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Set Graphs. III. Proof Pearl: Claw-free Graphs Mirrored into Transitive Hereditarily Finite Sets

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Abstract We report on the formalization of two classical results about claw-free graphs, which have been verified correct by the proof-checker Referee. We have proved formally that every connected claw-free graph admits (1) a near-perfect matching, (2) Hamiltonian cycles in its square. To take advantage of the set-theoretic foundation of Referee, we exploited set equivalents of the graph-theoretic notions involved in our experiment: edge, source, square, etc. To ease some proofs, we have often resorted to weak counterparts of well-established notions such as cycle, claw-freeness, longest directed path, etc.

Keywords Claw-free graph · Theory-based automated reasoning · Proof-checking · Referee

1 Introduction

In this paper we report about a computer-checked proof of two classical results on connected claw-free graphs [4,5], specifically the facts that any such graph

- owns a perfect matching if its number of vertices is even [23,26];
- has a Hamiltonian cycle in its square [17].

In our experiment, these results have been referred to a special class of digraphs, to be called *membership digraphs*, whose vertices are hereditarily finite

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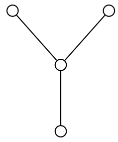


Fig. 1 The claw, $K_{1,3}$.

sets and whose edges reflect the membership relation between them. Ours is a legitimate change of perspective—one which will even lead us to more general results—in the light of [18], which showed how to translate connected claw-free graphs into membership digraphs.

Claw-free graphs, emerging in the 1960s as a generalization of line graphs [4, 5], are finite graphs that have no claw as induced subgraph: the *claw* being the complete bipartite graph $K_{1,3}$ depicted in Figure 1. As mentioned in [11], claw-free graphs caught the attention of the graph theory community once some basic graph properties regarding matchings and Hamiltonicity were discovered.

It is not rare that classes of graphs defined in terms of forbidden induced subgraphs are proposed as substitutes for interesting graph properties. For example, Berge's celebrated Strong Perfect Graph Conjecture [6] equates the notion of *perfectness* of a graph to forbidding two families of induced subgraphs from it. Quite worth of notice, although proved true by Chudnovsky et al [8] after a four decades' effort, Berge's conjecture was shown rather early to hold for claw-free graphs [20]. In the recent series of papers [9]–[10] Seymour and Chudnovsky also gave a structural characterization of claw-free graphs.

What we mean by a *set*, in the ongoing, is any of the values x_i specified by a system

$$\bigwedge_{i=0}^{n} x_i = \{x_{i1}, \ldots, x_{im_i}\}$$

of equations, within which every x_{ij} occurring in a right-hand side is one of the distinct unknowns $x_0, \ldots, x_{i-1}$.[1] A set is said to be *transitive* when each one of its elements is also a subset of it, as is the case with an x_i whose specifying equation in the system just considered is $x_i = \{x_0, \ldots, x_{i-1}\}$.

$$\begin{cases} x_0 = \{\} \\ x_1 = \{x_0\} \\ x_2 = \{x_0, x_1\} \\ x_3 = \{x_0, x_2\} \end{cases} \qquad \begin{cases} \mathbf{x}_0 \mapsto \emptyset \\ \mathbf{x}_1 \mapsto \{\emptyset\} \\ \mathbf{x}_2 \mapsto \{\emptyset, \{\emptyset\}\} \\ \mathbf{x}_3 \mapsto \{\emptyset, \{\emptyset, \{\emptyset\}\}\} \end{cases}$$

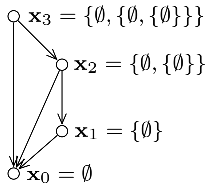


Fig. 2 Example of four sets; here set $\mathbf{x}_2$ is transitive and set $\mathbf{x}_3$ is not transitive.

[1] Traditionally, cf. [25, Section 7.5], the sets we are talking about have been called *hereditarily finite sets*.

It was shown in [18] that given a connected claw-free graph G, there exist a transitive set ν_G and an injection f from the vertices of G onto ν_G such that $\{x, y\}$ is an edge of G if and only if either $fx \in fy$ or $fy \in fx$. It ensues readily that claw-free graphs are the largest collection of graphs which is hereditary (i.e., closed under induced subgraphs) and whose connected graphs meet this set-representability property. Therefore, any computer language able to manipulate (finite, nested) sets, can represent a connected claw-free graph G simply as a set ν_G, so that the edge relation can be implicitly read off the membership relation between the elements of ν_G.

A convenient computerized system for *reasoning* about the entities of our discourse is the proof-checker Referee/ÆtnaNova [19, 22]. This system, in fact, consistently with its foundation which is the Zermelo-Fraenkel theory, ultimately represents every entity in the user's domains of discourse as a set; the framework it provides offers infinite sets also, but these are not relevant for our present purposes.

The correspondence between claw-free graphs and sets was exploited in [18] to put forth novel, simpler proofs of the two claims about connected claw-free graphs cited at the beginning. This paper is devoted to formalizing such proofs in Referee. On the one hand, by formalizing a connected claw-free graph as a 'claw-free set', we avoid explicitly defining graphs, together with the whole armamentarium of graph-theoretic notions that the original proofs required. On the other hand, we exploit Referee's built-in set manipulating operations to reflect with a minimum degree of encumbrance the two set-theoretic proofs.

The proof experiment on which we will report, available at [1], contains 16 definitions and 51 theorems, organized in 4 THEORYs. The overall number of proof lines is 663 and processing took less than 3 minutes.

The paper is structured as follows. Section 2, to be resumed at a closer-to-implementation level in Section 6.1, completes the discussion on the representation of graphs by sets undertaken above. Section 3 offers a quick overview of the proof-checker Referee on which the experimentation is based. Section 4 illustrates the experiment, highlighting—at the level of ordinary mathematical exposition—the major ideas implemented in the proofs; producing all salient formal definitions, and introducing the theorem claims which will enter into play. Some of these theorems will act as mere utilities; others will form two THEORYs (one generic, the other one application-specific) and we will explain the rationale lying behind these theorem-packaging design choices. Section 5 completes the picture, offering a detailed analysis of one of the two main proofs.

2 Representation of claw-free graphs by sets

Our proofs about the existence of perfect matchings and of Hamiltonian cycles will refer to special acyclic digraphs $D(s)$, each supported by a set s. The vertices of $D(s)$ are all sets belonging to s and, for v, w in s, there is an arc from v to w if and only if $w \in v$. We will call $D(s)$ a *membership digraph* when the set s is transitive: this requirement entails the *extensionality* of $D(s)$, namely that distinct vertices have different out-neighborhoods. $D(s)$ has the virtue that the in-/out-neighborhood of any v in s coincides, respectively, with $\{u \in s \,|\, v \in u\}$, and with $v \cap s$; here $v \cap s$ simplifies into v when s is transitive; moreover, the set of *sources* of $D(s)$ is $s \setminus \bigcup s$.[2]

As we will now discuss, our change of viewpoint is legitimatized by appropriate representation theorems.

