = \{\mathsf{mski}_\Theta \upharpoonright \mathsf{p}^{[2]} : \mathsf{p} \in \mathsf{d}_{0|\{\mathsf{x}_3\}}\}$ & $\mathsf{a} \neq \emptyset$

Consequently, by the weak extensionality of the digraph under consideration, we can find a z_0 such that $[\mathsf{x}_0, \mathsf{z}_0] \in \mathsf{d}_0 \neq [\mathsf{x}_3, \mathsf{z}_0] \in \mathsf{d}_0$. Hence $\{\mathsf{x}_0, \mathsf{x}_3, \mathsf{z}_0\} \subseteq \mathsf{v}_0$.

$\langle \mathsf{x}_0, \mathsf{x}_3 \rangle \hookrightarrow$ *Stat7*(*Stat6*$\star$) $\Rightarrow$ *Stat8* : $\langle \exists \mathsf{z} \mid ([\mathsf{x}_0, \mathsf{z}] \in \mathsf{d}_0 \leftrightarrow [\mathsf{x}_3, \mathsf{z}] \in \mathsf{d}_0) \rightarrow \mathsf{x}_0 = \mathsf{x}_3 \rangle$
$\langle \mathsf{z}_0 \rangle \hookrightarrow$ *Stat8*(*Stat6*$\star$) $\Rightarrow$ $[\mathsf{x}_0, \mathsf{z}_0] \in \mathsf{d}_0 \neq [\mathsf{x}_3, \mathsf{z}_0] \in \mathsf{d}_0$
Assump $\Rightarrow$ $\mathsf{v}_0 \times \mathsf{v}_0 \supseteq \mathsf{d}_0$
Suppose $\Rightarrow$ $\mathsf{z}_0 \notin \mathsf{v}_0$
TELEM $\Rightarrow$ $[\mathsf{x}_0, \mathsf{z}_0]^{[2]} = \mathsf{z}_0$ & $[\mathsf{x}_3, \mathsf{z}_0]^{[2]} = \mathsf{z}_0$
$\langle [\mathsf{x}_0, \mathsf{z}_0], \mathsf{v}_0, \mathsf{v}_0 \rangle \hookrightarrow T\mathsf{cartesian}_0 \Rightarrow$ AUTO
$\langle [\mathsf{x}_3, \mathsf{z}_0], \mathsf{v}_0, \mathsf{v}_0 \rangle \hookrightarrow T\mathsf{cartesian}_0 \Rightarrow$ AUTO
(*Stat8*$\star$)Discharge $\Rightarrow$ AUTO
Suppose $\Rightarrow$ $[\mathsf{x}_0, \mathsf{z}_0] \in \mathsf{d}_0$

Suppose first that $[\mathsf{x}_0, \mathsf{z}_0] \in \mathsf{d}_0$, so that $[\mathsf{x}_3, \mathsf{z}_0] \notin \mathsf{d}_0$. Then $\mathsf{mski}_\Theta \upharpoonright \mathsf{z}_0 \in \mathsf{a}$ and therefore there is a $\mathsf{p}_1 \in \mathsf{d}_{0|\{\mathsf{x}_3\}}$ such that $\mathsf{mski}_\Theta \upharpoonright \mathsf{p}_1^{[2]} = \mathsf{mski}_\Theta \upharpoonright \mathsf{z}_0$ and $\mathsf{mski}_\Theta \upharpoonright \mathsf{z}_0 \in \mathsf{a}$.

$\langle [\mathsf{x}_0, \mathsf{z}_0], \mathsf{x}_0, \mathsf{z}_0, \mathsf{d}_0, \mathsf{x}_0 \rangle \hookrightarrow T\mathsf{restr}_2$(*Stat8*$\star$) $\Rightarrow$ $[\mathsf{x}_0, \mathsf{z}_0] \in \mathsf{d}_{0|\{\mathsf{x}_0\}}$
Suppose $\Rightarrow$ *Stat9* : $\mathsf{mski}_\Theta \upharpoonright [\mathsf{x}_0, \mathsf{z}_0]^{[2]} \notin \{\mathsf{mski}_\Theta \upharpoonright \mathsf{p}^{[2]} : \mathsf{p} \in \mathsf{d}_{0|\{\mathsf{x}_0\}}\}$
$\langle [\mathsf{x}_0, \mathsf{z}_0] \rangle \hookrightarrow$ *Stat9*(*Stat8*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO
(*Stat3*$\star$)ELEM $\Rightarrow$ *Stat10* : $\mathsf{mski}_\Theta \upharpoonright \mathsf{z}_0 \in \{\mathsf{mski}_\Theta \upharpoonright \mathsf{p}^{[2]} : \mathsf{p} \in \mathsf{d}_{0|\{\mathsf{x}_3\}}\}$ & *Stat11* : $\mathsf{mski}_\Theta \upharpoonright \mathsf{z}_0 \notin \{\mathsf{mski}_\Theta \upharpoonright \mathsf{x} : \mathsf{x} \in \mathsf{v}_0 \mid \langle \exists \mathsf{x}' \in \mathsf{v}_0 \setminus \{\mathsf{x}\} \mid \mathsf{mski}_\Theta \upharpoonright \mathsf{x} = \mathsf{mski}_\Theta \upharpoonright \mathsf{x}' \rangle\}$

$\langle p_1 \rangle \hookrightarrow Stat10(Stat10\star) \Rightarrow \quad p_1 \in d_{0|\{x_3\}} \;\&\; mski_\Theta \upharpoonright p_1^{[2]} = mski_\Theta \upharpoonright z_0$

Since no $mski_\Theta \upharpoonright z_0 \in a$ can coincide with $mski_\Theta \upharpoonright z_1$ for any $z_1 \in v_0$ distinct from z_0, and since $p_1^{[2]} \in v_0$, we have $z_0 = p_1^{[2]}$.

$\langle d_0, \{x_3\} \rangle \hookrightarrow Trestr_0(Stat8\star) \Rightarrow \quad p_1 \in v_0 \times v_0$
$\langle p_1, v_0, v_0 \rangle \hookrightarrow Tcartesian_0(Stat10\star) \Rightarrow \quad p_1 = [p_1^{[1]}, p_1^{[2]}] \;\&\; p_1^{[2]} \in v_0$
$\langle p_1, p_1^{[1]}, p_1^{[2]}, d_0, x_3 \rangle \hookrightarrow Trestr_2(Stat10\star) \Rightarrow \quad x_3 = p_1^{[1]} \;\&\; [x_3, p_1^{[2]}] \in d_0$
$\langle p_1^{[2]} \rangle \hookrightarrow Stat11(Stat10\star) \Rightarrow \quad Stat12: \neg\langle \exists x' \in v_0 \setminus \{p_1^{[2]}\} \mid mski_\Theta \upharpoonright p_1^{[2]} = mski_\Theta \upharpoonright x' \rangle$
$\langle z_0 \rangle \hookrightarrow Stat12(Stat8\star) \Rightarrow \quad z_0 = p_1^{[2]}$

However, the left component of the pair p_1 is x_3, which contradicts the supposition that $[x_3, z_0] \notin \emptyset$.

EQUAL$(Stat10) \Rightarrow \quad [x_3, z_0] \in d_0$
$(Stat8\star)$Discharge $\Rightarrow \quad Stat13: [x_3, z_0] \in d_0 \;\&\; [x_0, z_0] \notin d_0$

Likewise we derive a contradiction, leading to the desired conclusion, from the other possibile case, namely from $[x_3, z_0] \in d_0$.

$\langle [x_3, z_0], x_3, z_0, d_0, x_3 \rangle \hookrightarrow Trestr_2(Stat13\star) \Rightarrow \quad [x_3, z_0] \in d_{0|\{x_3\}}$
Suppose $\Rightarrow \quad Stat14: mski_\Theta \upharpoonright [x_3, z_0]^{[2]} \notin \{mski_\Theta \upharpoonright p^{[2]} : p \in d_{0|\{x_3\}}\}$
$\langle [x_3, z_0] \rangle \hookrightarrow Stat14(Stat8\star) \Rightarrow \quad$ false; Discharge $\Rightarrow$ AUTO
$(Stat3\star)$ELEM $\Rightarrow \quad Stat15: mski_\Theta \upharpoonright z_0 \in \{mski_\Theta \upharpoonright p^{[2]} : p \in d_{0|\{x_0\}}\} \;\&\; Stat16: mski_\Theta \upharpoonright z_0 \notin \{mski_\Theta \upharpoonright x : x \in v_0 \mid \langle \exists x' \in v_0 \setminus \{x\} \mid mski_\Theta \upharpoonright x = mski_\Theta \upharpoonright x' \rangle\}$
$\langle p_2 \rangle \hookrightarrow Stat15(Stat15\star) \Rightarrow \quad p_2 \in d_{0|\{x_0\}} \;\&\; mski_\Theta \upharpoonright p_2^{[2]} = mski_\Theta \upharpoonright z_0$
$\langle d_0, \{x_0\} \rangle \hookrightarrow Trestr_0(Stat8\star) \Rightarrow \quad p_2 \in v_0 \times v_0$
$\langle p_2, v_0, v_0 \rangle \hookrightarrow Tcartesian_0(Stat15\star) \Rightarrow \quad p_2 = [p_2^{[1]}, p_2^{[2]}] \;\&\; p_2^{[2]} \in v_0$
$\langle p_2, p_2^{[1]}, p_2^{[2]}, d_0, x_0 \rangle \hookrightarrow Trestr_2(Stat15\star) \Rightarrow \quad x_0 = p_2^{[1]} \;\&\; [x_0, p_2^{[2]}] \in d_0$
$\langle p_2^{[2]} \rangle \hookrightarrow Stat16(Stat15\star) \Rightarrow \quad Stat17: \neg\langle \exists x' \in v_0 \setminus \{p_2^{[2]}\} \mid mski_\Theta \upharpoonright p_2^{[2]} = mski_\Theta \upharpoonright x' \rangle$
$\langle z_0 \rangle \hookrightarrow Stat17(Stat8\star) \Rightarrow \quad z_0 = p_2^{[2]}$
EQUAL$(Stat8) \Rightarrow \quad$ false; Discharge $\Rightarrow$ QED

The premiss in the claim of the following theorem is needed because of our convention that the application of a map to an element outside its domain yields the value $\emptyset$.

THM finMostowskiDecoration$_{10}$: [Under Mostowski's decoration, arcs are modeled by membership] $Y \in v_0 \to (mski_\Theta \upharpoonright Y \in mski_\Theta \upharpoonright X \leftrightarrow [X, Y] \in d_0)$. PROOF:
Suppose_not$(y_0, x_0) \Rightarrow$ AUTO
Assump $\Rightarrow \quad Stat1: v_0 \times v_0 \supseteq d_0$
Suppose $\Rightarrow \quad x_0 \in \mathbf{dom}(d_0)$

$\langle x_0 \rangle \hookrightarrow T$finMostowskiDecoration$_3(\star) \Rightarrow$ $\quad$ *Stat2* : $\mathsf{mski}_\Theta \upharpoonright y_0 \in \{\mathsf{mski}_\Theta \upharpoonright p^{[2]} : p \in d_{0|\{x_0\}}\} \neq [x_0, y_0] \in d_0$

Suppose $\Rightarrow$ $\quad$ *Stat3* : $\mathsf{mski}_\Theta \upharpoonright y_0 \in \{\mathsf{mski}_\Theta \upharpoonright p^{[2]} : p \in d_{0|\{x_0\}}\}$

$\langle p_0 \rangle \hookrightarrow$ *Stat3*(*Stat3*$\star$) $\Rightarrow$ $\quad$ $\mathsf{mski}_\Theta \upharpoonright y_0 = \mathsf{mski}_\Theta \upharpoonright p_0{}^{[2]}$ & $p_0 \in d_{0|\{x_0\}}$

$\langle d_0, \{x_0\} \rangle \hookrightarrow T$restr$_0$(*Stat1*$\star$) $\Rightarrow$ $\quad$ $p_0 \in v_0 \times v_0$

$\langle p_0, v_0, v_0 \rangle \hookrightarrow T$cartesian$_0$(*Stat3*$\star$) $\Rightarrow$ $\quad$ $p_0 = [p_0{}^{[1]}, p_0{}^{[2]}]$ & $p_0{}^{[2]} \in v_0$

$\langle p_0, p_0{}^{[1]}, p_0{}^{[2]}, d_0, x_0 \rangle \hookrightarrow T$restr$_2$(*Stat3*$\star$) $\Rightarrow$ $\quad$ $x_0 = p_0{}^{[1]}$ & $[x_0, p_0{}^{[2]}] \in d_0$

$\langle y_0, p_0{}^{[2]} \rangle \hookrightarrow T$finMostowskiDecoration$_9(\star) \Rightarrow$ $\quad$ $y_0 = p_0{}^{[2]}$

EQUAL(*Stat3*) $\Rightarrow$ $\quad$ $[x_0, y_0] \in d_0$

(*Stat2*$\star$)Discharge $\Rightarrow$ $\quad$ AUTO

(*Stat2*$\star$)ELEM $\Rightarrow$ $\quad$ *Stat4* : $\mathsf{mski}_\Theta \upharpoonright y_0 \notin \{\mathsf{mski}_\Theta \upharpoonright p^{[2]} : p \in d_{0|\{x_0\}}\}$ & $[x_0, y_0] \in d_0$

$\langle [x_0, y_0] \rangle \hookrightarrow$ *Stat4*(*Stat4*$\star$) $\Rightarrow$ $\quad$ $\neg(\mathsf{mski}_\Theta \upharpoonright y_0 = \mathsf{mski}_\Theta \upharpoonright [x_0, y_0]^{[2]}$ & $[x_0, y_0] \in d_{0|\{x_0\}})$

$\langle [x_0, y_0], x_0, y_0, d_0, x_0 \rangle \hookrightarrow T$restr$_2$(*Stat4*$\star$) $\Rightarrow$ $\quad$ $\mathsf{mski}_\Theta \upharpoonright y_0 \neq \mathsf{mski}_\Theta \upharpoonright [x_0, y_0]^{[2]}$

TELEM $\Rightarrow$ $\quad$ $y_0 = [x_0, y_0]^{[2]}$

EQUAL(*Stat4*) $\Rightarrow$ $\quad$ false; $\quad$ Discharge $\Rightarrow$ $\quad$ AUTO

Suppose $\Rightarrow$ $\quad$ $x_0 \in v_0$

$\langle x_0, y_0 \rangle \hookrightarrow T$finMostowskiDecoration$_7(\star) \Rightarrow$ $\quad$ *Stat5* : $[x_0, y_0] \in d_0$

$\langle x_0, y_0, d_0 \rangle \hookrightarrow T$domain$_3$(*Stat1*$\star$) $\Rightarrow$ $\quad$ false; $\quad$ Discharge $\Rightarrow$ $\quad$ AUTO

$\langle\ \rangle \hookrightarrow T$finMostowskiDecoration$_2$(*Stat1*$\star$) $\Rightarrow$ $\quad$ $x_0 \notin \mathbf{dom}(\mathsf{mski}_\Theta)$