On the one hand, Mostowski's collapse [16] ensures that to any extensional acyclic digraph $D = (V(D), \rightarrow_D)$ we can associate a transitive set ν_D and a bijection M from $V(D)$ to ν_D so that $w \rightarrow_D u$ if and only if $Mu \in Mw$. To see this, put recursively

$$Mw = \{Mu \,|\, u \in V(D) \wedge w \rightarrow_D u\},$$

a definition which makes sense thanks to the acyclicity of D. Taking $\nu_D = \{Mu \,|\, u \in V(D)\}$, we see that $M : V(D) \rightarrow \nu_D$ is surjective. The injectivity of M plainly ensues from the extensionality of D.

On the other hand, the set representation of undirected graphs can take advantage of the one just seen for digraphs since it is always possible to transform a connected claw-free graph into an extensional acyclic digraph through a suitable orientation of its edges [18]: for an example, see Figure 3. Hence, via Mostowski's collapse, we will have:

Proposition 1 *If $G = (V, E)$ is a connected claw-free graph, then there exists a finite transitive set ν_G and a bijection $f : V \rightarrow \nu_G$ so that $xy \in E$ if and only if either $fx \in fy$ or $fy \in fx$.*

We supply a new and simpler inductive proof of this proposition in the appendix.

Fig. 3 A connected claw-free graph and an extensional acyclic orientation of it. The vertices of its orientation are labeled with the sets associated to them by Mostowski's collapse.

Occasionally, in a situation like the one described in Proposition 1, we will refer to G as the graph *underlying* ν_G, and denote it as $G(\nu_G)$.

[2] For some basic nomenclature about graphs and digraphs, we refer the reader to [3] as a convenient standard. Among others: uv stands for an edge $\{u, v\}$; $N^-(u)$ and $N^+(u)$ denote the in-/out-neighborhood of a vertex u.

3 The Referee system in general

3.1 Ref's basic definition-handling and proof-checking abilities

The proof-checker Referee, or just 'Ref' for brevity, processes *proof scenarios* to establish whether or not they are formally correct. A scenario, typically written by a working mathematician or computer scientist, consists of definitions, theorem statements, proofs of the theorems, and 'theories' (see below); as shown in Fig. 4, one can intermix comments with these syntactical entities.

The deductive system underlying Ref is a variant of the Zermelo-Fraenkel set theory: this is evident from the syntax of the language, which borrows from the set-theoretic tradition many constructs, e.g. abstraction terms such as the set-former $\{u : v \in X,\ u \in v\}$ used as *definiens* for the union-set global operation $\bigcup X$; set theory also reflects into the semantics of the inference rules: for example, the inclusion $\{u : v \in x_0,\ u \in v\} \subseteq \{u : v \in x_0 \cup \{y_0\},\ u \in v\}$ can be proved in a single step as an application of the inference rule named Set_monot. Collectively, the inference rules embody almost every feature of the Zermelo-Fraenkel axioms: the only axiom of set theory which Ref maintains as an explicit assumption is, in fact, the one stating that there exist infinite sets.

Definitions often introduce abbreviating notation such as the union-set operation in the example just made, sometimes they bring into play sophisticated recursive notions such at the one of rank, to be seen in passing in Section 4.3.

Proofs are formed by two-component lines: the second component of each line is the claim being inferred, the first component hints at the inference rule being used to derive it. E.g., the hint Use_def($\bigcup$) suggests that one is expanding previous occurrences of the symbol $\bigcup$ inside the proof by the appropriate definition. Most often, the claim of a proof line is not sharply determined by the lines and the hint that precede it in the proof. Thus, for example, it is entirely a matter of taste whether to derive $\bigcup(m \cup \{x \cup \{y\}\}) \neq s$ or $\bigcup(m \cup \{x \cup \{y\}\}) \neq s \backslash \{s\}$ as the second step in the second proof of Fig. 4.

3.2 Proof encapsulation in Ref

> *Beyond this, definitions serve to 'instantiate', that is, to introduce the objects whose special properties are crucial to an intended argument. Like the selection of crucial lines, points, and circles from the infinity of geometric elements that might be considered in a Euclidean argument, definitions of this kind often carry a proof's most vital ideas.* (J. T. Schwartz, [22, p. 9])

The proof-checker Ref has a construct named THEORY, aimed at proof reuse, akin to a mechanism for parameterized specifications of the Clear specification language [7]. Besides providing theorems of which it holds the proofs, a THEORY has the ability to instantiate 'objects whose special properties are crucial to an intended argument'. Like procedures of a programming language, Ref's THEORYs have input formal parameters, in exchange of whose actualization they supply useful information. Actual input parameters must satisfy a conjunction of statements, called the *assumptions* of the THEORY. A THEORY

DEF unionset: [Family of all members of members of a set] $\bigcup X =_{\mathrm{Def}} \{u : v \in X, u \in v\}$

THM 2*e*: [Union of adjunction] $\bigcup(X \cup \{Y\}) = Y \cup \bigcup X$. PROOF:
Suppose_not$(x_0, y_0) \Rightarrow$ *Stat0* : $\bigcup(x_0 \cup \{y_0\}) \neq y_0 \cup \bigcup x_0$
$\langle a \rangle \hookrightarrow$ *Stat0* $\Rightarrow$ $a \in \bigcup(x_0 \cup \{y_0\}) \not\leftrightarrow a \in y_0 \cup \bigcup x_0$
Arguing by contradiction, let x_0, y_0 be a counterexample, so that in either one of $\bigcup(x_0 \cup \{y_0\})$ and $y_0 \cup \bigcup x_0$ there is an a not belonging to the other set. Taking the definition of $\bigcup$ into account, by monotonicity we must exclude the possibility that $a \in \bigcup x_0 \setminus \bigcup(x_0 \cup \{y_0\})$; through variable-substitution, we must also discard the possibility that $a \in \bigcup(x_0 \cup \{y_0\}) \setminus \bigcup x_0 \setminus y_0$.
Set_monot $\Rightarrow$ $\{u : v \in x_0, u \in v\} \subseteq \{u : v \in x_0 \cup \{y_0\}, u \in v\}$
Suppose $\Rightarrow$ *Stat1* : $a \in \{u : v \in x_0 \cup \{y_0\}, u \in v\}$ & $a \notin \{u : v \in x_0, u \in v\}$ & $a \notin y_0$
$\langle v_0, u_0, v_0, u_0 \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO
Use_def$(\bigcup) \Rightarrow$ *Stat2* : $a \notin \{u : v \in x_0 \cup \{y_0\}, u \in v\}$ & $a \in y_0$
The only possibility left, namely that $a \in y_0 \setminus \bigcup(x_0 \cup \{y_0\})$, is also manifestly absurd. This contradiction leads us to the desired conclusion.
$\langle y_0, a \rangle \hookrightarrow$ *Stat2* $\Rightarrow$ false; Discharge $\Rightarrow$ QED