Use_def($\mathbf{dom}(\mathsf{mski}_\Theta)$) $\Rightarrow$ $\quad$ AUTO

$\langle x_0, \mathsf{mski}_\Theta \rangle \hookrightarrow T$image$_3(\star) \Rightarrow$ $\quad$ *Stat12* : $[x_0, y_0] \in v_0 \times v_0$

$\langle [x_0, y_0], v_0, v_0 \rangle \hookrightarrow T$cartesian$_0$(*Stat12*) $\Rightarrow$ $\quad$ $x_0 \in v_0$

(*Stat1*$\star$)Discharge $\Rightarrow$ $\quad$ QED

ENTER_THEORY Set_theory

DISPLAY finMostowskiDecoration

THEORY finMostowskiDecoration(v_0, d_0)
$v_0 \times v_0 \supseteq d_0 \,\&\, v_0 \neq \emptyset \,\&\, \mathsf{Finite}(v_0)$
$\mathsf{Acyclic}(v_0, d_0)$
$\mathsf{WExtensional}(v_0, d_0)$
$\Rightarrow (\mathsf{mski}_\Theta)$
$\mathsf{Is\_map}(d_0) \,\&\, \mathbf{dom}(d_0) \subseteq v_0 \,\&\, \mathbf{range}(d_0) \subseteq v_0 \,\&\, \langle \forall x \in \mathbf{dom}(d_0), y \in \mathbf{dom}(d_0), \exists z \mid ([x,z] \in d_0 \leftrightarrow [y,z] \in d_0) \rightarrow x = y \rangle$
$\mathsf{Svm}(\mathsf{mski}_\Theta) \,\&\, \mathbf{dom}(\mathsf{mski}_\Theta) = v_0$
$\langle \forall w \mid W \in \mathbf{dom}(d_0) \rightarrow \mathsf{mski}_\Theta \restriction W = \{\mathsf{mski}_\Theta \restriction p^{[2]} : p \in d_{0|\{W\}}\} \,\&\, \mathsf{mski}_\Theta \restriction W \neq \emptyset \rangle$
$\emptyset \in \mathbf{range}(\mathsf{mski}_\Theta)$
$\langle \forall y \mid Y \in \mathbf{range}(\mathsf{mski}_\Theta) \rightarrow \mathsf{Finite}(Y) \rangle$
$\langle \forall x, y \mid \{X, Y\} \subseteq v_0 \,\&\, X \notin \mathbf{dom}(d_0) \rightarrow \mathsf{mski}_\Theta \restriction Y \notin \mathsf{mski}_\Theta \restriction X \rangle$
$\langle \forall x, y \mid \{X, Y\} \subseteq v_0 \,\&\, X \notin \mathbf{dom}(d_0) \,\&\, \mathsf{mski}_\Theta \restriction X = \mathsf{mski}_\Theta \restriction Y \rightarrow X = Y \rangle$
$\langle \forall x, y \mid \{X, Y\} \subseteq v_0 \,\&\, \mathsf{mski}_\Theta \restriction X = \mathsf{mski}_\Theta \restriction Y \rightarrow X = Y \rangle$
$\langle \forall y \mid Y \in v_0 \rightarrow (\mathsf{mski}_\Theta \restriction Y \in \mathsf{mski}_\Theta \restriction X \leftrightarrow [X, Y] \in d_0) \rangle$
END finMostowskiDecoration

3.3 Weak result about representing graphs as membership digraphs

Our next THEORY will combine the one just seen, namely finMostowskiDecoration, with Theorem $\mathsf{xtensionalization}_0$, in order to represent an undirected graph by way of a membership digraph. Specifically, we will label the vertices of a graph devoid of self-loops so that:

- the labeling be injective;
- each label be finite;
- the null set be the label of a vertex;
- each doubleton $\{V, W\}$ of vertices be and edge if and only if the label of one of the two vertices belongs to the label of the other vertex;
- the label of a vertex be a subset of the set of all labels whenever it intersects it.

THEORY finGraphRepr(v_0, e_0)
$e_0 \subseteq \{\{x, y\} : x \in v_0, y \in v_0 \setminus \{x\}\} \,\&\, v_0 \neq \emptyset \,\&\, \mathsf{Finite}(v_0)$
END finGraphRepr

ENTER_THEORY finGraphRepr

THM finGraphRepr$_0$: [Acyclic weakly extensional orientation of current graph]

$\langle \exists d \mid d \subseteq v_0 \times v_0 \;\&\; \mathsf{Orientates}(d, v_0, e_0) \;\&\; \mathsf{Acyclic}(v_0, d) \;\&\; \mathsf{WExtensional}(v_0, d) \;\&\; v_0 \neq \emptyset \;\&\; \mathsf{Finite}(v_0)\rangle$. PROOF:

Suppose_not() ⇒ $\mathit{Stat0}: \neg\langle \exists d \mid d \subseteq v_0 \times v_0 \;\&\; \mathsf{Orientates}(d, v_0, e_0) \;\&\; \mathsf{Acyclic}(v_0, d) \;\&\; \mathsf{WExtensional}(v_0, d) \;\&\; v_0 \neq \emptyset \;\&\; \mathsf{Finite}(v_0)\rangle$

Assump ⇒ $\mathsf{Finite}(v_0) \;\&\; \mathbf{arb}(v_0) \in v_0$

Loc_def ⇒ $s_0 = \mathbf{arb}(v_0)$

$\langle v_0, s_0, e_0\rangle \hookrightarrow T$xtensionalization$_0(\star)$ ⇒ $\mathit{Stat1}: \langle \exists d \mid \mathsf{Orientates}(d, v_0, e_0) \;\&\; \mathsf{Acyclic}(v_0, d) \;\&\; \mathsf{WExtensional}(v_0, d) \;\&\; s_0 \notin \mathbf{range}(d)\rangle$

$\langle d_0\rangle \hookrightarrow \mathit{Stat1}(\mathit{Stat1}\star)$ ⇒ $\mathsf{Orientates}(d_0, v_0, e_0) \;\&\; \mathsf{Acyclic}(v_0, d_0) \;\&\; \mathsf{WExtensional}(v_0, d_0) \;\&\; s_0 \notin \mathbf{range}(d_0)$

$\langle d_0 \cap (v_0 \times v_0)\rangle \hookrightarrow \mathit{Stat0}(\star)$ ⇒ $\neg\Big(\mathsf{Orientates}\big(d_0 \cap (v_0 \times v_0), v_0, e_0\big) \;\&\; \mathsf{Acyclic}\big(v_0, d_0 \cap (v_0 \times v_0)\big) \;\&\; \mathsf{WExtensional}\big(v_0, d_0 \cap (v_0 \times v_0)\big)\Big)$

$\langle d_0, v_0, e_0\rangle \hookrightarrow T$orientation$_2$ ⇒ AUTO

$\langle v_0, d_0, v_0, d_0 \cap (v_0 \times v_0)\rangle \hookrightarrow T$acyclicity$_1$ ⇒ AUTO

$\langle v_0, d_0\rangle \hookrightarrow T$weaXtensionality$_0$ ⇒ AUTO

$(\mathit{Stat1}\star)$Discharge ⇒ QED

APPLY $\langle v1_\Theta : \mathsf{wskiArcs}\rangle$ Skolem⇒

THM finGraphRepr$_{0a}$. $\mathsf{wskiArcs} \subseteq v_0 \times v_0 \;\&\; \mathsf{Orientates}(\mathsf{wskiArcs}, v_0, e_0) \;\&\; \mathsf{Acyclic}(v_0, \mathsf{wskiArcs}) \;\&\; \mathsf{WExtensional}(v_0, \mathsf{wskiArcs}) \;\&\; v_0 \neq \emptyset \;\&\; \mathsf{Finite}(v_0)$.

APPLY $\langle \mathsf{mski}_\Theta : \mathsf{wski}_\Theta\rangle$ finMostowskiDecoration$(v_0 \mapsto v_0, d_0 \mapsto \mathsf{wskiArcs})$⇒

THM finGraphRepr$_{0b}$. $\mathsf{Svm}(\mathsf{wski}_\Theta) \;\&\; \mathbf{dom}(\mathsf{wski}_\Theta) = v_0 \;\&\; \langle \forall w \mid w \in \mathbf{dom}(\mathsf{wskiArcs}) \rightarrow \mathsf{wski}_\Theta\restriction w = \{\mathsf{wski}_\Theta\restriction p^{[2]} : p \in \mathsf{wskiArcs}_{|\{w\}}\} \;\&\; \mathsf{wski}_\Theta\restriction w \neq \emptyset\rangle \;\&$
$\emptyset \in \mathbf{range}(\mathsf{wski}_\Theta) \;\&\; \langle \forall y \mid Y \in \mathbf{range}(\mathsf{wski}_\Theta) \rightarrow \mathsf{Finite}(Y)\rangle \;\&\; \langle \forall x, y \mid \{X, Y\} \subseteq v_0 \;\&\; \mathsf{wski}_\Theta\restriction X = \mathsf{wski}_\Theta\restriction Y \rightarrow X = Y\rangle \;\&$
$\langle \forall y, x \mid Y \in v_0 \rightarrow (\mathsf{wski}_\Theta\restriction Y \in \mathsf{wski}_\Theta\restriction X \leftrightarrow [X, Y] \in \mathsf{wskiArcs})\rangle$.

THM finGraphRepr$_1$. $[X, Y] \in \mathsf{wskiArcs} \vee [Y, X] \in \mathsf{wskiArcs} \leftrightarrow \{X, Y\} \in e_0$. PROOF:

Suppose_not(x_0, y_0) ⇒ AUTO

TfinGraphRepr$_{0a}(\star)$ ⇒ $\mathit{Stat1}: \mathsf{wskiArcs} \subseteq v_0 \times v_0 \;\&\; \mathsf{Orientates}(\mathsf{wskiArcs}, v_0, e_0)$

Assump ⇒ $e_0 \subseteq \{\{x, y\} : x \in v_0, y \in v_0 \setminus \{x\}\}$

Suppose ⇒ $\mathit{Stat2}: \{\{p^{[1]}, p^{[2]}\} : p \in \mathsf{wskiArcs} \mid p = [p^{[1]}, p^{[2]}]\} \neq \{\{p^{[1]}, p^{[2]}\} : p \in \mathsf{wskiArcs}\}$

$\langle p_0\rangle \hookrightarrow \mathit{Stat2}(\mathit{Stat1}\star)$ ⇒ $p_0 \in v_0 \times v_0 \;\&\; p_0 \neq [{p_0}^{[1]}, {p_0}^{[2]}]$

$\langle p_0, v_0, v_0\rangle \hookrightarrow T$cartesian$_0(\mathit{Stat2}\star)$ ⇒ false; Discharge ⇒ AUTO

Use_def(Orientates) ⇒ $\mathit{Stat3}: [x_0, y_0] \in \mathsf{wskiArcs} \vee [y_0, x_0] \in \mathsf{wskiArcs} \neq \{x_0, y_0\} \in \{\{p^{[1]}, p^{[2]}\} : p \in \mathsf{wskiArcs}\}$

Suppose ⇒ $\mathit{Stat4}: \{x_0, y_0\} \in \{\{p^{[1]}, p^{[2]}\} : p \in \mathsf{wskiArcs}\} \;\&\; \neg([x_0, y_0] \in \mathsf{wskiArcs} \vee [y_0, x_0] \in \mathsf{wskiArcs})$

$\langle p_1\rangle \hookrightarrow \mathit{Stat4}(\mathit{Stat1}\star)$ ⇒ $\mathit{Stat5}: \{x_0, y_0\} = \{{p_1}^{[1]}, {p_1}^{[2]}\} \;\&\; p_1 \in v_0 \times v_0 \;\&\; p_1 \in \mathsf{wskiArcs}$

$\langle p_1, v_0, v_0\rangle \hookrightarrow T$cartesian$_0(\mathit{Stat5}\star)$ ⇒ $p_1 = [{p_1}^{[1]}, {p_1}^{[2]}] \;\&\; (x_0 = {p_1}^{[1]} \;\&\; y_0 = {p_1}^{[2]}) \vee (x_0 = {p_1}^{[2]} \;\&\; y_0 = {p_1}^{[1]})$

Suppose ⇒ $x_0 = {p_1}^{[1]} \;\&\; y_0 = {p_1}^{[2]}$

EQUAL ⇒ false; Discharge ⇒ $y_0 = {p_1}^{[1]} \;\&\; x_0 = {p_1}^{[2]}$

EQUAL ⇒ false; Discharge ⇒ AUTO

Suppose ⇒ $\mathit{Stat6}: \{x_0, y_0\} \notin \{\{p^{[1]}, p^{[2]}\} : p \in \mathsf{wskiArcs}\} \;\&\; [x_0, y_0] \in \mathsf{wskiArcs}$

$\langle [x_0, y_0] \rangle \hookrightarrow Stat6(Stat6\star) \Rightarrow$ false; Discharge $\Rightarrow$ $Stat7: \{x_0, y_0\} \notin \{\{p^{[1]}, p^{[2]}\} : p \in wskiArcs\} \;\&\; [y_0, x_0] \in wskiArcs$

$\langle [y_0, x_0] \rangle \hookrightarrow Stat7(Stat7\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

THM finGraphRepr$_2$. $\{X, Y\} \subseteq v_0 \rightarrow (wski_\Theta \restriction Y \in wski_\Theta \restriction X \vee wski_\Theta \restriction X \in wski_\Theta \restriction Y \leftrightarrow \{X, Y\} \in e_0)$. PROOF:

Suppose_not$(x_0, y_0) \Rightarrow$ AUTO

TfinGraphRepr$_{0b}(\star) \Rightarrow$ $Stat1: \langle \forall y, x \,|\, y \in v_0 \rightarrow (wski_\Theta \restriction y \in wski_\Theta \restriction x \leftrightarrow [x, y] \in wskiArcs)\rangle$

$\langle y_0, x_0 \rangle \hookrightarrow Stat1(\star) \Rightarrow$ $wski_\Theta \restriction y_0 \in wski_\Theta \restriction x_0 \leftrightarrow [x_0, y_0] \in wskiArcs$

$\langle x_0, y_0 \rangle \hookrightarrow Stat1(\star) \Rightarrow$ $wski_\Theta \restriction x_0 \in wski_\Theta \restriction y_0 \leftrightarrow [y_0, x_0] \in wskiArcs$

$\langle x_0, y_0 \rangle \hookrightarrow T$finGraphRepr$_1(\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

THM finGraphRepr$_3$. $wski_\Theta \restriction X \cap \mathbf{range}(wski_\Theta) \neq \emptyset \rightarrow wski_\Theta \restriction X \subseteq \mathbf{range}(wski_\Theta)$. PROOF:

Suppose_not$(x_0) \Rightarrow$ AUTO

Use_def$(\mathbf{range}(wski_\Theta)) \Rightarrow$ AUTO

ELEM $\Rightarrow$ $Stat1: wski_\Theta \restriction x_0 \cap \{p^{[2]} : p \in wski_\Theta\} \neq \emptyset \;\&\; wski_\Theta \restriction x_0 \not\subseteq \{p^{[2]} : p \in wski_\Theta\}$

$\langle z_0 \rangle \hookrightarrow Stat1(Stat1\star) \Rightarrow$ $Stat2: z_0 \in \{p^{[2]} : p \in wski_\Theta\} \;\&\; z_0 \in wski_\Theta \restriction x_0$