THM 31*h*: [Less-one lemma for union set]
$\bigcup M = T \setminus \{C\}$ & $S = T \cup X \cup \{V\}$ & $Y = V \vee (C = Y$ & $Y \in S) \rightarrow$
$\langle \exists d \mid \bigcup(M \cup \{X \cup \{Y\}\}) = S \setminus \{d\} \rangle$. PROOF:
Suppose_not$(m, t, c, s, x, v, y) \Rightarrow$ *Stat0* : $\neg\langle \exists d \mid \bigcup(m \cup \{x \cup \{y\}\}) = s \setminus \{d\} \rangle$
& $\bigcup m = t \setminus \{c\}$ & $s = t \cup x \cup \{v\}$ & $y = v \vee (c = y$ & $y \in s)$
For, supposing the contrary, $\bigcup(m \cup \{x \cup \{y\}\})$ would differ from each of $s \setminus \{s\}$, $s \setminus \{c\}$, and $s \setminus \{v\}$, the first of which equals s. Thanks to THM 2*e*, we can rewrite $\bigcup(m \cup \{x \cup \{y\}\})$ as $x \cup \{y\} \cup \bigcup m$; but then the decision algorithm for a fragment of set theory known as 'multi-level syllogistic with singleton' yields an immediate contradiction.
$\langle s \rangle \hookrightarrow$ *Stat0* $\Rightarrow$ $\bigcup(m \cup \{x \cup \{y\}\}) \neq s$
$\langle c \rangle \hookrightarrow$ *Stat0* $\Rightarrow$ $\bigcup(m \cup \{x \cup \{y\}\}) \neq s \setminus \{c\}$
$\langle v \rangle \hookrightarrow$ *Stat0* $\Rightarrow$ $\bigcup(m \cup \{x \cup \{y\}\}) \neq s \setminus \{v\}$
$\langle m, x \cup \{y\} \rangle \hookrightarrow$ *T2e* $\Rightarrow$ AUTO
EQUAL $\Rightarrow$ *Stat1* : $x \cup \{y\} \cup \bigcup m \neq s \setminus \{c\}$ &
$x \cup \{y\} \cup \bigcup m \neq s \setminus \{v\}$ & $x \cup \{y\} \cup \bigcup m \neq s$
(*Stat0*, *Stat1*)Discharge $\Rightarrow$ QED

Fig. 4 Tiny scenario for Ref.

usually encapsulates the definitions of entities related to the input parameters and it supplies, along with some consequences of the assumptions, theorems talking about these internally defined entities, which the THEORY returns as output parameters.[3] After having been derived by the user once and for all inside the THEORY, the consequences of the assumptions, as well as the claims involving the output parameters, are available to be exploited repeatedly.

A simple yet significant example is the THEORY finiteInduction displayed in Fig. 5, which receives a finite set s_0 along with a property P such that $P(s_0)$ holds; in exchange, it will return a 'minimal witness' of P, i.e., a finite set fin_Θ satisfying $P(fin_\Theta)$ none of whose strict subsets t satisfies $P(t)$.

[3] As a visible countersign, the formal output parameters of a THEORY must carry the Greek letter Θ as a subscript.

```
THEORY finiteInduction(s₀, P(S))
    Finite(s₀) & P(s₀)
⇒ (fin_Θ)
    ⟨∀S | S ⊆ fin_Θ → Finite(S) & (P(S) ↔ S = fin_Θ)⟩
END finiteInduction
```

Fig. 5 A finite induction mechanism.

4 The Ref system in action

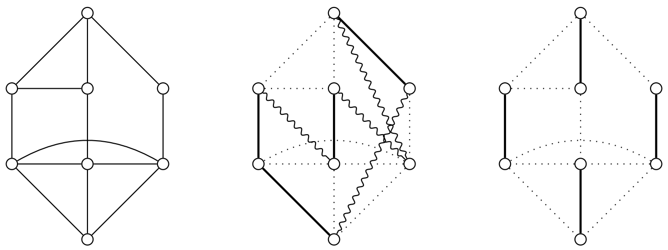
4.1 Perfect matchings and Hamiltonian cycles for claw-free graphs

Let us start with some graph-theoretic definitions, which we will specify in Ref's language later on.

Definition 1 A *perfect matching* in a graph G is a subset M of its edges such that no two edges in M have a common endpoint and every vertex of G is endpoint of some edge in M.

A *Hamiltonian cycle* in a graph G is a subset C of its edges which forms a cycle such that every vertex of G is endpoint of an edge in C.

The *square* of a graph G, denoted G^2, is the graph with the same vertex set as G in which two vertices are adjacent if either they are adjacent in G, or they have a common adjacent vertex in G.



(a) a connected claw-free graph G (b) a Hamiltonian cycle in G^2 (c) a perfect matching in G

We briefly recall here, for convenience of the reader, the proofs of two classical results about connected claw-free graphs, which we have recast formally as the proof-scenario checked by Ref on which we are about to report.

Proposition 2 ([23, 26]) *If G is a connected claw-free graph with $2n$ vertices, then G has a perfect matching.*

Proof ([18]) We reason by induction on n. If $n > 1$, let D be an acyclic orientation of G having a unique sink. Let $x_0, x_1, \ldots, x_r$ be a directed path of maximum length in D, and let $x = x_0$ and $y = x_1$. Observe that no two vertices of $N^-(y)$ are adjacent, due to the 'pivotal' choice of y. Since G is claw-free, it follows that $|N^-(y)| \leqslant 2$, for otherwise $\{y\} \cup N^-(y)$ would induce a claw in G.

If $N^-(y) = \{x\}$, then $G - \{x, y\}$ is connected, else D would have two sinks. From the inductive hypothesis, $G - \{x, y\}$ has a perfect matching, which, with the adjunction of the edge xy, constitutes a perfect matching for G.

If $N^-(y) = \{x, z\}$ then, similarly, $G - \{x, z\}$ is connected. Let yw be an edge of the perfect matching of $G - \{x, z\}$, obtained from the inductive assumption. Since G is claw-free, assume w.l.o.g. that $xw \in E(G)$. Obtain a perfect matching for G from the one for $G - \{x, z\}$ by replacing the edge yw by the edges xw and zy. □

In [17] it was proved that a connected claw-free graph with at least three vertices has a Hamiltonian square. We will consider in the ongoing the slightly stronger result of [18].

Proposition 3 *If G is a connected claw-free graph with at least three vertices, and $S \subseteq V(G)$ is the set of sources of an acyclic orientation of G endowed with exactly one sink, then G^2 has a Hamiltonian cycle C such that for every $s \in S$, at least one edge of C incident to s belongs to $E(G)$.*

Proof Arguing as in the preceding proof, unless G has 3 or 4 vertices, we select a 'pivotal' pair x, y, so that $|N^-(y)| \leqslant 2$; moreover, $G - N^-(y)$ has at least 3 vertices and is connected, since otherwise D would have two sinks. In applying the inductive hypothesis to $H = G - N^-(y)$, take the orientation induced by D, so that y is a source, and consider a Hamiltonian cycle C of H containing an edge $yw \in E(G)$.

If $N^-(y) = \{x\}$, notice that $xw \in E(G^2)$. Obtain a Hamiltonian cycle for G by replacing the edge yw in C by the path yxw (so that $xy \in E(G)$). If $N^-(y) = \{x, z\}$, due to the claw-freeness of G at least one of the edges xw or zw, say xw, belongs to G. Moreover, $zx \in E(G^2)$. Obtain a Hamiltonian cycle for G by replacing the edge yw in C with the path $yzxw$ (so that $yz, xw \in E(G)$). □

Our formal specification of the above stated theorems will refer to claw-free and transitive sets, instead of to claw-free graphs. Thus, as explained in Section 2, the orientation of edges can be left as implicit; moreover, the unique-sink assumption will readily ensue from extensionality.