$\langle p_0 \rangle \hookrightarrow Stat2(Stat2\star) \Rightarrow$ $p_0 \in wski_\Theta \;\&\; z_0 = p_0^{[2]}$

TfinGraphRepr$_{0b}(Stat2\star) \Rightarrow$ $Stat3: \langle \forall y, x \,|\, y \in v_0 \rightarrow (wski_\Theta \restriction y \in wski_\Theta \restriction x \leftrightarrow [x, y] \in wskiArcs)\rangle \;\&$
$\langle \forall w \,|\, w \in \mathbf{dom}(wskiArcs) \rightarrow wski_\Theta \restriction w = \{wski_\Theta \restriction p^{[2]} : p \in wskiArcs_{|\{w\}}\} \;\&\; wski_\Theta \restriction w \neq \emptyset\rangle \;\&\; Svm(wski_\Theta) \;\&\; \mathbf{dom}(wski_\Theta) = v_0$

Loc_def $\Rightarrow$ $y_0 = p_0^{[1]}$

$\langle p_0, wski_\Theta \rangle \hookrightarrow T$domain$_2(Stat2\star) \Rightarrow$ $y_0 \in v_0$

$\langle wski_\Theta, p_0 \rangle \hookrightarrow T$image$_4(Stat2\star) \Rightarrow$ $p_0 = [p_0^{[1]}, wski_\Theta \restriction p_0^{[1]}]$

TELEM $\Rightarrow$ $[p_0^{[1]}, wski_\Theta \restriction p_0^{[1]}]^{[2]} = wski_\Theta \restriction p_0^{[1]}$

EQUAL$(Stat2) \Rightarrow$ $z_0 = wski_\Theta \restriction y_0$

$\langle x_0, y_0, wskiArcs \rangle \hookrightarrow T$domain$_3 \Rightarrow$ AUTO

$\langle y_0, x_0, x_0 \rangle \hookrightarrow Stat3(\star) \Rightarrow$ $Stat5: \{wski_\Theta \restriction p^{[2]} : p \in wskiArcs_{|\{x_0\}}\} \not\subseteq \mathbf{range}(wski_\Theta)$

TfinGraphRepr$_{0a}(Stat5\star) \Rightarrow$ $wskiArcs \subseteq v_0 \times v_0$

$\langle wskiArcs, \{x_0\} \rangle \hookrightarrow T$restr$_0(Stat5\star) \Rightarrow$ $Stat6: wskiArcs_{|\{x_0\}} \subseteq v_0 \times v_0$

$\langle c \rangle \hookrightarrow Stat5(Stat5\star) \Rightarrow$ $Stat7: c \in \{wski_\Theta \restriction p^{[2]} : p \in wskiArcs_{|\{x_0\}}\} \;\&\; c \notin \mathbf{range}(wski_\Theta)$

$\langle p_1 \rangle \hookrightarrow Stat7(Stat5\star) \Rightarrow$ $Stat8: p_1 \in v_0 \times v_0 \;\&\; wski_\Theta \restriction p_1^{[2]} \notin \mathbf{range}(wski_\Theta)$

$\langle p_1, v_0, v_0 \rangle \hookrightarrow T$cartesian$_0(Stat8\star) \Rightarrow$ $p_1^{[2]} \in v_0$

$\langle wski_\Theta \rangle \hookrightarrow T$image$_5(Stat3, Stat3\star) \Rightarrow$ $wski_\Theta = \{[x, wski_\Theta \restriction x] : x \in \mathbf{dom}(wski_\Theta)\}$

EQUAL$(Stat3) \Rightarrow$ $wski_\Theta = \{[x, wski_\Theta \restriction x] : x \in v_0\}$

Suppose $\Rightarrow$ $Stat9: [p_1^{[2]}, wski_\Theta \restriction p_1^{[2]}] \notin \{[x, wski_\Theta \restriction x] : x \in v_0\}$

$\langle p_1^{[2]} \rangle \hookrightarrow Stat9(Stat8\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO

Use_def$(\mathbf{range}(wski_\Theta)) \Rightarrow$ AUTO

$(Stat8\star)$ELEM $\Rightarrow$ $Stat10: wski_\Theta \restriction p_1^{[2]} \notin \{p^{[2]} : p \in wski_\Theta\} \;\&\; [p_1^{[2]}, wski_\Theta \restriction p_1^{[2]}] \in wski_\Theta$

$\langle [p_1^{[2]}, wski_\Theta \restriction p_1^{[2]}] \rangle \hookrightarrow Stat10(Stat10\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

ENTER_THEORY Set_theory
DISPLAY finGraphRepr

THEORY $\mathsf{finGraphRepr}(v_0, e_0)$
$e_0 \subseteq \{\{x,y\} : x \in v_0, y \in v_0 \setminus \{x\}\} \ \& \ v_0 \neq \emptyset \ \& \ \mathsf{Finite}(v_0)$
$\Rightarrow (\mathsf{wski}_\Theta)$
$\mathsf{Svm}(\mathsf{wski}_\Theta) \ \& \ \mathbf{dom}(\mathsf{wski}_\Theta) = v_0$
$\emptyset \in \mathbf{range}(\mathsf{wski}_\Theta)$
$\langle \forall y \mid Y \in \mathbf{range}(\mathsf{wski}_\Theta) \rightarrow \mathsf{Finite}(Y)\rangle$
$\langle \forall x, y \mid \{X, Y\} \subseteq v_0 \ \& \ \mathsf{wski}_\Theta \upharpoonright X = \mathsf{wski}_\Theta \upharpoonright Y \rightarrow X = Y\rangle$
$\langle \forall x, y \mid \{X, Y\} \subseteq v_0 \rightarrow (\mathsf{wski}_\Theta \upharpoonright Y \in \mathsf{wski}_\Theta \upharpoonright X \vee \mathsf{wski}_\Theta \upharpoonright X \in \mathsf{wski}_\Theta \upharpoonright Y \leftrightarrow \{X, Y\} \in e_0)\rangle$
$\langle \forall x \mid \mathsf{wski}_\Theta \upharpoonright X \cap \mathbf{range}(\mathsf{wski}_\Theta) \neq \emptyset \rightarrow \mathsf{wski}_\Theta \upharpoonright X \subseteq \mathbf{range}(\mathsf{wski}_\Theta)\rangle$
END finGraphRepr

4 Representation of a connected, claw-free graph as the membership digraph of a transitive set

4.1 Extensional acyclic orientation of a connected, claw-free graph

DEF clawFreeGraph: [Claw-freeness, as a property of graphs] $\mathsf{ClawFreeG}(V, E) \leftrightarrow_{\mathrm{Def}}$
$\langle \forall w, x, y, z \mid \{w, x, y, z\} \subseteq V \ \& \ \{w, y\}, \{y, x\}, \{y, z\} \in E \rightarrow \neg(x \neq z \ \& \ w \notin \{z, x\} \ \& \ \{x, z\} \notin E \ \& \ \{z, w\} \notin Y \ \& \ \{w, x\} \notin E)\rangle$

THM $\mathsf{cClawFreeG}_0$: [Hereditarity of the claw-freeness property of graphs] $\mathsf{ClawFreeG}(V, E) \ \& \ W \subseteq V \rightarrow \mathsf{ClawFreeG}(W, \{a \in E \mid a \subseteq W\})$. PROOF:
$\mathsf{Suppose\_not}(v_0, e_0, w_0) \Rightarrow$ *Stat0*: $\mathsf{ClawFreeG}(v_0, e_0) \ \& \ w_0 \subseteq v_0 \ \& \ \neg\mathsf{ClawFreeG}(w_0, \{a \in e_0 \mid a \subseteq w_0\})$
Loc_def $\Rightarrow$ *Stat1*: $e_1 = \{a \in e_0 \mid a \subseteq w_0\}$
EQUAL $\Rightarrow$ $\neg\mathsf{ClawFreeG}(w_0, e_1)$
Set_monot $\Rightarrow$ $\{a \in e_0 \mid a \subseteq w_0\} \subseteq \{a \in e_0 \mid \mathsf{true}\}$
Use_def(ClawFreeG) $\Rightarrow$ *Stat2*:
$\neg\langle \forall w, x, y, z \mid \{w, x, y, z\} \subseteq w_0 \ \& \ \{w, y\}, \{y, x\}, \{y, z\} \in e_1 \rightarrow \neg(x \neq z \ \& \ w \notin \{z, x\} \ \& \ \{x, z\} \notin e_1 \ \& \ \{z, w\} \notin e_1 \ \& \ \{w, x\} \notin e_1)\rangle \ \&$
$\langle \forall w, x, y, z \mid \{w, x, y, z\} \subseteq v_0 \ \& \ \{w, y\}, \{y, x\}, \{y, z\} \in e_0 \rightarrow \neg(x \neq z \ \& \ w \notin \{z, x\} \ \& \ \{x, z\} \notin e_0 \ \& \ \{z, w\} \notin e_0 \ \& \ \{w, x\} \notin e_0)\rangle$
(*Stat0*⋆)ELEM $\Rightarrow$ $w_0 \subseteq v_0 \ \& \ e_1 \subseteq e_0$
$\langle w', x', y', z', w', x', y', z'\rangle \hookrightarrow$ *Stat2*(*Stat2*⋆) $\Rightarrow$ *Stat3*:
$\{w', x', y', z'\} \subseteq w_0 \ \& \ \{x', z'\} \notin e_1 \ \& \ \{z', w'\} \notin e_1 \ \& \ \{w', x'\} \notin e_1 \ \& \ \neg(\{x', z'\} \notin e_0 \ \& \ \{z', w'\} \notin e_0 \ \& \ \{w', x'\} \notin e_0)$
(*Stat3*, *Stat1*⋆)ELEM $\Rightarrow$ *Stat4*: $\{x', z'\} \notin \{a \in e_0 \mid a \subseteq w_0\}$ & *Stat5*: $\{z', w'\} \notin \{a \in e_0 \mid a \subseteq w_0\}$ & *Stat6*: $\{w', x'\} \notin \{a \in e_0 \mid a \subseteq w_0\}$
$\langle\rangle \hookrightarrow$ *Stat4*(*Stat3*, *Stat7*⋆) $\Rightarrow$ *Stat7*: $\{x', z'\} \notin e_0$
$\langle\rangle \hookrightarrow$ *Stat5*(*Stat3*, *Stat8*⋆) $\Rightarrow$ *Stat8*: $\{z', w'\} \notin e_0$
$\langle\rangle \hookrightarrow$ *Stat6*(*Stat3*, *Stat7*, *Stat8*⋆) $\Rightarrow$ false; Discharge $\Rightarrow$ QED

THM $\mathsf{cClawFreeG}_1$: [Preservation of acyclicity and extensionality under adjunction of an outer vertex to a digraph whose underlying graph is connected and claw free]

$\mathsf{W = V \cup \{U\}}$ & $\mathsf{U \notin V}$ & $\{\mathsf{s \in V \mid D_{|\{s\}} = \emptyset}$ & $\mathsf{\{s,U\} \in E}\} = \emptyset$ & $\mathsf{E \subseteq \{\{x,y\} : x \in Z, y \in Z \setminus \{x\}\}}$ & $\mathsf{ClawFreeG(W,E)}$ & $\mathsf{HasSpanningTree(W,E)}$ & $\mathsf{Orientates(D,V,E)}$ & $\mathsf{Acyclic(V,D)}$ & $\mathsf{Extensional(V,D)}$ & $\mathsf{D \subseteq V \times V}$ & $\mathsf{Dp = D \cup \{U\} \times \{t \in V \mid \{U,t\} \in E\}} \rightarrow$
$\mathsf{Orientates(Dp,W,E)}$ & $\mathsf{Acyclic(W,Dp)}$ & $\mathsf{Extensional(W,Dp)}$ & $\mathsf{Dp \subseteq W \times W}$. PROOF:

Suppose_not$(\mathsf{v_1,v_0,x_0,d_0,e_2,v_2,d_1}) \Rightarrow$ *Stat0*: $\neg(\mathsf{Orientates(d_1,v_1,e_2)}$ & $\mathsf{Acyclic(v_1,d_1)}$ & $\mathsf{Extensional(v_1,d_1)}$ & $\mathsf{d_1 \subseteq v_1 \times v_1})$ & $\mathsf{x_0 \in v_1}$ & $\mathsf{v_0 = v_1 \setminus \{x_0\}}$ & $\{\mathsf{s \in v_0 \mid d_{0|\{s\}} = \emptyset}$ & $\mathsf{\{s,x_0\} \in e_2}\} = \emptyset$ & $\mathsf{e_2 \subseteq \{\{x,y\} : x \in v_2, y \in v_2 \setminus \{x\}\}}$ & $\mathsf{ClawFreeG(v_1,e_2)}$ & $\mathsf{HasSpanningTree(v_1,e_2)}$ & $\mathsf{Orientates(d_0,v_0,e_2)}$ & $\mathsf{Acyclic(v_0,d_0)}$ & $\mathsf{Extensional(v_0,d_0)}$ & $\mathsf{d_0 \subseteq v_0 \times v_0}$ & $\mathsf{d_1 = d_0 \cup \{x_0\} \times \{t \in v_0 \mid \{x_0,t\} \in e_2\}}$

Suppose that $\mathsf{v_1,v_0,x_0,d_0,e_2,v_2,d_1}$ constitute a counterexample to the claim. We readily exclude the case that $\neg\mathsf{Acyclic(v_1,d_1)}$ and the case that $\neg\mathsf{Orientates(d_1,v_1,e_2)}$. Also, through Theorem $\mathsf{xtensionalization_1}$, we exclude that $\mathsf{d_1 \not\subseteq v_1 \times v_1}$ and reduce the only possibility left, namely $\neg\mathsf{Extensional(v_1,d_1)}$, to the condition $\langle \exists \mathsf{x} \in \mathsf{v_0}, \forall \mathsf{z} \mid [\mathsf{x_0,z}] \in \mathsf{d_1} \leftrightarrow [\mathsf{x,z}] \in \mathsf{d_0}\rangle$. So, let $\mathsf{x_1 \in v_0}$ be such that $\langle \forall \mathsf{z} \mid [\mathsf{x_0,z}] \in \mathsf{d_1} \leftrightarrow [\mathsf{x_1,z}] \in \mathsf{d_0}\rangle$.

Set_monot $\Rightarrow$ $\mathsf{\{t \in v_0 \mid true\} \supseteq \{t \in v_0 \mid \{x_0,t\} \in e_2\}}$
(*Stat0*$\star$)ELEM $\Rightarrow$ *Stat1*: $\mathsf{v_1 = v_0 \cup \{x_0\}}$ & $\mathsf{v_0 \supseteq \{t \in v_0 \mid \{x_0,t\} \in e_2\}}$
$\langle \mathsf{v_0,d_0,x_0,\{t \in v_0 \mid \{x_0,t\} \in e_2\}}\rangle \hookrightarrow T\mathsf{acyclicity_0}\ (\star) \Rightarrow$ $\mathsf{Acyclic(v_0 \cup \{x_0\}, d_0 \cup \{x_0\} \times \{t \in v_0 \mid \{x_0,t\} \in e_2\})}$
EQUAL $\Rightarrow$ $\mathsf{Acyclic(v_1,d_1)}$
$\langle \mathsf{d_0,v_0,e_2,v_1,x_0,d_1}\rangle \hookrightarrow T\mathsf{orientation_3}\ (\star) \Rightarrow$ *Stat2*: $\mathsf{Orientates(d_1,v_1,e_2)}$
$\langle \mathsf{v_1,v_0,x_0,d_0,e_2,v_2,d_1}\rangle \hookrightarrow T\mathsf{xtensionalization_1}\ (\star) \Rightarrow$ *Stat3*: $\langle \exists \mathsf{x} \in \mathsf{v_0}, \forall \mathsf{z} \mid [\mathsf{x_0,z}] \in \mathsf{d_1} \leftrightarrow [\mathsf{x,z}] \in \mathsf{d_0}\rangle$
$\langle \mathsf{x_1}\rangle \hookrightarrow$ *Stat3*$(\star) \Rightarrow$ *Stat4*: $\langle \forall \mathsf{z} \mid [\mathsf{x_0,z}] \in \mathsf{d_1} \leftrightarrow [\mathsf{x_1,z}] \in \mathsf{d_0}\rangle$ & $\mathsf{x_1 \in v_1 \setminus \{x_0\}}$

The digraph $\mathsf{v_1,d_1}$ is connected, hence $\mathsf{x_0}$ has at least one successor in $\mathsf{v_1,d_1}$, witnessing that $\mathsf{x_1}$ has successors in $\mathsf{v_0,d_0}$.

Suppose $\Rightarrow$ *Stat5*: $\mathsf{\{z \in v_0 \mid [x_1,z] \in d_0\} = \emptyset}$
(*Stat0*, *Stat4*$\star$)ELEM $\Rightarrow$ $\mathsf{HasSpanningTree(v_1,e_2)}$ & $\mathsf{x_0 \in v_1}$ & $\mathsf{v_1 \setminus \{x_0\} \neq \emptyset}$ & $\mathsf{e_2 \subseteq \{\{x,y\} : x \in v_2, y \in v_2 \setminus \{x\}\}}$
$\langle \mathsf{v_1,e_2,v_2,x_0}\rangle \hookrightarrow T\mathsf{connectivity_1}$ (*Stat0*, *Stat4*$\star$) $\Rightarrow$ *Stat6*: $\langle \exists \mathsf{w} \in \mathsf{v_1 \setminus \{x_0\}} \mid \mathsf{\{x_0,w\} \in e_2}\rangle$
$\langle \mathsf{y_1}\rangle \hookrightarrow$ *Stat6*(*Stat6*$\star$) $\Rightarrow$ *Stat7*: $\mathsf{y_1 \in v_1 \setminus \{x_0\}}$ & $\mathsf{\{x_0,y_1\} \in e_2}$
Suppose $\Rightarrow$ *Stat8*: $[\mathsf{y_1,x_0}] \in \mathsf{d_1}$
$\langle [\mathsf{y_1,x_0}], \mathsf{\{x_0\}, \{t \in v_0 \mid \{x_0,t\} \in e_2\}}\rangle \hookrightarrow T\mathsf{cartesian_0}$ (*Stat8*, *Stat0*, *Stat7*$\star$) $\Rightarrow$ *Stat9*: $[\mathsf{y_1,x_0}] \in \mathsf{d_0}$
$\langle \mathsf{d_0,v_0,e_2,y_1,x_0}\rangle \hookrightarrow T\mathsf{orientation_6}$ (*Stat0*, *Stat9*$\star$) $\Rightarrow$ false
Discharge $\Rightarrow$ AUTO
$\langle \mathsf{d_1,v_1,e_2,x_0,y_1}\rangle \hookrightarrow T\mathsf{orientation_7}$ (*Stat1*$\star$) $\Rightarrow$ $[\mathsf{x_0,y_1}] \in \mathsf{d_1}$
$\langle \mathsf{y_1}\rangle \hookrightarrow$ *Stat4*(*Stat6*$\star$) $\Rightarrow$ $[\mathsf{x_1,y_1}] \in \mathsf{d_0}$
$\langle \mathsf{y_1}\rangle \hookrightarrow$ *Stat5* $\Rightarrow$ AUTO
(*Stat0*$\star$)Discharge $\Rightarrow$ AUTO

Since $\mathsf{v_0,d_0}$ is acyclic, at least one successor $\mathsf{y_0}$ of $\mathsf{x_1}$ in $\mathsf{v_0,d_0}$ has no successors in common with $\mathsf{x_1}$ in $\mathsf{v_0,d_0}$.

Use_def(Acyclic) $\Rightarrow$ *Stat11*: $\langle \forall \mathsf{w} \subseteq \mathsf{v_0} \mid \mathsf{w} \neq \emptyset \rightarrow \langle \exists \mathsf{t} \in \mathsf{w} \mid \emptyset = \{\mathsf{y} \in \mathsf{w} \mid [\mathsf{t,y}] \in \mathsf{d_0}\}\rangle\rangle$

Set_monot $\langle Stat11 \rangle \Rightarrow$ $\{z \in v_0 \mid [x_1, z] \in d_0\} \subseteq \{z \in v_0 \mid \text{true}\}$
$\langle \{z \in v_0 \mid [x_1, z] \in d_0\} \rangle \hookrightarrow Stat11(Stat4\star) \Rightarrow$ $Stat12: \langle \exists t \in \{z \in v_0 \mid [x_1, z] \in d_0\} \mid \emptyset = \{y \in \{z \in v_0 \mid [x_1, z] \in d_0\} \mid [t, y] \in d_0\} \rangle$
$\langle y_0 \rangle \hookrightarrow Stat12(Stat12\star) \Rightarrow$ $y_0 \in \{z \in v_0 \mid [x_1, z] \in d_0\}$ & $\emptyset = \{y \in \{z \in v_0 \mid [x_1, z] \in d_0\} \mid [y_0, y] \in d_0\}$
SIMPLF $\langle Stat12 \rangle \Rightarrow$ $Stat13: y_0 \in \{z \in v_0 \mid [x_1, z] \in d_0\}$ & $\{y \in v_0 \mid [x_1, y], [y_0, y] \in d_0\} = \emptyset$
$\langle\ \rangle \hookrightarrow Stat13(Stat13\star) \Rightarrow$ $Stat14: y_0 \in v_0$ & $[x_1, y_0] \in d_0$

Being a successor of x_1 in d_0, y_0 must also be a successor of x_0 in d_1; but then it must, in its turn, have a successor z_0 in d_0: this is because we have supposed at the outset that x_0 is adjacent to no sink of d_0.

$\langle y_0 \rangle \hookrightarrow Stat4(Stat14\star) \Rightarrow$ $Stat15: [x_0, y_0] \in d_1$
Suppose $\Rightarrow$ $Stat16: \neg\langle \exists z \in v_0 \mid [y_0, z] \in d_0 \rangle$
$(Stat0\star)$ELEM $\Rightarrow$ $Stat17: y_0 \notin \{s \in v_0 \mid d_{0|\{s\}} = \emptyset$ & $\{s, x_0\} \in e_2\}$
$\langle d_1, v_1, e_2, x_0, y_0 \rangle \hookrightarrow T$orientation$_6$ $(Stat2, Stat15\star) \Rightarrow$ $\{y_0, x_0\} \in e_2$
$\langle y_0 \rangle \hookrightarrow Stat17(Stat14\star) \Rightarrow$ $Stat18: d_{0|\{y_0\}} \neq \emptyset$
$\langle d_0, \{y_0\} \rangle \hookrightarrow T$restr$_0$ $(Stat0, Stat0\star) \Rightarrow$ $d_{0|\{y_0\}} \subseteq v_0 \times v_0$
$\langle p_2 \rangle \hookrightarrow Stat18(Stat18\star) \Rightarrow$ $p_2 \in d_{0|\{y_0\}}$ & $p_2 \in v_0 \times v_0$
$\langle p_2, v_0, v_0 \rangle \hookrightarrow T$cartesian$_0$ $(Stat18\star) \Rightarrow$ $p_2 = [p_2^{[1]}, p_2^{[2]}]$ & $p_2^{[2]} \in v_0$
$\langle p_2, p_2^{[1]}, p_2^{[2]}, d_0, y_0 \rangle \hookrightarrow T$restr$_2$ $(Stat18\star) \Rightarrow$ $[y_0, p_2^{[2]}] \in d_0$
$\langle p_2^{[2]} \rangle \hookrightarrow Stat16 \Rightarrow$ AUTO
$(Stat18\star)$Discharge $\Rightarrow$ $Stat19: \langle \exists z \in v_0 \mid [y_0, z] \in d_0 \rangle$
$\langle z_0 \rangle \hookrightarrow Stat19(Stat19\star) \Rightarrow$ $Stat20: z_0 \in v_0$ & $[y_0, z_0] \in d_0$

We will show next $\{x_0, x_1, y_0, z_0\}$ form a claw of v_1, e_2, which is untenable. To reach this (sought) contradiction we are to show (in addition to the already established facts that $\{y_0, x_1\}$, $\{x_0, y_0\}$, and $\{y_0, z_0\}$ are edges, to the already known inequalities $x_1 \neq x_0$, $y_0 \neq x_0$, $z_0 \neq x_0$, and to the obvious consequences $y_0 \neq x_1$, $y_0 \neq z_0$ of the acyclicity of d_0) that none of $\{x_0, x_1\}$, $\{x_1, z_0\}$, $\{z_0, x_0\}$ is an edge and that $x_1 \neq z_0$.

$(Stat0, Stat4, Stat14, Stat20\star)$ELEM $\Rightarrow$ $Stat21: \{x_0, x_1, y_0, z_0\} \subseteq v_1$ & $x_0 \notin \{z_0, x_1\}$

In particular, why $\{x_0, x_1\}$ cannot be an edge? Suppose the contrary. It follows from $[x_1, x_1] \notin d_0$ that $[x_0, x_1] \notin d_0$; therefore, since Orientates(d_1, v_1, e_2) holds, we must have $[x_1, x_0] \in d_1$. Since all pairs in $d_1 \backslash d_0$ have first component x_0, this implies $[x_1, x_0] \in d_0$. However, because of Orientates(d_0, v_0, e_2), this implies $\{x_1, x_0\} \subseteq v_0$, whereas we know that $x_0 \notin v_0$.

$\langle d_0, v_0, e_2, x_1 \rangle \hookrightarrow T$orientation$_8$ $(Stat0, Stat0\star) \Rightarrow$ $[x_1, x_1] \notin d_0$
$\langle x_1 \rangle \hookrightarrow Stat4(Stat21\star) \Rightarrow$ $[x_0, x_1] \notin d_1$
Suppose $\Rightarrow$ $\{x_0, x_1\} \in e_2$

(*Stat2, Stat0, Stat4*⋆)ELEM ⇒ $\mathsf{Orientates}(d_1, v_1, e_2)\ \&\ x_0 \in v_1 \backslash v_0\ \&\ x_1 \in v_1 \backslash \{x_0\}\ \&\ \mathsf{Orientates}(d_0, v_0, e_2)$

$\langle d_1, v_1, e_2, x_0, x_1 \rangle \hookrightarrow T\mathsf{orientation}_7\ (Stat15\star) \Rightarrow\quad Stat22:\ [x_1, x_0] \in d_1$

$\left\langle [x_1, x_0], \{x_0\}, \{t \in v_0 \mid \{x_0, t\} \in e_2\} \right\rangle \hookrightarrow T\mathsf{cartesian}_0\ (Stat4, Stat0, Stat22\star) \Rightarrow\quad [x_1, x_0] \in d_0$

$\langle d_0, v_0, e_2, x_1, x_0 \rangle \hookrightarrow T\mathsf{orientation}_6 \Rightarrow$ Auto

(*Stat15*⋆)Discharge ⇒ Auto

$\langle d_1, v_1, e_2, x_1, y_0 \rangle \hookrightarrow T\mathsf{orientation}_6\ (Stat14, Stat0, Stat1, Stat2\star) \Rightarrow\quad \{y_0, x_1\} \in e_2$

$\langle d_1, v_1, e_2, y_0, z_0 \rangle \hookrightarrow T\mathsf{orientation}_6\ (Stat20, Stat0, Stat1, Stat2\star) \Rightarrow\quad \{y_0, z_0\} \in e_2$

$\langle d_1, v_1, e_2, x_0, y_0 \rangle \hookrightarrow T\mathsf{orientation}_6\ (Stat15, Stat2\star) \Rightarrow\quad \{x_0, y_0\} \in e_2$

Suppose ⇒ $x_1 = z_0$

EQUAL $\langle Stat14 \rangle \Rightarrow\quad Stat23:\ [z_0, y_0] \in d_0$

$\langle v_0, d_0, y_0, z_0 \rangle \hookrightarrow T\mathsf{acyclicity}_2\ (Stat0, Stat14, Stat20, Stat23\star) \Rightarrow$ false

Discharge ⇒ Auto

Use_def(ClawFreeG) ⇒ $Stat24:\ \langle \forall w, x, y, z \mid \{w, x, y, z\} \subseteq v_1\ \&\ \{w, y\}, \{y, x\}, \{y, z\} \in e_2 \rightarrow \neg(x \neq z\ \&\ w \notin \{z, x\}\ \&\ \{x, z\} \notin e_2\ \&\ \{z, w\} \notin e_2\ \&\ \{w, x\} \notin e_2) \rangle$

$\langle x_0, x_1, y_0, z_0 \rangle \hookrightarrow Stat24(Stat21\star) \Rightarrow\quad Stat25:\ \{x_1, z_0\} \in e_2 \vee \{z_0, x_0\} \in e_2$

Suppose ⇒ $Stat26:\ [x_0, z_0] \in d_1$

$\langle z_0 \rangle \hookrightarrow Stat4(Stat26, Stat13\star) \Rightarrow\quad Stat27:\ z_0 \notin \{y \in v_0 \mid [x_1, y], [y_0, y] \in d_0\}\ \&\ [x_1, z_0] \in d_0$

$\langle z_0 \rangle \hookrightarrow Stat27 \Rightarrow$ Auto

(*Stat20*⋆)Discharge ⇒ $Stat28:\ [x_0, z_0] \notin d_1$

Suppose ⇒ $Stat29:\ \{x_1, z_0\} \in e_2$

$\langle d_0, v_0, e_2, x_1, z_0 \rangle \hookrightarrow T\mathsf{orientation}_7\ (Stat0, Stat4, Stat20, Stat29\star) \Rightarrow\quad x_1 = z_0 \vee [x_1, z_0] \in d_0 \vee [z_0, x_1] \in d_0$

Suppose ⇒ $x_1 = z_0$

$\langle v_0, d_0, y_0, x_1 \rangle \hookrightarrow T\mathsf{acyclicity}_2\ (Stat0, Stat4, Stat14\star) \Rightarrow\quad [y_0, x_1] \notin d_0$

EQUAL $\langle Stat20 \rangle \Rightarrow$ false; Discharge ⇒ Auto

$\langle z_0 \rangle \hookrightarrow Stat4(Stat28\star) \Rightarrow\quad [z_0, x_1] \in d_0$

$\langle v_0, d_0, x_1, y_0, z_0 \rangle \hookrightarrow T\mathsf{acyclicity}_5\ (Stat0, Stat20, Stat14, Stat4\star) \Rightarrow\quad [z_0, x_1] \notin d_0$

(*Stat29*⋆)Discharge ⇒ $Stat30:\ \{z_0, x_0\} \in e_2$

$\langle d_1, v_1, e_2, z_0, x_0 \rangle \hookrightarrow T\mathsf{orientation}_7\ (Stat2, Stat0, Stat20, Stat21, Stat30, Stat28\star) \Rightarrow\quad Stat31:\ [z_0, x_0] \in d_1$

$\left\langle [z_0, x_0], \{x_0\}, \{t \in v_0 \mid \{x_0, t\} \in e_2\} \right\rangle \hookrightarrow T\mathsf{cartesian}_0\ (Stat21, Stat0, Stat31\star) \Rightarrow\quad Stat32:\ [z_0, x_0] \in d_0$

$\langle d_0, v_0, e_2, z_0, x_0 \rangle \hookrightarrow T\mathsf{orientation}_6\ (Stat32, Stat0\star) \Rightarrow$ false; Discharge ⇒ Qed

Thm cClawFreeG$_2$: [Acyclic extensional orientability of a connected and claw-free (finite) graph] $\mathsf{Finite}(V)\ \&\ \mathsf{HasSpanningTree}(V, E)\ \&\ E \subseteq \{\{x, y\} : x \in V, y \in V \backslash \{x\}\}\ \&\ \mathsf{ClawFreeG}(V, E) \rightarrow \langle \exists d \subseteq V \times V \mid \mathsf{Orientates}(d, V, E)\ \&\ \mathsf{Acyclic}(V, d)\ \&\ \mathsf{Extensional}(V, d) \rangle$. Proof:

Suppose_not(v_2, e_2) ⇒ Auto

> Arguing by contradiction, suppose that there is a counterexample v_2, e_2 to the claim. Then, thanks to the finiteness hypothesis, we can take a minimal counterexample v_1, e_1 with $v_1 \subseteq v_2$ and $e_1 = e_2 \cap \{\{x, y\} : x \in v_1, y \in v_1\}$.