4.2 Down-to-earth notions for our experiment

In the first place we must define the notions of finiteness and transitivity of a set, for the former of which we can rely on [24]. Both notions presuppose the power-set operation, which we also specify here—its companion union-set operation has been introduced in Section 3.1.

Thm 2*a*: [Union of doubletons and singletons] $\mathsf{Z} = \{\mathsf{X}, \mathsf{Y}\} \rightarrow \bigcup \mathsf{Z} = \mathsf{X} \cup \mathsf{Y}$
Thm 2*c*: [Additivity and monotonicity of monadic union]
$\bigcup(\mathsf{X} \cup \mathsf{Y}) = \bigcup \mathsf{X} \cup \bigcup \mathsf{Y} \;\&\; (\mathsf{Y} \supseteq \mathsf{X} \rightarrow \bigcup \mathsf{Y} \supseteq \bigcup \mathsf{X})$
Thm 2*e*: [Union of adjunction] $\bigcup(\mathsf{X} \cup \{\mathsf{Y}\}) = \mathsf{Y} \cup \bigcup \mathsf{X}$
Thm 3*a*: [The unionset of a transitive set is included in it] $\mathsf{Trans}(\mathsf{T}) \leftrightarrow \mathsf{T} \supseteq \bigcup \mathsf{T}$
Thm 3*c*: [For a transitive set, elements are also subsets] $\mathsf{Trans}(\mathsf{T}) \;\&\; \mathsf{X} \in \mathsf{T} \rightarrow \mathsf{X} \subseteq \mathsf{T}$
Thm 3*d*: [Trapping phenomenon for trivial sets] $\mathsf{Trans}(\mathsf{S}) \;\&\; \mathsf{X}, \mathsf{Z} \in \mathsf{S} \;\&\; \mathsf{X} \notin \mathsf{Z} \;\&\; \mathsf{Z} \notin \mathsf{X} \;\&$
$\mathsf{S} \setminus \{\mathsf{X}, \mathsf{Z}\} \subseteq \{\emptyset, \{\emptyset\}\} \rightarrow \mathsf{S} \subseteq \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$
Thm 4*b*: [$\emptyset$ belongs to any nonnull transitive set t, $\{\emptyset\}$ also does if $t \not\subseteq \{\emptyset\}$, and so on]
$\mathsf{Trans}(\mathsf{T}) \;\&\; \mathsf{N} \in \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \;\&\; \mathsf{T} \not\subseteq \mathsf{N} \rightarrow$
$\mathsf{N} \subseteq \mathsf{T} \;\&\; \big(\mathsf{N} \in \mathsf{T} \vee (\mathsf{N} = \{\emptyset, \{\emptyset\}\} \;\&\; \{\{\emptyset\}\} \in \mathsf{T})\big)$
Thm 4*c*: [Source removal from a transitive set does not disrupt transitivity]
$\mathsf{Trans}(\mathsf{S}) \;\&\; \mathsf{S} \supseteq \mathsf{T} \;\&\; (\mathsf{S} \setminus \mathsf{T}) \cap \bigcup \mathsf{S} = \emptyset \rightarrow \mathsf{Trans}(\mathsf{T})$
Thm 24: [Monotonicity of finiteness] $\mathsf{Y} \supseteq \mathsf{X} \;\&\; \mathsf{Finite}(\mathsf{Y}) \rightarrow \mathsf{Finite}(\mathsf{X})$
Thm 31*d*: [Unionset of $\emptyset$ and $\{\emptyset\}$] $\mathsf{Y} \subseteq \{\emptyset\} \leftrightarrow \bigcup \mathsf{Y} = \emptyset$
Thm 31*f*: [Unionset of a set obtained through removal followed by adjunction]
$\bigcup \mathsf{M} \supseteq \mathsf{P} \;\&\; \mathsf{Q} \cup \mathsf{R} = \mathsf{P} \cup \mathsf{S} \rightarrow \bigcup(\mathsf{M} \setminus \{\mathsf{P}\} \cup \{\mathsf{Q}, \mathsf{R}\}) = \bigcup \mathsf{M} \cup \mathsf{S}$
Thm 31*h*: [Less-one lemma for unionset]
$\bigcup \mathsf{M} = \mathsf{T} \setminus \{\mathsf{C}\} \;\&\; \mathsf{S} = \mathsf{T} \cup \mathsf{X} \cup \{\mathsf{V}\} \;\&\; \mathsf{Y} = \mathsf{V} \vee (\mathsf{C} = \mathsf{Y} \;\&\; \mathsf{Y} \in \mathsf{S}) \rightarrow$
$\big\langle \exists \mathsf{d} \mid \bigcup(\mathsf{M} \cup \{\mathsf{X} \cup \{\mathsf{Y}\}\}) = \mathsf{S} \setminus \{\mathsf{d}\} \big\rangle$
Thm 32: [Finite, nonnull sets, own sources] $\mathsf{Finite}(\mathsf{F}) \;\&\; \mathsf{F} \neq \emptyset \rightarrow \mathsf{F} \setminus \bigcup \mathsf{F} \neq \emptyset$

Fig. 6 Basic laws about $\bigcup$, Trans and Finite.

Def $\mathscr{P}$: [Family of all subsets of a given set] $\mathscr{P}\mathsf{S} =_{\mathrm{Def}} \{\mathsf{x} : \mathsf{x} \subseteq \mathsf{S}\}$
Def Fin: [Finiteness] $\mathsf{Finite}(\mathsf{F}) \leftrightarrow_{\mathrm{Def}} \big\langle \forall \mathsf{g} \in \mathscr{P}(\mathscr{P}\mathsf{F}) \setminus \{\emptyset\}, \exists \mathsf{m} \mid \mathsf{g} \cap \mathscr{P}\mathsf{m} = \{\mathsf{m}\} \big\rangle$
Def transitivity: [Transitive set] $\mathsf{Trans}(\mathsf{T}) \leftrightarrow_{\mathrm{Def}} \{\mathsf{y} \in \mathsf{T} \mid \mathsf{y} \not\subseteq \mathsf{T}\} = \emptyset$

Pre-existing ancillary properties about these constructs were available for reuse or readaptation in a shared common Ref scenario, cf. Fig. 6.

Next come our definitions of claws and claw-free *sets*. In the second of these, the assumption that S is transitive is omitted and left pending to be introduced explicitly in the pertaining theorems.

Def claw: [Pair characterizing a claw, possibly endowed with more than 3 el'ts]
$\mathsf{Claw}(\mathsf{Y}, \mathsf{F}) \leftrightarrow_{\mathrm{Def}} \mathsf{F} \cap \bigcup \mathsf{F} = \emptyset \;\&$
$\big\langle \exists \mathsf{x}, \mathsf{z}, \mathsf{w} \mid \mathsf{F} \supseteq \{\mathsf{x}, \mathsf{z}, \mathsf{w}\} \;\&\; \mathsf{x} \neq \mathsf{z} \;\&\; \mathsf{w} \notin \{\mathsf{x}, \mathsf{z}\} \;\&\; \{\mathsf{w}\} \cap \mathsf{Y} \supseteq \{\mathsf{v} \in \mathsf{F} \mid \mathsf{Y} \notin \mathsf{v}\} \big\rangle$
Def clawFreeness: [Claw-freeness, for a membership digraph]
$\mathsf{ClawFree}(\mathsf{S}) \leftrightarrow_{\mathrm{Def}} \big\langle \forall \mathsf{y} \in \mathsf{S}, \mathsf{e} \subseteq \mathsf{S} \mid \neg \mathsf{Claw}(\mathsf{y}, \mathsf{e}) \big\rangle$

A claw is thereby defined to be a pair y, F of sets such that:

1. F has at least three elements,
2. no element of F belongs to any other element of F,
3. either y belongs to all elements of F or there is a $w \in y$ such that y belongs to all elements of $F \setminus \{w\}$.

Accordingly, a *claw-free set* will be one which does not include a claw. For that, it suffices that it does not contain a claw y, F with $|F| = 3$, like the one shown in Fig. 7.

On the basis of these definitions, one easily proves the monotonicity of claw-freeness, along with two slightly less obvious properties:

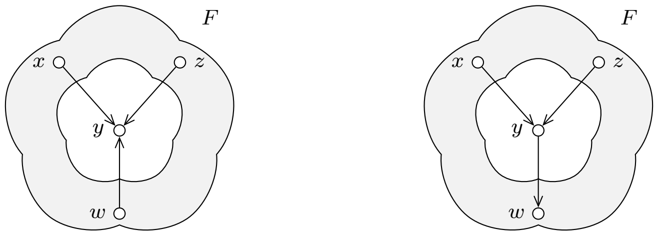


Fig. 7 The forbidden orientations of a claw in a claw-free set.

Thm $\mathsf{clawFreeness_a}$: [Subsets of claw-free sets are claw-free]
$\mathsf{ClawFree(S)}$ & $\mathsf{T \subseteq S \rightarrow ClawFree(T)}$
Thm $\mathsf{clawFreeness_b}$: [In a claw-free set, any potential claw must have a bypass]
$\mathsf{ClawFree(S)}$ & $\mathsf{S \supseteq \{Y, X, Z, W\}}$ & $\mathsf{Y \in X \cap Z}$ & $\mathsf{W \in Y}$ & $\mathsf{X \notin Z \cup \{Z\}}$ & $\mathsf{Z \notin X \rightarrow}$
$\mathsf{W \in X \cup Z}$
Thm $\mathsf{clawFreeness_0}$: [Pivots in a claw-free set own at most two predecessors therein]
$\mathsf{ClawFree(S)}$ & $\mathsf{X \in S}$ & $\mathsf{Y \in X \cap S \backslash \bigcup(S \cap \bigcup S) \rightarrow}$
$\mathsf{\langle \exists z \in S \mid \{v \in S \mid Y \in v\} = \{X, z\}}$ & $\mathsf{Y \in z \rangle}$

To comment on the third of these, we switch back to our view of a set s as being a digraph $D(s)$ with sources $s \setminus \bigcup s$. Relevant for what is to follow, we will focus on the *pivots* of $D(s)$, which we define to be the elements of $(\bigcup s) \setminus \bigcup(s \cap \bigcup s)$; in graph-theoretic terms, $y \in s$ is a pivot of $D(s)$ if y is an out-neighbor of a source of $D(s)$, but is not at the end of any directed path included in s whose length exceeds 1. The salient property of a pivot y is that if x, z are in-neighbors of y, then neither $x \in z$ nor $z \in x$ holds, thanks to the claim

Thm $31g$. $\mathsf{Y \in X}$ & $\mathsf{X \in Z}$ & $\mathsf{X, Z \in S \rightarrow Y \in \bigcup(S \cap \bigcup S)}$

whose contrapositive ensures, when $y \notin \bigcup(s \cap \bigcup s)$, the incomparability of x and z.

When s is transitive, the set of pivots reduces to $(\bigcup s) \setminus \bigcup \bigcup s$; but in order to state Thm $\mathsf{clawFreeness_0}$ in its most basic form, we avoid this assumption here. The claim hence is that in a claw-free set the in-neighbors of a pivot $y \in x \in s$ form a set $\{x, z\}$, possibly singleton. This is straightforward: should y have three in-neighbors x, z, w, the pair $y, \{x, z, w\}$ would be a claw.

4.3 A crucial auxiliary theory

Two instantiating mechanisms play a key role in our proof-pearl scenario. One relates to the finiteness of the graphs under study here: this assumption conveniently reflects into the induction principle discussed in Section 3.

The other theory more specifically reflects our claw-freeness and transitivity assumptions; it factors out a mathematical insight which is common to the two main proofs on which we are reporting. Essentially, it says that in a transitive claw-free set $\mathsf{s_0} \nsubseteq \{\emptyset\}$ we can always select a pivot $\mathsf{y_\Theta}$ and its in-neighborhood $\{x_\Theta, z_\Theta\}$. Along with $y_\Theta, x_\Theta, z_\Theta$, this THEORY returns the set $t_\Theta = s_0 \setminus \{x_\Theta, z_\Theta\} = \{v \in s_0 \mid y_\Theta \notin v\}$, strictly included in s_0; in its turn, t_Θ is proved to be claw-free and transitive.

THEORY pivotsForClawFreeness(s_0)
  ClawFree(s_0) & Trans(s_0) & Finite(s_0)
  $s_0 \not\subseteq \{\emptyset\}$
⇒ ($x_\Theta, y_\Theta, z_\Theta, t_\Theta$)
  $\langle \forall x \in s_0, y \in x \setminus \bigcup\bigcup s_0 \mid \langle \exists z \in s_0 \mid \{v \in s_0 \mid y \in v\} = \{x, z\} \ \& \ y \in z \rangle \rangle$
  $\{x_\Theta, y_\Theta, z_\Theta\} \subseteq s_0$
  $x_\Theta \notin z_\Theta \ \& \ z_\Theta \notin x_\Theta \ \& \ y_\Theta \in x_\Theta \cap z_\Theta \setminus \bigcup\bigcup s_0$
  $y_\Theta \in t_\Theta \setminus \bigcup t_\Theta \ \& \ t_\Theta = s_0 \setminus \{x_\Theta, z_\Theta\} \ \& \ t_\Theta = \{v \in s_0 \mid y_\Theta \notin v\}$
  ClawFree(t_Θ) & Trans(t_Θ)
END pivotsForClawFreeness

Fig. 8 A key quadruple associated with a claw-free set.