$\langle e_2, v_2, v_2 \rangle \hookrightarrow T\mathsf{vertexInduced}_0\ (\star) \Rightarrow\quad Stat1:\ e_2 \cap \{\{x, y\} : x \in v_2, y \in v_2\} = e_2\ \&\ e_2 = e_2 \cap \{\{x, y\} : x \in v_2, y \in v_2 \backslash \{x\}\}$

EQUAL $\Rightarrow$ $\mathsf{HasSpanningTree}(v_2, e_2 \cap \{\{x,y\} : x \in v_2, y \in v_2\})$ & $\mathsf{ClawFreeG}(v_2, e_2 \cap \{\{x,y\} : x \in v_2, y \in v_2\})$ &
$\neg\langle \exists d \subseteq v_2 \times v_2 \mid \mathsf{Orientates}(d, v_2, e_2)$ & $\mathsf{Acyclic}(v_2, d)$ & $\mathsf{Extensional}(v_2, d)\rangle$
APPLY $\langle \mathsf{fin}_\Theta : v_1\rangle$ $\mathsf{finiteInduction}\big(s_0 \mapsto v_2,\ \Rightarrow$
$P(S) \mapsto \big(\mathsf{HasSpanningTree}(S, e_2 \cap \{\{x,y\} : x \in S, y \in S\})$ & $\mathsf{ClawFreeG}(S, e_2 \cap \{\{x,y\} : x \in S, y \in S\})$ &
$\neg\langle \exists d \subseteq S \times S \mid \mathsf{Orientates}(d, S, e_2)$ & $\mathsf{Acyclic}(S, d)$ & $\mathsf{Extensional}(S, d)\rangle\big)\big)$
Stat3 : $\langle \forall S \mid S \subseteq v_1 \rightarrow \mathsf{Finite}(S)$ & $\big(\mathsf{HasSpanningTree}(S, e_2 \cap \{\{x,y\} : x \in S, y \in S\})$ & $\mathsf{ClawFreeG}(S, e_2 \cap \{\{x,y\} : x \in S, y \in S\})$ &
$\neg\langle \exists d \subseteq S \times S \mid \mathsf{Orientates}(d, S, e_2)$ & $\mathsf{Acyclic}(S, d)$ & $\mathsf{Extensional}(S, d)\rangle \leftrightarrow S = v_1\big)\rangle$
$\langle v_1\rangle \hookrightarrow$ *Stat3(Stat3⋆)* $\Rightarrow$ *Stat4* : $\neg\langle \exists d \subseteq v_1 \times v_1 \mid \mathsf{Orientates}(d, v_1, e_2)$ & $\mathsf{Acyclic}(v_1, d)$ & $\mathsf{Extensional}(v_1, d)\rangle$ &
$\mathsf{HasSpanningTree}(v_1, e_2 \cap \{\{x,y\} : x \in v_1, y \in v_1\})$ & $\mathsf{ClawFreeG}(v_1, e_2 \cap \{\{x,y\} : x \in v_1, y \in v_1\})$

We exclude that v_1 can be a singleton, else a contradiction would arise. In this case, in fact, an extensional acyclic orientation of v_1, e_1 is the null set of edges.

Suppose $\Rightarrow$ $v_1 = \{\mathbf{arb}(v_1)\}$
$\langle \emptyset\rangle \hookrightarrow$ *Stat4* $\Rightarrow$ AUTO; $\langle v_1, \mathbf{arb}(v_1), e_2\rangle \hookrightarrow T\mathsf{voidgraph}_1$ $\Rightarrow$ AUTO; $\langle v_1\rangle \hookrightarrow T\mathsf{voidgraph}_2$ $\Rightarrow$ AUTO
(*Stat4⋆*)Discharge $\Rightarrow$ AUTO

Since v_1 is not a singleton, thanks to Theorem $\mathsf{connectivity}_2$, we can consider a non-cut vertex x_0 of v_1, e_1.

$\langle e_2, v_2, v_1\rangle \hookrightarrow T\mathsf{vertexInduced}_0$ (*Stat1⋆*) $\Rightarrow$ $e_2 \cap \{\{x,y\} : x \in v_1, y \in v_1\} \subseteq \{\{x,y\} : x \in v_1, y \in v_1 \setminus \{x\}\}$
$\langle v_1, e_2 \cap \{\{x,y\} : x \in v_1, y \in v_1\}\rangle \hookrightarrow T\mathsf{connectivity}_2$ (*Stat4⋆*) $\Rightarrow$ *Stat10* : $\langle \exists u \in v_1 \mid \mathsf{HasSpanningTree}(v_1 \setminus \{u\}, \{a \in e_2 \cap \{\{x,y\} : x \in v_1, y \in v_1\} \mid u \notin a\})\rangle$
$\langle x_0\rangle \hookrightarrow$ *Stat10(Stat10⋆)* $\Rightarrow$ $x_0 \in v_1$ & $\mathsf{HasSpanningTree}(v_1 \setminus \{x_0\}, \{a \in e_2 \cap \{\{x,y\} : x \in v_1, y \in v_1\} \mid x_0 \notin a\})$

Now consider the graph v_0, e_0 induced by v_1, e_1 on the strict subset $v_1 \setminus \{x_0\}$ of the set of vertices. Before we can utilize the induction hypothesis, which trivially applies to this subgraph, in order to get an acyclic and extensional orientation d_0 of its vertices, we must specify the set of edges of the induced subgraph in two convenient, equivalent ways.

Suppose $\Rightarrow$ *Stat11* : $\{a \in e_2 \cap \{\{x,y\} : x \in v_1, y \in v_1\} \mid x_0 \notin a\} \neq e_2 \cap \{\{x,y\} : x \in v_1 \setminus \{x_0\}, y \in v_1 \setminus \{x_0\}\}$
$\langle a_1\rangle \hookrightarrow$ *Stat11* $\Rightarrow$ AUTO
Suppose $\Rightarrow$ *Stat12* : $a_1 \in \{a \in e_2 \cap \{\{x,y\} : x \in v_1, y \in v_1\} \mid x_0 \notin a\}$
$\langle\ \rangle \hookrightarrow$ *Stat12(Stat11⋆)* $\Rightarrow$ *Stat13* :
$a_1 \in \{\{x,y\} : x \in v_1, y \in v_1\}$ & $a_1 \notin \{\{x,y\} : x \in v_1 \setminus \{x_0\}, y \in v_1 \setminus \{x_0\}\}$ & $x_0 \notin a_1$
$\langle x_4, y_4, x_4, y_4\rangle \hookrightarrow$ *Stat13(Stat13⋆)* $\Rightarrow$ false
(*Stat12⋆*)Discharge $\Rightarrow$ *Stat14* : $a_1 \in \{\{x,y\} : x \in v_1 \setminus \{x_0\}, y \in v_1 \setminus \{x_0\}\}$ & $a_1 \notin \{a \in e_2 \cap \{\{x,y\} : x \in v_1, y \in v_1\} \mid x_0 \notin a\}$ & $a_1 \in e_2$
$\langle x_5, y_5, \{x_5, y_5\}\rangle \hookrightarrow$ *Stat14(Stat14⋆)* $\Rightarrow$ *Stat15* :
$\{x_5, y_5\} \notin \{\{x,y\} : x \in v_1, y \in v_1\}$ & $x_5, y_5 \in v_1 \setminus \{x_0\}$
$\langle x_5, y_5\rangle \hookrightarrow$ *Stat15(Stat15⋆)* $\Rightarrow$ false; Discharge $\Rightarrow$ $\{a \in e_2 \cap \{\{x,y\} : x \in v_1, y \in v_1\} \mid x_0 \notin a\} = e_2 \cap \{\{x,y\} : x \in v_1 \setminus \{x_0\}, y \in v_1 \setminus \{x_0\}\}$

Suppose $\Rightarrow$ *Stat16*: $\{a \in e_2 \cap \{\{x,y\} : x \in v_1, y \in v_1\} \mid x_0 \notin a\} \neq \{a \in e_2 \cap \{\{x,y\} : x \in v_1, y \in v_1\} \mid a \subseteq v_1 \setminus \{x_0\}\}$
$\langle a_3 \rangle \hookrightarrow$ *Stat16(Stat16⋆)* $\Rightarrow$ *Stat17*: $a_3 \in \{\{x,y\} : x \in v_1, y \in v_1\}$ & $x_0 \notin a_3$ & $a_3 \not\subseteq v_1 \setminus \{x_0\}$
$\langle x_6, y_6 \rangle \hookrightarrow$ *Stat17(Stat17⋆)* $\Rightarrow$ false; Discharge $\Rightarrow$ $\{a \in e_2 \cap \{\{x,y\} : x \in v_1, y \in v_1\} \mid x_0 \notin a\} = \{a \in e_2 \cap \{\{x,y\} : x \in v_1, y \in v_1\} \mid a \subseteq v_1 \setminus \{x_0\}\}$

Suppose $\Rightarrow$ $\neg(\mathsf{HasSpanningTree}(v_1 \setminus \{x_0\}, e_2 \cap \{\{x,y\} : x \in v_1 \setminus \{x_0\}, y \in v_1 \setminus \{x_0\}\})$ & $\mathsf{ClawFreeG}(v_1 \setminus \{x_0\}, e_2 \cap \{\{x,y\} : x \in v_1 \setminus \{x_0\}, y \in v_1 \setminus \{x_0\}\}))$
$\langle v_1, e_2 \cap \{\{x,y\} : x \in v_1, y \in v_1\}, v_1 \setminus \{x_0\} \rangle \hookrightarrow T\mathsf{cClawFreeG}_0$ *(Stat4⋆)* $\Rightarrow$
$\mathsf{ClawFreeG}(v_1 \setminus \{x_0\}, \{a \in e_2 \cap \{\{x,y\} : x \in v_1, y \in v_1\} \mid a \subseteq v_1 \setminus \{x_0\}\})$
EQUAL $\langle$*Stat10*$\rangle$ $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO
$\langle v_1 \setminus \{x_0\} \rangle \hookrightarrow$ *Stat3(Stat10⋆)* $\Rightarrow$ *Stat18*:
$\langle \exists d \subseteq (v_1 \setminus \{x_0\}) \times (v_1 \setminus \{x_0\}) \mid \mathsf{Orientates}(d, v_1 \setminus \{x_0\}, e_2)$ & $\mathsf{Acyclic}(v_1 \setminus \{x_0\}, d)$ & $\mathsf{Extensional}(v_1 \setminus \{x_0\}, d) \rangle$
$\langle d_0 \rangle \hookrightarrow$ *Stat18(Stat18⋆)* $\Rightarrow$ $\mathsf{Orientates}(d_0, v_1 \setminus \{x_0\}, e_2)$ & $\mathsf{Acyclic}(v_1 \setminus \{x_0\}, d_0)$ & $\mathsf{Extensional}(v_1 \setminus \{x_0\}, d_0)$ & $d_0 \subseteq (v_1 \setminus \{x_0\}) \times (v_1 \setminus \{x_0\})$

We first deal with the case when the acyclic digraph $v_1 \setminus \{x_0\}, d_0$ has no sink adjacent to x_0 through e_1 In this case, as suggested by Theorem $\mathsf{cClawFreeG}_1$, we orient the edges incident to x_0 as out-going from x_0, to obtain an extensional acyclic orientation for v_1, e_1. Note that the neighbors of x_0 through e_1 are $\{t \in v_1 \mid \{x_0, t\} \in e_1\}$, hence $d_1 = d_0 \cup \{x_0\} \times \{t \in v_1 \mid \{x_0, t\} \in e_1\}$, although our specification of d_1 will not be so transparent.

Suppose $\Rightarrow$ $\{s \in v_1 \setminus \{x_0\} \mid d_{0|\{s\}} = \emptyset$ & $\{s, x_0\} \in e_2 \cap \{\{x,y\} : x \in v_1, y \in v_1\}\} = \emptyset$
$\langle v_1, v_1 \setminus \{x_0\}, d_0, e_2 \rangle \hookrightarrow T\mathsf{orientation}_0$ *(Stat18⋆)* $\Rightarrow$
$\mathsf{Orientates}(d_0, v_1 \setminus \{x_0\}, e_2 \cap \{\{x,y\} : x \in v_1, y \in v_1\})$
Loc_def $\Rightarrow$ $d_1 = d_0 \cup \{x_0\} \times \{t \in v_1 \setminus \{x_0\} \mid \{x_0, t\} \in e_2 \cap \{\{x,y\} : x \in v_1, y \in v_1\}\}$
*(Stat1⋆)*ELEM $\Rightarrow$ $e_2 \subseteq \{\{x,y\} : x \in v_2, y \in v_2 \setminus \{x\}\}$ & $v_1 = v_1 \setminus \{x_0\} \cup \{x_0\}$
$\langle v_1, v_1 \setminus \{x_0\}, x_0, d_0, e_2 \cap \{\{x,y\} : x \in v_1, y \in v_1\}, v_2, d_1 \rangle \hookrightarrow T\mathsf{cClawFreeG}_1$ *(Stat4⋆)* $\Rightarrow$ *Stat21*:
$d_1 \subseteq v_1 \times v_1$ & $\mathsf{Orientates}(d_1, v_1, e_2 \cap \{\{x,y\} : x \in v_1, y \in v_1\})$ & $\mathsf{Acyclic}(v_1, d_1)$ & $\mathsf{Extensional}(v_1, d_1)$
$\langle v_1, v_1, d_1, e_2 \rangle \hookrightarrow T\mathsf{orientation}_0$ *(Stat18⋆)* $\Rightarrow$ $\mathsf{Orientates}(d_1, v_1, e_2) \leftrightarrow \mathsf{Orientates}(d_1, v_1, e_2 \cap \{\{x,y\} : x \in v_1, y \in v_1\})$
$\langle d_1 \rangle \hookrightarrow$ *Stat4(Stat21⋆)* $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat22*: $\{s \in v_1 \setminus \{x_0\} \mid d_{0|\{s\}} = \emptyset$ & $\{s, x_0\} \in e_2 \cap \{\{x,y\} : x \in v_1, y \in v_1\}\} \neq \emptyset$

Next we deal with the case when the acyclic digraph $v_1 \setminus \{x_0\}, d_0$ has its sink s_1 adjacent to x_0 through e_2. In this case, as suggested by Theorem $\mathsf{xtensionalization}_2$, we orient the edges incident to x_0 as in-coming to x_0.