For an intuition of how the quadruple $x_\Theta, y_\Theta, z_\Theta, t_\Theta$ can be obtained, referring to the classical notion of *rank* recursively definable as

$$\mathsf{rk}(s) \ =_{\mathrm{Def}} \begin{cases} 0 & \text{if } s = \emptyset \\ \max\{\, \mathsf{rk}(t) + 1 : \ t \in s \,\} & \text{otherwise,} \end{cases}$$

observe that a transitive set s_0 not included in $\{\emptyset\}$ must have rank $r \geqslant 2$, and hence must have elements x_Θ, y_Θ such that $y_\Theta \in x_\Theta$ and $\mathsf{rk}(y_\Theta) = r - 2$.

Although a recursive definition such as the one of rk just seen is supported by Ref (as a benefit originating from the assumption that set membership is a well-founded relation), we preferred to avoid it in order to circumvent any possible complication that might ensue from an explicit handling of numbers.

As a surrogate for the rank notion, we conceal inside this THEORY the definition of the *frontier* of a set s: this consists of those elements s to which a pivot of s belongs:

DEF frontier: [Frontier of a set] $\mathsf{front}(S) =_{\mathrm{Def}} \{x \in S \mid x \cap S \setminus \bigcup (S \cap \bigcup S) \neq \emptyset\}$.

Aided by this notion, we get x_Θ and y_Θ by drawing arbitrarily the former from $\mathsf{front}(s_0)$, the latter from $x_\Theta \setminus \bigcup\bigcup s_0$. This presupposes, of course, a proof that $\mathsf{front}(s_0) \neq \emptyset$, a fact simply ensuing from the more general proposition

THM frontier$_1$. $\mathsf{Finite}(S \cap \bigcup S) \ \& \ S \cap \bigcup S \neq \emptyset \rightarrow \mathsf{front}(S) \neq \emptyset$,

applicable to s_0 thanks to the assumption $s_0 \not\subseteq \{\emptyset\}$ of the THEORY at hand. To conclude the development of this THEORY, one must show that $t_\Theta = \{v \in s_0 \mid y_\Theta \notin v\}$ is transitive, as follows from

THM frontier$_2$. $\mathsf{Trans}(S) \ \& \ X \in \mathsf{front}(S) \ \& \ Y \in X \setminus \bigcup\bigcup S \ \& \ T = \{z \in S \mid Y \notin z\} \rightarrow \mathsf{Trans}(T) \ \& \ T \subsetneq S \ \& \ X \notin T \ \& \ Y \in T \setminus \bigcup T$,

in view of THM 4c. from Fig. 6.

4.4 Preparatory lemmas

Since an edge of a graph is represented as membership between two sets, we define a *perfect matching* to be a set of disjoint doubletons $\{x, y\}$ such that $y \in x$ holds.

DEF perfect_matching: [set of disjoint membership pairs]

$$\mathsf{perfectMatching}(\mathsf{M}) \leftrightarrow_{\mathrm{Def}} \langle \forall \mathsf{p} \in \mathsf{M}, \exists \mathsf{x} \in \mathsf{p}, \mathsf{y} \in \mathsf{x}, \forall \mathsf{q} \in \mathsf{M} \mid \mathsf{x} \in \mathsf{q} \vee \mathsf{y} \in \mathsf{q} \rightarrow \{\mathsf{x}, \mathsf{y}\} = \mathsf{q} \rangle$$

The following theorems about perfect matchings admit straightforward proofs.

THM perfectMatching$_0$: [The null set is a perfect matching] $\mathsf{perfectMatching}(\emptyset)$

THM perfectMatching$_2$: [All subsets of a perfect matching are perfect matchings]
$\mathsf{perfectMatching}(\mathsf{M})\ \&\ \mathsf{M} \supseteq \mathsf{N} \rightarrow \mathsf{perfectMatching}(\mathsf{N})$

THM perfectMatching$_3$: [Bottom-up assembly of a finite perfect matching]
$\mathsf{perfectMatching}(\mathsf{M})\ \&\ \mathsf{X} \notin \bigcup \mathsf{M}\ \&\ \mathsf{Y} \notin \bigcup \mathsf{M}\ \&\ \mathsf{Y} \in \mathsf{X} \rightarrow \mathsf{perfectMatching}(\mathsf{M} \cup \{\{\mathsf{X}, \mathsf{Y}\}\})$

THM perfectMatching$_4$: [Deviated perfect matching] $\mathsf{perfectMatching}(\mathsf{M})\ \&\ \{\mathsf{Y}, \mathsf{W}\} \in \mathsf{M}\ \&\ \mathsf{X} \notin \bigcup \mathsf{M}\ \&\ \mathsf{Z} \notin \bigcup \mathsf{M}\ \&\ \mathsf{Y} \in \mathsf{Z}\ \&\ \mathsf{Y} \neq \mathsf{X}\ \&\ \mathsf{X} \neq \mathsf{Z}\ \&\ \mathsf{W} \in \mathsf{X} \rightarrow \mathsf{perfectMatching}(\mathsf{M} \setminus \{\{\mathsf{Y}, \mathsf{W}\}\} \cup \{\{\mathsf{Y}, \mathsf{Z}\}, \{\mathsf{X}, \mathsf{W}\}\})$

The last two of these reflect our proof strategy: THM perfectMatching$_3$ claims that we can extend a matching by insertion of a doubleton of new sets, while THM perfectMatching$_4$ states conditions under which we can break a pair $\{y, w\}$ of a matching M into two doubletons $\{y, z\}$ and $\{x, w\}$ (see Fig. 9).

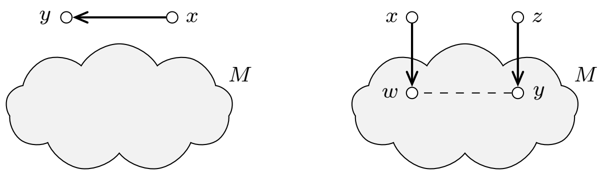


Fig. 9 Two strategies for extending a perfect matching.

Next come our definitions pertaining to Hamiltonian cycles. These notions must refer to the edges in the square of a claw-free set, which will be formalized as unstructured doubletons. In order to define a Hamiltonian cycle, we can avoid speaking of sequences of vertices of a graph, and refer only to subsets of edges forming a cycle. This is done in two steps: we define $\mathsf{Hank}(H)$ to hold, for an $H \neq \emptyset$, if every element $x \in e \in H$ is a member of another element $q \neq e$ of H. Roughly speaking, this says that every end point of an edge of H has degree at least 2 in H; but notice that for the time being we are not insisting that H is formed by doubletons. Next, we define $\mathsf{Cycle}(C)$ to hold if $\mathsf{Hank}(C)$ holds and C is inclusion-minimal with this property (cf. [12, p. 288]).

Let us briefly digress to show that whenever C is a non-null subset of edges of a graph G and $\mathsf{Cycle}(C)$ holds, the subgraph $G[C]$ of G induced by the edges

DEF cycle_0: [Collection of edges whose endpoints have degree greater than 1]
$\mathsf{Hank}(\mathsf{H}) \leftrightarrow_{\mathrm{Def}} \emptyset \notin \mathsf{H} \,\&\, \langle \forall \mathsf{e} \in \mathsf{H} \mid \mathsf{e} \subseteq \bigcup(\mathsf{H} \setminus \{\mathsf{e}\}) \rangle$
DEF cycle_1: [Cycle (unless null)]
$\mathsf{Cycle}(\mathsf{C}) \leftrightarrow_{\mathrm{Def}} \mathsf{Hank}(\mathsf{C}) \,\&\, \langle \forall \mathsf{d} \subseteq \mathsf{C} \mid \mathsf{Hank}(\mathsf{d}) \,\&\, \mathsf{d} \neq \emptyset \rightarrow \mathsf{d} = \mathsf{C} \rangle$

of C is a cycle in the customary sense. We argue first that $G[C]$ is not a forest and that it must contain a cycle (not necessarily induced). Otherwise, let P be the longest path in $G[C]$ and let its successive vertices be $x_1, \ldots, x_k$. Since $\mathsf{Hank}(C)$ holds, C must have an edge $x_1 x'$ with $x' \neq x_2$, also belonging to G. From the maximality of P we have that $x' \in P$, contradicting the supposed acyclicity of $G[C]$. If $G[C]$ is not a cycle, then we can find a strictly included induced cycle C' by picking a minimal-length closed walk of $G[C]$. Therefore $\mathsf{Hank}(C')$ holds, contradicting the fact that $\mathsf{Cycle}(C)$ holds.

DEF $\mathsf{hamiltonian}_1$: [Hamiltonian cycle, in graph without isolated vertices]
$\mathsf{Hamiltonian}(\mathsf{H}, \mathsf{S}, \mathsf{E}) \leftrightarrow_{\mathrm{Def}} \mathsf{Cycle}(\mathsf{H}) \,\&\, \bigcup \mathsf{H} = \mathsf{S} \,\&\, \mathsf{H} \subseteq \mathsf{E}$
DEF $\mathsf{hamiltonian}_2$: [Edges in squared membership]
$\mathsf{sqEdges}(\mathsf{S}) =_{\mathrm{Def}} \{\{\mathsf{x}, \mathsf{y}\} : \mathsf{x} \in \mathsf{S}, \mathsf{y} \in \mathsf{S}, \mathsf{z} \in \mathsf{S} \mid$
$\mathsf{x} \in \mathsf{y} \vee (\mathsf{x} \in \mathsf{z} \,\&\, \mathsf{z} \in \mathsf{y}) \vee (\mathsf{z} \in \mathsf{x} \cap \mathsf{y} \,\&\, \mathsf{x} \neq \mathsf{y})\}$
DEF $\mathsf{hamiltonian}_3$: [Restraining condition for Hamiltonian cycles]
$\mathsf{SqHamiltonian}(\mathsf{H}, \mathsf{S}) \leftrightarrow_{\mathrm{Def}} \mathsf{Hamiltonian}(\mathsf{H}, \mathsf{S}, \mathsf{sqEdges}(\mathsf{S})) \,\&$
$\langle \forall \mathsf{x} \in \mathsf{S} \setminus \bigcup \mathsf{S}, \exists \mathsf{y} \in \mathsf{x} \mid \{\mathsf{x}, \mathsf{y}\} \in \mathsf{H} \rangle$

Given an undirected graph (S, E), we say that $H \subseteq E$ is a *Hamiltonian cycle* of it if $\mathsf{Cycle}(H)$ holds, and each vertex v of S is *covered* by an edge e of H, in the sense that $v \in e$. Given a set s, we characterize the set of *square edges* of s by allowing only three of the four possible membership alignments of two sets x, y whose distance in the graph $G(s)$ underlying s is 1 or 2 (see Fig. 10). These three configurations suffice in a proof of the announced theorem. To complete our setup, we need the notion of $\mathsf{SqHamiltonian}$, which describes a Hamiltonian cycle H of a set s reflecting the claim of Proposition 3: in the first place, we require H to be Hamiltonian in the square of the underlying graph $G(s)$; secondly, H must cover each source of s by an edge of $G(s)$.

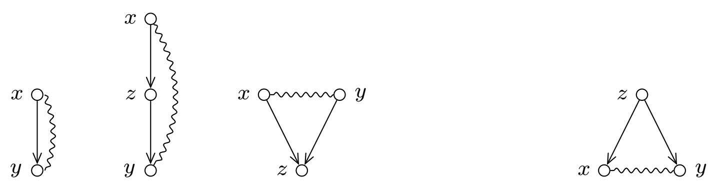


Fig. 10 Four orientations of a path of length 1 or 2 between two vertices x and y; the last of these is not taken into account by our definition $\mathsf{sqEdges}$.

The following theorems about Hamiltonian cycles admit straightforward proofs.

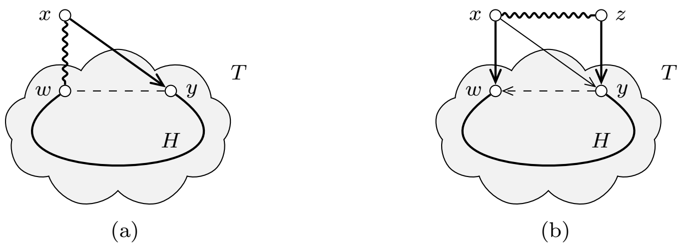
THM hamiltonian$_1$: [Enriched Hamiltonian cycles]
$\mathsf{S} = \mathsf{T} \cup \{\mathsf{X}\}$ & $\mathsf{X} \notin \mathsf{T}$ & $\mathsf{Y} \in \mathsf{X}$ & $\mathsf{SqHamiltonian}(\mathsf{H},\mathsf{T})$ &
$\{\mathsf{W},\mathsf{Y}\} \in \mathsf{H}$ & $\mathsf{W} \in \mathsf{Y} \vee (\mathsf{Y} \in \mathsf{W}$ & $\mathsf{K} \neq \mathsf{Y}$ & $\{\mathsf{W},\mathsf{K}\} \in \mathsf{H}$ & $\mathsf{K} \in \mathsf{W}) \rightarrow$
$\mathsf{SqHamiltonian}(\mathsf{H} \setminus \{\{\mathsf{W},\mathsf{Y}\}\} \cup \{\{\mathsf{W},\mathsf{X}\},\{\mathsf{X},\mathsf{Y}\}\},\mathsf{S})$

THM hamiltonian$_2$: [Doubly enriched Hamiltonian cycles]
$\mathsf{S} = \mathsf{T} \cup \{\mathsf{X},\mathsf{Z}\}$ & $\{\mathsf{X},\mathsf{Z}\} \cap \mathsf{T} = \emptyset$ & $\mathsf{X} \neq \mathsf{Z}$ & $\mathsf{Y} \in \mathsf{X} \cap \mathsf{Z}$ &
$\mathsf{SqHamiltonian}(\mathsf{H},\mathsf{T})$ & $\{\mathsf{W},\mathsf{Y}\} \in \mathsf{H}$ & $\mathsf{W} \in \mathsf{Y} \cap \mathsf{X} \rightarrow$
$\mathsf{SqHamiltonian}(\mathsf{H} \setminus \{\{\mathsf{W},\mathsf{Y}\}\} \cup \{\{\mathsf{W},\mathsf{X}\},\{\mathsf{X},\mathsf{Z}\},\{\mathsf{Z},\mathsf{Y}\}\},\mathsf{S})$