$\langle s_1 \rangle \hookrightarrow$ *Stat22(Stat22⋆)* $\Rightarrow$ *Stat23*: $\{s_1, x_0\} \in e_2$ & $s_1 \in v_1 \setminus \{x_0\}$ & $d_{0|\{s_1\}} = \emptyset$
Suppose $\Rightarrow$ *Stat24*: $s_1 \notin \{t \in v_1 \mid \{x_0, t\} \in e_2\}$
$\langle s_1 \rangle \hookrightarrow$ *Stat24(Stat23⋆)* $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO
$\langle v_1, v_1 \setminus \{x_0\}, x_0, s_1, d_0, e_2, v_2 \rangle \hookrightarrow T\mathsf{xtensionalization}_2$ *(⋆)* $\Rightarrow$ *Stat25*: $\{y \in v_1 \setminus \{x_0\} \mid [s_1, y] \in d_0\} \neq \emptyset$
$\langle y_7 \rangle \hookrightarrow$ *Stat25(Stat25⋆)* $\Rightarrow$ $y_7 \in v_1 \setminus \{x_0\}$ & $[s_1, y_7] \in d_0$
$\langle [s_1, y_7], s_1, y_7, d_0, s_1 \rangle \hookrightarrow T\mathsf{restr}_2$ *(Stat23⋆)* $\Rightarrow$ false; Discharge $\Rightarrow$ QED

4.2 Result about representing claw-free graphs as transitive sets

DEF heredFinite: [Hereditarily finite] $\mathsf{HerFin}(S) \leftrightarrow_{\mathrm{Def}} \mathsf{Finite}(S) \,\&\, \langle \forall x \in S \mid \mathsf{HerFin}(x) \rangle$

> Our next THEORY will combine the THEORY finMostowskiDecoration seen above with THM cClawFreeG$_2$, in order to represent a claw-free graph by way of a membership digraph whose set of support is a transitive set. Specifically, we will label the vertices of a claw-free graph devoid of self-loops so that:
>
> - the labeling be injective;
> - the null set be the label of a vertex;
> - each doubleton $\{V, W\}$ of vertices be and edge if and only if the label of one of the two vertices belongs to the label of the other vertex;
> - the label of each vertex be a subset of the set of all labels;
> - the set of all labels be transitive;
> - each label be a hereditarily finite set.

THEORY herfinCCFGraphRepr(v_0, e_0)
 $e_0 \subseteq \{\{x, y\} : x \in v_0, y \in v_0 \setminus \{x\}\} \,\&\, \mathsf{Finite}(v_0)$
 $\mathsf{HasSpanningTree}(v_0, e_0) \,\&\, \mathsf{ClawFreeG}(v_0, e_0)$
END herfinCCFGraphRepr

ENTER_THEORY herfinCCFGraphRepr

THM herfinCCFGraphRepr$_0$: [Acyclic extensional orientation of current graph] $\langle \exists d \mid d \subseteq v_0 \times v_0 \,\&\, \mathsf{Orientates}(d, v_0, e_0) \,\&\, \mathsf{Acyclic}(v_0, d) \,\&\, \mathsf{Extensional}(v_0, d) \rangle$. PROOF:
Suppose_not() ⇒ AUTO
 Assump ⇒ $\mathsf{Finite}(v_0) \,\&\, \mathsf{HasSpanningTree}(v_0, e_0) \,\&\, e_0 \subseteq \{\{x, y\} : x \in v_0, y \in v_0 \setminus \{x\}\} \,\&\, \mathsf{ClawFreeG}(v_0, e_0)$
 $\langle v_0, e_0 \rangle \hookrightarrow T$cClawFreeG$_2(\star)$ ⇒ *Stat1* :
  $\langle \exists d \subseteq v_0 \times v_0 \mid \mathsf{Orientates}(d, v_0, e_0) \,\&\, \mathsf{Acyclic}(v_0, d) \,\&\, \mathsf{Extensional}(v_0, d) \rangle \,\&\, \neg \langle \exists d \mid d \subseteq v_0 \times v_0 \,\&\, \mathsf{Orientates}(d, v_0, e_0) \,\&\, \mathsf{Acyclic}(v_0, d) \,\&\, \mathsf{Extensional}(v_0, d) \rangle$
 $\langle d_0, d_0 \rangle \hookrightarrow$ *Stat1*$(\star)$ ⇒ false; Discharge ⇒ QED

APPLY $\langle v1_\Theta : \mathsf{wskiArcs} \rangle$ Skolem⇒

THM herfinCCFGraphRepr$_{0a}$. $\mathsf{wskiArcs} \subseteq v_0 \times v_0 \,\&\, \mathsf{Orientates}(\mathsf{wskiArcs}, v_0, e_0) \,\&\, \mathsf{Acyclic}(v_0, \mathsf{wskiArcs}) \,\&\, \mathsf{Extensional}(v_0, \mathsf{wskiArcs})$.

THM herfinCCFGraphRepr$_{0b}$. $\mathsf{wskiArcs} \subseteq v_0 \times v_0 \,\&\, \mathsf{Orientates}(\mathsf{wskiArcs}, v_0, e_0) \,\&\, \mathsf{Acyclic}(v_0, \mathsf{wskiArcs}) \,\&\, \mathsf{WExtensional}(v_0, \mathsf{wskiArcs}) \,\&\, v_0 \neq \emptyset \,\&\, \mathsf{Finite}(v_0)$. PROOF:

$\mathsf{Suppose\_not}() \Rightarrow$ AUTO

$\mathsf{Assump} \Rightarrow \mathsf{Finite}(v_0) \;\&\; \mathsf{HasSpanningTree}(v_0, e_0)$

$\langle v_0, e_0\rangle \hookrightarrow T\mathsf{connectivity}_0(\star) \Rightarrow v_0 \neq \emptyset$

$\langle\,\rangle \hookrightarrow T\mathsf{herfinCCFGraphRepr}_{0a}(\star) \Rightarrow \mathsf{Extensional}(v_0, \mathsf{wskiArcs}) \;\&\; v_0 \times v_0 \supseteq \mathsf{wskiArcs} \;\&\; \neg\mathsf{WExtensional}(v_0, \mathsf{wskiArcs})$

$\langle v_0, \mathsf{wskiArcs}\rangle \hookrightarrow T\mathsf{extensionality}_0(\star) \Rightarrow \mathsf{false};$ Discharge $\Rightarrow$ QED

APPLY $\langle \mathsf{mski}_\Theta : \mathsf{trans}_\Theta\rangle$ $\mathsf{finMostowskiDecoration}(v_0 \mapsto v_0, d_0 \mapsto \mathsf{wskiArcs}) \Rightarrow$

THM herfinCCFGraphRepr$_{0c}$. $\mathsf{Svm}(\mathsf{trans}_\Theta) \;\&\; \mathbf{dom}(\mathsf{trans}_\Theta) = v_0 \;\&\; \langle \forall w \mid w \in \mathbf{dom}(\mathsf{wskiArcs}) \rightarrow \mathsf{trans}_\Theta{\upharpoonright}w = \{\mathsf{trans}_\Theta{\upharpoonright}p^{[2]} : p \in \mathsf{wskiArcs}_{|\{w\}}\} \;\&\; \mathsf{trans}_\Theta{\upharpoonright}w \neq \emptyset\rangle \;\&\; \emptyset \in \mathbf{range}(\mathsf{trans}_\Theta) \;\&\; \langle \forall y \mid Y \in \mathbf{range}(\mathsf{trans}_\Theta) \rightarrow \mathsf{Finite}(Y)\rangle \;\&\; \langle \forall x, y \mid \{X, Y\} \subseteq v_0 \;\&\; \mathsf{trans}_\Theta{\upharpoonright}X = \mathsf{trans}_\Theta{\upharpoonright}Y \rightarrow X = Y\rangle \;\&\; \langle \forall y, x \mid Y \in v_0 \rightarrow (\mathsf{trans}_\Theta{\upharpoonright}Y \in \mathsf{trans}_\Theta{\upharpoonright}X \leftrightarrow [X, Y] \in \mathsf{wskiArcs})\rangle.$

THM herfinCCFGraphRepr$_{0d}$: [Recursive characterization of the edges-to-membership translation] $\mathsf{trans}_\Theta{\upharpoonright}W = \{\mathsf{trans}_\Theta{\upharpoonright}p^{[2]} : p \in \mathsf{wskiArcs}_{|\{W\}}\}$. PROOF:

$\mathsf{Suppose\_not}(w_0) \Rightarrow$ *Stat0*: $\mathsf{trans}_\Theta{\upharpoonright}w_0 \neq \{\mathsf{trans}_\Theta{\upharpoonright}p^{[2]} : p \in \mathsf{wskiArcs}_{|\{w_0\}}\}$

$\langle\,\rangle \hookrightarrow T\mathsf{herfinCCFGraphRepr}_{0c}(\star) \Rightarrow$ *Stat1*:
$\langle \forall w \mid w \in \mathbf{dom}(\mathsf{wskiArcs}) \rightarrow \mathsf{trans}_\Theta{\upharpoonright}w = \{\mathsf{trans}_\Theta{\upharpoonright}p^{[2]} : p \in \mathsf{wskiArcs}_{|\{w\}}\} \;\&\; \mathsf{trans}_\Theta{\upharpoonright}w \neq \emptyset\rangle \;\&\; \mathbf{dom}(\mathsf{trans}_\Theta) = v_0 \;\&\; \emptyset \in \mathbf{range}(\mathsf{trans}_\Theta) \;\&\; \mathsf{Svm}(\mathsf{trans}_\Theta)$

$\mathsf{Suppose} \Rightarrow w_0 \in \mathbf{dom}(\mathsf{wskiArcs})$

$\langle w_0\rangle \hookrightarrow \mathit{Stat1}(\star) \Rightarrow \mathsf{false};$ Discharge $\Rightarrow$ AUTO

$\mathsf{Suppose} \Rightarrow \mathsf{trans}_\Theta{\upharpoonright}w_0 = \emptyset$

$(\mathit{Stat0}\star)\mathsf{ELEM} \Rightarrow$ *Stat2*: $\{\mathsf{trans}_\Theta{\upharpoonright}p^{[2]} : p \in \mathsf{wskiArcs}_{|\{w_0\}}\} \neq \emptyset$

$\langle p_0\rangle \hookrightarrow \mathit{Stat2}(\mathit{Stat2}\star) \Rightarrow p_0 \in \mathsf{wskiArcs}_{|\{w_0\}}$

$\langle \mathsf{wskiArcs}, \{w_0\}\rangle \hookrightarrow T\mathsf{restr}_0(\mathit{Stat2}\star) \Rightarrow p_0 \in \mathsf{wskiArcs}$

$\langle p_0, \mathsf{wskiArcs}, \{w_0\}\rangle \hookrightarrow T\mathsf{restr}_1(\mathit{Stat2}\star) \Rightarrow p_0{}^{[1]} = w_0$

$\langle p_0, \mathsf{wskiArcs}\rangle \hookrightarrow T\mathsf{domain}_2(\mathit{Stat1}\star) \Rightarrow \mathsf{false};$ Discharge $\Rightarrow$ AUTO

$\langle w_0, \mathsf{trans}_\Theta\rangle \hookrightarrow T\mathsf{image}_3(\star) \Rightarrow$ *Stat4*: $\mathsf{trans}_\Theta{\upharpoonright}w_0 \neq \emptyset \;\&\; w_0 \in \mathbf{dom}(\mathsf{trans}_\Theta)$

$\mathsf{Use\_def}(\mathsf{Extensional}(v_0, \mathsf{wskiArcs})) \Rightarrow$ AUTO

$\mathsf{Use\_def}(\mathbf{range}) \Rightarrow$ *Stat5*: $\emptyset \in \{p^{[2]} : p \in \mathsf{trans}_\Theta\}$

$\langle \mathsf{trans}_\Theta\rangle \hookrightarrow T\mathsf{image}_5(\mathit{Stat1}\star) \Rightarrow \mathsf{trans}_\Theta = \{[x, \mathsf{trans}_\Theta{\upharpoonright}x] : x \in \mathbf{dom}(\mathsf{trans}_\Theta)\}$

$\mathsf{EQUAL}(\mathit{Stat5}) \Rightarrow \emptyset \in \{p^{[2]} : p \in \{[x, \mathsf{trans}_\Theta{\upharpoonright}x] : x \in \mathbf{dom}(\mathsf{trans}_\Theta)\}\}$

$\mathsf{SIMPLF}(\mathit{Stat5}) \Rightarrow$ *Stat6*: $\emptyset \in \left\{[x, \mathsf{trans}_\Theta{\upharpoonright}x]^{[2]} : x \in \mathbf{dom}(\mathsf{trans}_\Theta)\right\}$

$\langle x_0\rangle \hookrightarrow \mathit{Stat6}(\mathit{Stat6}\star) \Rightarrow \mathsf{trans}_\Theta{\upharpoonright}x_0 = \emptyset \;\&\; x_0 \in \mathbf{dom}(\mathsf{trans}_\Theta)$

$\mathsf{Suppose} \Rightarrow x_0 \in \mathbf{dom}(\mathsf{wskiArcs})$

$\langle x_0\rangle \hookrightarrow \mathit{Stat1}(\mathit{Stat6}\star) \Rightarrow \mathsf{false};$ Discharge $\Rightarrow$ AUTO

$\langle\,\rangle \hookrightarrow T\mathsf{herfinCCFGraphRepr}_{0a}(\mathit{Stat6}\star) \Rightarrow \mathsf{Extensional}(v_0, \mathsf{wskiArcs})$

$\mathsf{Suppose} \Rightarrow x_0 = w_0$

$\mathsf{EQUAL}(\mathit{Stat4}) \Rightarrow \mathsf{false};$ Discharge $\Rightarrow$ AUTO

$\langle v_0, \mathsf{wskiArcs}, x_0, w_0\rangle \hookrightarrow T\mathsf{extensionality}_2(\mathit{Stat1}\star) \Rightarrow$ *Stat7*: $\neg(\langle \forall z \mid [x_0, z] \notin \mathsf{wskiArcs}\rangle \;\&\; \langle \forall z \mid [w_0, z] \notin \mathsf{wskiArcs}\rangle)$

$\mathsf{Suppose} \Rightarrow$ *Stat8*: $\neg\langle \forall z \mid [w_0, z] \notin \mathsf{wskiArcs}\rangle$

$\langle z_0 \rangle \hookrightarrow \mathit{Stat8}(\mathit{Stat8}\star) \Rightarrow \quad [w_0, z_0] \in \mathsf{wskiArcs}$
$\langle w_0, z_0, \mathsf{wskiArcs} \rangle \hookrightarrow T\mathsf{domain}_3(\mathit{Stat1}\star) \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \text{AUTO}$
$(\mathit{Stat7}\star)\mathsf{ELEM} \Rightarrow \quad \mathit{Stat9}: \neg\langle \forall z \mid [x_0, z] \notin \mathsf{wskiArcs} \rangle$
$\langle z_1 \rangle \hookrightarrow \mathit{Stat9}(\mathit{Stat9}\star) \Rightarrow \quad [x_0, z_1] \in \mathsf{wskiArcs}$
$\langle x_0, z_1, \mathsf{wskiArcs} \rangle \hookrightarrow T\mathsf{domain}_3(\mathit{Stat6}\star) \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \text{QED}$

THM herfinCCFGraphRepr$_1$. $[X, Y] \in \mathsf{wskiArcs} \vee [Y, X] \in \mathsf{wskiArcs} \leftrightarrow \{X, Y\} \in e_0$. PROOF:
$\mathsf{Suppose\_not}(x_0, y_0) \Rightarrow \quad \text{AUTO}$
$\langle\,\rangle \hookrightarrow T\mathsf{herfinCCFGraphRepr}_{0a}(\star) \Rightarrow \quad \mathit{Stat1}: \mathsf{wskiArcs} \subseteq v_0 \times v_0 \;\&\; \mathsf{Orientates}(\mathsf{wskiArcs}, v_0, e_0)$
$\mathsf{Assump} \Rightarrow \quad e_0 \subseteq \{\{x, y\} : x \in v_0, y \in v_0 \setminus \{x\}\}$
$\mathsf{Suppose} \Rightarrow \quad \mathit{Stat2}: \{\{p^{[1]}, p^{[2]}\} : p \in \mathsf{wskiArcs} \mid p = [p^{[1]}, p^{[2]}]\} \neq \{\{p^{[1]}, p^{[2]}\} : p \in \mathsf{wskiArcs}\}$
$\langle p_0 \rangle \hookrightarrow \mathit{Stat2}(\mathit{Stat1}\star) \Rightarrow \quad p_0 \in v_0 \times v_0 \;\&\; p_0 \neq [p_0{}^{[1]}, p_0{}^{[2]}]$
$\langle p_0, v_0, v_0 \rangle \hookrightarrow T\mathsf{cartesian}_0(\mathit{Stat2}\star) \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \text{AUTO}$
$\mathsf{Use\_def}(\mathsf{Orientates}) \Rightarrow \quad \mathit{Stat3}: [x_0, y_0] \in \mathsf{wskiArcs} \vee [y_0, x_0] \in \mathsf{wskiArcs} \neq \{x_0, y_0\} \in \{\{p^{[1]}, p^{[2]}\} : p \in \mathsf{wskiArcs}\}$
$\mathsf{Suppose} \Rightarrow \quad \mathit{Stat4}: \{x_0, y_0\} \in \{\{p^{[1]}, p^{[2]}\} : p \in \mathsf{wskiArcs}\} \;\&\; \neg([x_0, y_0] \in \mathsf{wskiArcs} \vee [y_0, x_0] \in \mathsf{wskiArcs})$
$\langle p_1 \rangle \hookrightarrow \mathit{Stat4}(\mathit{Stat1}\star) \Rightarrow \quad \mathit{Stat5}: \{x_0, y_0\} = \{p_1{}^{[1]}, p_1{}^{[2]}\} \;\&\; p_1 \in v_0 \times v_0 \;\&\; p_1 \in \mathsf{wskiArcs}$
$\langle p_1, v_0, v_0 \rangle \hookrightarrow T\mathsf{cartesian}_0(\mathit{Stat5}\star) \Rightarrow \quad p_1 = [p_1{}^{[1]}, p_1{}^{[2]}] \;\&\; (x_0 = p_1{}^{[1]} \;\&\; y_0 = p_1{}^{[2]}) \vee (x_0 = p_1{}^{[2]} \;\&\; y_0 = p_1{}^{[1]})$
$\mathsf{Suppose} \Rightarrow \quad x_0 = p_1{}^{[1]} \;\&\; y_0 = p_1{}^{[2]}$
$\mathsf{EQUAL} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad y_0 = p_1{}^{[1]} \;\&\; x_0 = p_1{}^{[2]}$
$\mathsf{EQUAL} \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \text{AUTO}$
$\mathsf{Suppose} \Rightarrow \quad \mathit{Stat6}: \{x_0, y_0\} \notin \{\{p^{[1]}, p^{[2]}\} : p \in \mathsf{wskiArcs}\} \;\&\; [x_0, y_0] \in \mathsf{wskiArcs}$
$\langle [x_0, y_0] \rangle \hookrightarrow \mathit{Stat6}(\mathit{Stat6}\star) \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \mathit{Stat7}: \{x_0, y_0\} \notin \{\{p^{[1]}, p^{[2]}\} : p \in \mathsf{wskiArcs}\} \;\&\; [y_0, x_0] \in \mathsf{wskiArcs}$
$\langle [y_0, x_0] \rangle \hookrightarrow \mathit{Stat7}(\mathit{Stat7}\star) \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \text{QED}$

THM herfinCCFGraphRepr$_2$: [Edges between vertices are modeled by membership under the translation]
$\{X, Y\} \subseteq v_0 \rightarrow (\mathsf{trans}_\Theta \upharpoonright Y \in \mathsf{trans}_\Theta \upharpoonright X \vee \mathsf{trans}_\Theta \upharpoonright X \in \mathsf{trans}_\Theta \upharpoonright Y \leftrightarrow \{X, Y\} \in e_0)$. PROOF:
$\mathsf{Suppose\_not}(x_0, y_0) \Rightarrow \quad \text{AUTO}$
$\langle\,\rangle \hookrightarrow T\mathsf{herfinCCFGraphRepr}_{0c}(\star) \Rightarrow \quad \mathit{Stat1}: \langle \forall y, x \mid y \in v_0 \rightarrow (\mathsf{trans}_\Theta \upharpoonright y \in \mathsf{trans}_\Theta \upharpoonright x \leftrightarrow [x, y] \in \mathsf{wskiArcs}) \rangle$
$\langle y_0, x_0 \rangle \hookrightarrow \mathit{Stat1}(\star) \Rightarrow \quad \mathsf{trans}_\Theta \upharpoonright y_0 \in \mathsf{trans}_\Theta \upharpoonright x_0 \leftrightarrow [x_0, y_0] \in \mathsf{wskiArcs}$
$\langle x_0, y_0 \rangle \hookrightarrow \mathit{Stat1}(\star) \Rightarrow \quad \mathsf{trans}_\Theta \upharpoonright x_0 \in \mathsf{trans}_\Theta \upharpoonright y_0 \leftrightarrow [y_0, x_0] \in \mathsf{wskiArcs}$
$\langle x_0, y_0 \rangle \hookrightarrow T\mathsf{herfinCCFGraphRepr}_1(\star) \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad \text{QED}$

> The following theorem sets the ground for a proof that the range of the edges-to-membership translation is a transitive set.

THM herfinCCFGraphRepr$_3$: [Every image of the edges-to-membership translation is included in its range] $X \in \mathbf{dom}(\mathsf{trans}_\Theta) \rightarrow \mathsf{trans}_\Theta \upharpoonright X \subseteq \mathbf{range}(\mathsf{trans}_\Theta)$. PROOF:
$\mathsf{Suppose\_not}(x_0) \Rightarrow \quad \mathit{Stat0}: \mathsf{trans}_\Theta \upharpoonright x_0 \not\subseteq \mathbf{range}(\mathsf{trans}_\Theta) \;\&\; x_0 \in \mathbf{dom}(\mathsf{trans}_\Theta)$
$\langle\,\rangle \hookrightarrow T\mathsf{herfinCCFGraphRepr}_{0c}(\star) \Rightarrow \quad \mathit{Stat1}: \mathsf{Svm}(\mathsf{trans}_\Theta) \;\&\; \mathbf{dom}(\mathsf{trans}_\Theta) = v_0$

$\langle x_0 \rangle \hookrightarrow T\text{herfinCCFGraphRepr}_{0d} \Rightarrow$ AUTO

$\langle x_1 \rangle \hookrightarrow Stat0(Stat0\star) \Rightarrow$ $Stat2: x_1 \in \{\text{trans}_\Theta \upharpoonright p^{[2]} : p \in \text{wskiArcs}_{|\{x_0\}}\} \ \&\ x_1 \notin \mathbf{range}(\text{trans}_\Theta)$

$\langle \text{wskiArcs}, \{x_0\} \rangle \hookrightarrow T\text{restr}_0(Stat2\star) \Rightarrow$ $\text{wskiArcs}_{|\{x_0\}} \subseteq \text{wskiArcs}$

$\langle\ \rangle \hookrightarrow T\text{herfinCCFGraphRepr}_{0a}(Stat2\star) \Rightarrow$ $\text{wskiArcs} \subseteq v_0 \times v_0$

$\langle p_1 \rangle \hookrightarrow Stat2(Stat2\star) \Rightarrow$ $p_1 \in v_0 \times v_0 \ \&\ \text{trans}_\Theta \upharpoonright p_1^{[2]} \notin \mathbf{range}(\text{trans}_\Theta)$

$\langle p_1, v_0, v_0 \rangle \hookrightarrow T\text{cartesian}_0(Stat2\star) \Rightarrow$ $p_1^{[2]} \in v_0$

$\langle p_1^{[2]}, \text{trans}_\Theta \upharpoonright p_1^{[2]}, \text{trans}_\Theta \rangle \hookrightarrow T\text{domain}_3(Stat1\star) \Rightarrow$ $Stat3: [p_1^{[2]}, \text{trans}_\Theta \upharpoonright p_1^{[2]}] \notin \text{trans}_\Theta$

$\langle \text{trans}_\Theta \rangle \hookrightarrow T\text{image}_5(Stat1, Stat1\star) \Rightarrow$ $\text{trans}_\Theta = \{[x, \text{trans}_\Theta \upharpoonright x] : x \in \mathbf{dom}(\text{trans}_\Theta)\}$

EQUAL$(Stat3) \Rightarrow$ $Stat4: [p_1^{[2]}, \text{trans}_\Theta \upharpoonright p_1^{[2]}] \notin \{[x, \text{trans}_\Theta \upharpoonright x] : x \in \mathbf{dom}(\text{trans}_\Theta)\}$

$\langle p_1^{[2]} \rangle \hookrightarrow Stat4(Stat1\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

|| The following theorem states that the range of the edges-to-membership translation is a transitive set.

THM herfinCCFGraphRepr$_4$: [Transitivity of the range of the edges-to-membership translation] $\{y \in \mathbf{range}(\text{trans}_\Theta) \mid y \not\subseteq \mathbf{range}(\text{trans}_\Theta)\} = \emptyset$. PROOF:

Suppose_not() $\Rightarrow$ $Stat0: \{y \in \mathbf{range}(\text{trans}_\Theta) \mid y \not\subseteq \mathbf{range}(\text{trans}_\Theta)\} \neq \emptyset$

$\langle y_0 \rangle \hookrightarrow Stat0(\star) \Rightarrow$ $y_0 \in \mathbf{range}(\text{trans}_\Theta) \ \&\ y_0 \not\subseteq \mathbf{range}(\text{trans}_\Theta)$

$\langle\ \rangle \hookrightarrow T\text{herfinCCFGraphRepr}_{0c}(\star) \Rightarrow$ $\text{Svm}(\text{trans}_\Theta) \ \&\ \mathbf{dom}(\text{trans}_\Theta) = v_0$

$\langle \text{trans}_\Theta \rangle \hookrightarrow T\text{image}_5(\star) \Rightarrow$ $\text{trans}_\Theta = \{[x, \text{trans}_\Theta \upharpoonright x] : x \in \mathbf{dom}(\text{trans}_\Theta)\}$

Use_def$(\mathbf{range}) \Rightarrow$ $y_0 \in \{p^{[2]} : p \in \text{trans}_\Theta\}$

EQUAL $\Rightarrow$ $y_0 \in \{p^{[2]} : p \in \{[x, \text{trans}_\Theta \upharpoonright x] : x \in \mathbf{dom}(\text{trans}_\Theta)\}\}$

SIMPLF $\Rightarrow$ $Stat1: y_0 \in \{[x, \text{trans}_\Theta \upharpoonright x]^{[2]} : x \in \mathbf{dom}(\text{trans}_\Theta)\}$

$\langle x_0 \rangle \hookrightarrow Stat1(Stat1) \Rightarrow$ $y_0 = \text{trans}_\Theta \upharpoonright x_0 \ \&\ x_0 \in \mathbf{dom}(\text{trans}_\Theta)$

$\langle x_0 \rangle \hookrightarrow T\text{herfinCCFGraphRepr}_3(\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

|| The following theorem recapitulates some of the facts already proved in THM herfinCCFGraphRepr$_{0c}$ (which cannot be exported from the present THEORY, because its claim involves the private symbol 'wskiArcs'); it also says something new, namely that the range of the edges-to-membership translation is a hereditarily finite set.

THM herfinCCFGraphRepr$_5$: [Compendium of properties of the edges-to-membership translation]
$\text{Svm}(\text{trans}_\Theta) \ \&\ \mathbf{dom}(\text{trans}_\Theta) = v_0 \ \&\ \emptyset \neq \mathbf{range}(\text{trans}_\Theta) \ \&\ \text{HerFin}(\mathbf{range}(\text{trans}_\Theta))$. PROOF:

Suppose_not() $\Rightarrow$ AUTO

Use_def$(\text{HerFin}(\mathbf{range}(\text{trans}_\Theta))) \Rightarrow$ AUTO

$\langle\ \rangle \hookrightarrow T\text{herfinCCFGraphRepr}_{0c}(\star) \Rightarrow$ $Stat1: \langle \forall y \mid y \in \mathbf{range}(\text{trans}_\Theta) \rightarrow \text{Finite}(y) \rangle \ \&\ \text{Svm}(\text{trans}_\Theta) \ \&\ \mathbf{dom}(\text{trans}_\Theta) = v_0 \ \&$
$\neg(\text{Finite}(\mathbf{range}(\text{trans}_\Theta)) \ \&\ \langle \forall y \in \mathbf{range}(\text{trans}_\Theta) \mid \text{HerFin}(y) \rangle)$

$\langle \text{trans}_\Theta \rangle \hookrightarrow T\text{image}_5(Stat1\star) \Rightarrow$ $\text{trans}_\Theta = \{[x, \text{trans}_\Theta \upharpoonright x] : x \in \mathbf{dom}(\text{trans}_\Theta)\}$

|| The following TELEM step makes implicit use of the THEORY isSvm seen at the beginning of this scenario.

TELEM $\Rightarrow$ $\mathbf{range}(\{[x, trans_\Theta \restriction x] : x \in \mathbf{dom}(trans_\Theta)\}) = \{trans_\Theta \restriction x : x \in \mathbf{dom}(trans_\Theta)\}$
EQUAL(*Stat1*) $\Rightarrow$ *Stat2* : $\neg(\text{Finite}(\{trans_\Theta \restriction x : x \in v_0\}) \,\&\, \langle \forall y \in \{trans_\Theta \restriction x : x \in v_0\} \mid \text{HerFin}(y)\rangle)$
Assump $\Rightarrow$ $\text{Finite}(v_0)$
APPLY $\langle\ \rangle$ finiteImage$(s_0 \mapsto v_0, f(X) \mapsto trans_\Theta \restriction X)$ $\Rightarrow$ $\text{Finite}(\{trans_\Theta \restriction x : x \in v_0\})$
(*Stat2*⋆)ELEM $\Rightarrow$ *Stat3* : $\neg\langle \forall y \in \{trans_\Theta \restriction x : x \in v_0\} \mid \text{HerFin}(y)\rangle$
$\langle y_0 \rangle \hookrightarrow$ *Stat3*(*Stat3*⋆) $\Rightarrow$ *Stat4* : $y_0 \in \{trans_\Theta \restriction x : x \in v_0\} \,\&\, \neg\text{HerFin}(y_0)$
$\langle x_0 \rangle \hookrightarrow$ *Stat4*(*Stat4*⋆) $\Rightarrow$ $x_0 \in v_0 \,\&\, y_0 = trans_\Theta \restriction x_0 \,\&\, \neg\text{HerFin}(y_0)$
EQUAL(*Stat4*) $\Rightarrow$ $\neg\text{HerFin}(trans_\Theta \restriction x_0)$
Loc_def $\Rightarrow$ *Stat5* : $a_0 = \mathbf{arb}(\{trans_\Theta \restriction x : x \in v_0 \mid \neg\text{HerFin}(trans_\Theta \restriction x)\})$
Suppose $\Rightarrow$ *Stat6* : $\{trans_\Theta \restriction x : x \in v_0 \mid \neg\text{HerFin}(trans_\Theta \restriction x)\} = \emptyset$
$\langle x_0 \rangle \hookrightarrow$ *Stat6*(*Stat4*⋆) $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO
(*Stat5*)ELEM $\Rightarrow$ *Stat7* : $a_0 \in \{trans_\Theta \restriction x : x \in v_0 \mid \neg\text{HerFin}(trans_\Theta \restriction x)\} \,\&\, a_0 \cap \{trans_\Theta \restriction x : x \in v_0 \mid \neg\text{HerFin}(trans_\Theta \restriction x)\} = \emptyset$
$\langle x_1 \rangle \hookrightarrow$ *Stat7*(*Stat7*⋆) $\Rightarrow$ $x_1 \in v_0 \,\&\, a_0 = trans_\Theta \restriction x_1 \,\&\, \neg\text{HerFin}(trans_\Theta \restriction x_1)$
Use_def(HerFin(a_0)) $\Rightarrow$ AUTO
EQUAL(*Stat7*) $\Rightarrow$ *Stat8* : $\neg(\text{Finite}(a_0) \,\&\, \langle \forall x \in a_0 \mid \text{HerFin}(x)\rangle)$
Suppose $\Rightarrow$ $\neg\text{Finite}(a_0)$
$\langle a_0 \rangle \hookrightarrow$ *Stat1*(*Stat1*⋆) $\Rightarrow$ $a_0 \notin \mathbf{range}(trans_\Theta)$
$\langle x_1, a_0, trans_\Theta \rangle \hookrightarrow T\text{domain}_3$(*Stat8*⋆) $\Rightarrow$ $[x_1, a_0] \notin trans_\Theta$
$\langle trans_\Theta \rangle \hookrightarrow T\text{image}_5$(*Stat1*, *Stat1*⋆) $\Rightarrow$ $trans_\Theta = \{[x, trans_\Theta \restriction x] : x \in \mathbf{dom}(trans_\Theta)\}$
EQUAL(*Stat7*) $\Rightarrow$ *Stat9* : $[x_1, trans_\Theta \restriction x_1] \notin \{[x, trans_\Theta \restriction x] : x \in \mathbf{dom}(trans_\Theta)\}$
$\langle x_1 \rangle \hookrightarrow$ *Stat9*(*Stat1*⋆) $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO
$\langle x_1 \rangle \hookrightarrow T\text{herfinCCFGraphRepr}_{0d}$(*Stat7*⋆) $\Rightarrow$ $a_0 = \{trans_\Theta \restriction p^{[2]} : p \in \text{wskiArcs}_{|\{x_1\}}\}$
$\langle \text{wskiArcs}, \{x_1\} \rangle \hookrightarrow T\text{restr}_0$(*Stat8*⋆) $\Rightarrow$ $\text{wskiArcs}_{|\{x_1\}} \subseteq \text{wskiArcs}$
$\langle\ \rangle \hookrightarrow T\text{herfinCCFGraphRepr}_{0a}$(*Stat8*⋆) $\Rightarrow$ $\text{wskiArcs} \subseteq v_0 \times v_0$
(*Stat8*)ELEM $\Rightarrow$ *Stat10* : $\neg\langle \forall x \in a_0 \mid \text{HerFin}(x)\rangle$
$\langle x_2 \rangle \hookrightarrow$ *Stat10*(*Stat7*⋆) $\Rightarrow$ *Stat11* : $x_2 \in \{trans_\Theta \restriction p^{[2]} : p \in \text{wskiArcs}_{|\{x_1\}}\} \,\&\, x_2 \notin \{trans_\Theta \restriction x : x \in v_0 \mid \neg\text{HerFin}(trans_\Theta \restriction x)\} \,\&\, \neg\text{HerFin}(x_2)$
$\langle p_1, p_1^{[2]} \rangle \hookrightarrow$ *Stat11*(*Stat11*⋆) $\Rightarrow$ $x_2 = trans_\Theta \restriction p_1^{[2]} \,\&\, p_1 \in \text{wskiArcs}_{|\{x_1\}} \,\&\, \neg\text{HerFin}(x_2) \,\&\, \neg(p_1^{[2]} \in v_0 \,\&\, \neg\text{HerFin}(trans_\Theta \restriction p_1^{[2]}))$
EQUAL(*Stat11*) $\Rightarrow$ $\neg\text{HerFin}(trans_\Theta \restriction p_1^{[2]})$
(*Stat8*⋆)ELEM $\Rightarrow$ $p_1 \in v_0 \times v_0 \,\&\, p_1^{[2]} \notin v_0$
$\langle p_1, v_0, v_0 \rangle \hookrightarrow T\text{cartesian}_0$(*Stat11*⋆) $\Rightarrow$ false; Discharge $\Rightarrow$ QED

ENTER_THEORY Set_theory
DISPLAY herfinCCFGraphRepr

THEORY herfinCCFGraphRepr$(\mathsf{v}_0, \mathsf{e}_0)$

$\mathsf{e}_0 \subseteq \{\{\mathsf{x},\mathsf{y}\} : \mathsf{x} \in \mathsf{v}_0, \mathsf{y} \in \mathsf{v}_0 \setminus \{\mathsf{x}\}\}$ & $\mathsf{Finite}(\mathsf{v}_0)$

$\mathsf{HasSpanningTree}(\mathsf{v}_0, \mathsf{e}_0)$ & $\mathsf{ClawFreeG}(\mathsf{v}_0, \mathsf{e}_0)$

$\Rightarrow (\mathsf{trans}_\Theta)$

$\mathsf{Svm}(\mathsf{trans}_\Theta)$ & $\mathbf{dom}(\mathsf{trans}_\Theta) = \mathsf{v}_0$

$\langle \forall \mathsf{x}, \mathsf{y} \mid \{\mathsf{X}, \mathsf{Y}\} \subseteq \mathsf{v}_0$ & $\mathsf{trans}_\Theta \restriction \mathsf{X} = \mathsf{trans}_\Theta \restriction \mathsf{Y} \rightarrow \mathsf{X} = \mathsf{Y} \rangle$

$\langle \forall \mathsf{x}, \mathsf{y} \mid \{\mathsf{X}, \mathsf{Y}\} \subseteq \mathsf{v}_0 \rightarrow (\mathsf{trans}_\Theta \restriction \mathsf{Y} \in \mathsf{trans}_\Theta \restriction \mathsf{X} \vee \mathsf{trans}_\Theta \restriction \mathsf{X} \in \mathsf{trans}_\Theta \restriction \mathsf{Y} \leftrightarrow \{\mathsf{X}, \mathsf{Y}\} \in \mathsf{e}_0) \rangle$

$\{\mathsf{y} \in \mathbf{range}(\mathsf{trans}_\Theta) \mid \mathsf{y} \not\subseteq \mathbf{range}(\mathsf{trans}_\Theta)\} = \emptyset$

$\mathbf{range}(\mathsf{trans}_\Theta) \neq \emptyset$ & $\mathsf{HerFin}(\mathbf{range}(\mathsf{trans}_\Theta))$

END herfinCCFGraphRepr

A Synopsis of all definitions

DEF pair_1: [Ordered pair] $[X, Y] =_{\text{Def}}$

$\{\{X\}, \{\{X\}, \{\{Y\}, Y\}\}\}$

DEF pair_2: [First component of ordered pair] $P^{[1]} =_{\text{Def}}$

$\mathbf{arb}(\mathbf{arb}(P))$

DEF pair_3: [Second component of ordered pair] $P^{[2]} =_{\text{Def}}$

$\mathbf{arb}(\mathbf{arb}(\mathbf{arb}(P \setminus \{\mathbf{arb}(P)\}) \setminus \{\mathbf{arb}(P)\}))$

DEF maps_1: [Map domain, i.e. first components of pairs in map] $\mathbf{dom}(F) =_{\text{Def}}$

$\{x^{[1]} : x \in F\}$

DEF maps_2: [Map restriction] $F_{|A} =_{\text{Def}}$

$\{p \in F \mid p^{[1]} \in A\}$

DEF maps_3: [Value of single-valued function] $F \upharpoonright X =_{\text{Def}}$

$\mathbf{arb}\big(F_{|\{X\}}\big)^{[2]}$

DEF maps_4: [Map range, i.e. second components of pairs in map] $\mathbf{range}(F) =_{\text{Def}}$

$\{p^{[2]} : p \in F\}$

DEF maps_5: [Map predicate] $\mathsf{Is\_map}(F) \leftrightarrow_{\text{Def}}$

$\langle \forall p \in F \mid p = [p^{[1]}, p^{[2]}] \rangle$

DEF maps_6: [Single-valued map] $\mathsf{Svm}(F) \leftrightarrow_{\text{Def}}$

$\mathsf{Is\_map}(F) \,\&\, \langle \forall p \in F, q \in F \mid p^{[1]} = q^{[1]} \rightarrow p = q \rangle$

DEF pow: [Family of all subsets of a given set] $\mathcal{P}S =_{\text{Def}}$

$\{x : x \subseteq S\}$

DEF Finite : [Finitude] $\mathsf{Finite}(F) \leftrightarrow_{\text{Def}}$

$\langle \forall g \in \mathcal{P}(\mathcal{P}F) \setminus \{\emptyset\}, \exists m \mid g \cap \mathcal{P}m = \{m\} \rangle$

DEF cartesianProduct: [Cartesian product] $S \times T =_{\text{Def}}$

$\{[x, y] : x \in S, y \in T\}$

DEF orientation : [Orientation of a graph] $\mathsf{Orientates}(D, V, E) \leftrightarrow_{\mathrm{Def}}$

$$E \cap \{\{x, y\} : x \in V, y \in V \setminus \{x\}\} = \{\{p^{[1]}, p^{[2]}\} : p \in D \mid p = [p^{[1]}, p^{[2]}]\}$$

DEF xtensionality$_0$: [Extensionality] $\mathsf{Extensional}(V, D) \leftrightarrow_{\mathrm{Def}}$

$$\langle \forall x \in V, y \in V, \exists z \mid ([x, z] \in D \leftrightarrow [y, z] \in D) \to x = y \rangle$$

DEF xtensionality$_1$: [Weak extensionality] $\mathsf{WExtensional}(V, D) \leftrightarrow_{\mathrm{Def}}$

$$\mathsf{Extensional}(V \cap \mathbf{dom}(D \cap (V \times V)), D \cap (V \times V))$$

DEF acyclicity: [Acyclicity] $\mathsf{Acyclic}(V, D) \leftrightarrow_{\mathrm{Def}}$

$$\langle \forall w \subseteq V \mid w \neq \emptyset \to \langle \exists t \in w \mid \emptyset = \{y \in w \mid [t, y] \in D\} \rangle \rangle$$

DEF unionset: [Family of all members of members of a set] $\bigcup S =_{\mathrm{Def}}$

$$\{u : v \in S, u \in v\}$$

DEF hank_free: [A hank-free graph is one whose edges do not include a hank] $\mathsf{HankFree}(T) \leftrightarrow_{\mathrm{Def}}$

$$\langle \forall e \subseteq T \mid e = \emptyset \vee \langle \exists a \in e \mid a \not\subseteq \bigcup (e \setminus \{a\}) \rangle \rangle$$

DEF tree$_1$: [Hank-free graph whose edges cannot be partitioned into multiple vertex-disjoint blocks] $\mathsf{Is\_tree}(T) \leftrightarrow_{\mathrm{Def}}$

$$\langle \forall p \mid \bigcup p = T \,\&\, \langle \forall b \in p, c \in p \mid \bigcup b \cap \bigcup c \neq \emptyset \leftrightarrow b = c \rangle \to p = \{T\} \rangle \,\&\, \mathsf{HankFree}(T)$$

DEF connectivity$_1$: [A graph endowed with a spanning tree] $\mathsf{HasSpanningTree}(V, E) \leftrightarrow_{\mathrm{Def}}$

$$\langle \exists t \mid \mathsf{Is\_tree}(t) \,\&\, \bigcup t = V \,\&\, (V = \{\mathbf{arb}(V)\} \vee t \subseteq E) \rangle$$

DEF clawFreeGraph: [Claw-freeness, as a property of graphs] $\mathsf{ClawFreeG}(V, E) \leftrightarrow_{\mathrm{Def}}$

$$\langle \forall w, x, y, z \mid \{w, x, y, z\} \subseteq V \,\&\, \{w, y\}, \{y, x\}, \{y, z\} \in E \to \neg(x \neq z \,\&\, w \notin \{z, x\} \,\&\, \{x, z\} \notin E \,\&\, \{z, w\} \notin E \,\&\, \{w, x\} \notin E) \rangle$$

DEF heredFinite: [Hereditary finitude] $\mathsf{HerFin}(S) \leftrightarrow_{\mathrm{Def}}$

$$\mathsf{Finite}(S) \,\&\, \langle \forall x \in S \mid \mathsf{HerFin}(x) \rangle$$