THM hamiltonian$_3$: [Trivial Hamiltonian cycles]
$\mathsf{S} = \{\mathsf{X},\mathsf{Y},\mathsf{Z}\}$ & $\mathsf{X} \in \mathsf{Y}$ & $\mathsf{Y} \in \mathsf{Z} \rightarrow \mathsf{SqHamiltonian}(\{\{\mathsf{X},\mathsf{Y}\},\{\mathsf{Y},\mathsf{Z}\},\{\mathsf{Z},\mathsf{X}\}\},\mathsf{S})$

THM hamiltonian$_4$. [Any nontrivial transitive set whose square is devoid of Hamiltonian cycles must strictly comprise certain sets]
$\mathsf{Trans}(\mathsf{S})$ & $\mathsf{S} \not\subseteq \{\emptyset,\{\emptyset\}\}$ & $\neg\langle \exists \mathsf{h} \mid \mathsf{SqHamiltonian}(\mathsf{h},\mathsf{S})\rangle \rightarrow$
$\mathsf{S} \neq \{\emptyset,\{\emptyset\},\{\{\emptyset\}\}\}$ & $\mathsf{S} \neq \{\emptyset,\{\emptyset\},\{\emptyset,\{\emptyset\}\}\}$ &
$\mathsf{S} \neq \{\emptyset,\{\emptyset\},\{\{\emptyset\}\},\{\emptyset,\{\emptyset\}\}\}$ & $\mathsf{S} \supseteq \{\emptyset,\{\emptyset\}\}$ &
$(\{\{\emptyset\}\} \in \mathsf{S} \vee \{\emptyset,\{\emptyset\}\} \in \mathsf{S})$

The last two of these will serve as base case for the proof we are after, namely the case when a transitive set s has 3 or 4 elements. In particular, if $s = \{x,y,z\}$ is a transitive tripleton, then its elements are $x = \emptyset$, $y = \{\emptyset\}$, $z = \{\{\emptyset\}\} \vee z = \{\emptyset,\{\emptyset\}\}$, and x,z form a square edge; therefore $\{x,y\}$, $\{y,z\}$, $\{z,x\}$ form a hank, and then clearly a cycle, because hanks of cardinality 1 or 2 do not exist. When s has 5 elements of more, then, mimicking the proof of Proposition 3 seen above, we will proceed differently, depending on whether the selected pivot of s belongs to a single element of s, or to two: THM hamiltonian$_2$ will serve us when s has two such predecessors, and THM hamiltonian$_1$ will settle the other case.



5 Specifications of the Hamiltonicity proof and of the perfect matching theorem

We will now examine in detail our formal reconstruction of Proposition 3, as readjusted for membership digraphs and certified correct with Ref.

Assuming the contrary, let s_1 be a finite transitive claw-free set with at least three elements, i.e. $\mathsf{s}_1 \not\subseteq \{\emptyset,\{\emptyset\}\}$, which does not have a Hamiltonian cycle in its square (step **1**). By the finiteInduction THEORY, there would exist an

Non-trivial claw-free transitive sets have Hamiltonian squares

THM $\mathsf{clawFreeness}_1$.
$\mathsf{Finite(S)} \;\&\; \mathsf{Trans(S)} \;\&\; \mathsf{ClawFree(S)} \;\&\; \mathsf{S} \not\subseteq \{\emptyset, \{\emptyset\}\} \rightarrow \langle \exists \mathsf{h} \mid \mathsf{SqHamiltonian(h, S)} \rangle$. PROOF:

1 $\mathsf{Suppose\_not}(\mathsf{s}_1) \Rightarrow$ AUTO

2 APPLY $\langle \mathsf{fin}_\Theta : \mathsf{s}_0 \rangle$ $\mathsf{finiteInduction}\big(\mathsf{s}_0 \mapsto \mathsf{s}_1,$
$\mathsf{P(S)} \mapsto \big(\mathsf{Trans(S)} \;\&\; \mathsf{ClawFree(S)} \;\&\; \mathsf{S} \not\subseteq \{\emptyset, \{\emptyset\}\} \;\&\; \neg \langle \exists \mathsf{h} \mid \mathsf{SqHamiltonian(h, S)} \rangle\big)\big) \Rightarrow$
Stat1 : $\langle \forall \mathsf{s} \mid \mathsf{s} \subseteq \mathsf{s}_0 \rightarrow \mathsf{Finite(s)} \;\&\; (\mathsf{Trans(s)} \;\&\; \mathsf{ClawFree(s)} \;\&\; \mathsf{s} \not\subseteq \{\emptyset, \{\emptyset\}\} \;\&$
$\neg \langle \exists \mathsf{h} \mid \mathsf{SqHamiltonian(h, s)} \rangle \leftrightarrow \mathsf{s} = \mathsf{s}_0) \rangle$

3 $\langle \mathsf{s}_0 \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ *Stat2* : $\neg \langle \exists \mathsf{h} \mid \mathsf{SqHamiltonian(h, s_0)} \rangle \;\&$
$\mathsf{Finite(s_0)} \;\&\; \mathsf{Trans(s_0)} \;\&\; \mathsf{ClawFree(s_0)} \;\&\; \mathsf{s}_0 \not\subseteq \{\emptyset, \{\emptyset\}\}$

4 APPLY $\langle \mathsf{x}_\Theta : \mathsf{x}, \mathsf{y}_\Theta : \mathsf{y}, \mathsf{z}_\Theta : \mathsf{z}, \mathsf{t}_\Theta : \mathsf{t} \rangle$ $\mathsf{pivotsForClawFreeness}(\mathsf{s}_0 \mapsto \mathsf{s}_0) \Rightarrow$
$\{\mathsf{v} \in \mathsf{s}_0 \mid \mathsf{y} \in \mathsf{v}\} = \{\mathsf{x}, \mathsf{z}\} \;\&\; \mathsf{x}, \mathsf{y}, \mathsf{z} \in \mathsf{s}_0 \;\&\; \mathsf{y} \in \mathsf{x} \cap \mathsf{z} \setminus \bigcup\bigcup \mathsf{s}_0 \;\&$
$\mathsf{y} \in \mathsf{t} \setminus \bigcup \mathsf{t} \;\&\; \mathsf{t} = \mathsf{s}_0 \setminus \{\mathsf{x}, \mathsf{z}\} \;\&\; \mathsf{t} = \{\mathsf{u} \in \mathsf{s}_0 \mid \mathsf{y} \notin \mathsf{u}\} \;\&\; \mathsf{s}_0 \supseteq \mathsf{t} \;\&$
$\mathsf{Trans(t)} \;\&\; \mathsf{ClawFree(t)} \;\&\; \mathsf{x} \notin \mathsf{t} \;\&\; \mathsf{x} \notin \mathsf{z} \;\&\; \mathsf{z} \notin \mathsf{x}$

5 Suppose $\Rightarrow$ $\mathsf{t} \subseteq \{\emptyset, \{\emptyset\}\}$

6 $\langle \mathsf{s}_0, \mathsf{x}, \mathsf{z} \rangle \hookrightarrow$ *T3d* $\Rightarrow$ $\mathsf{s}_0 \subseteq \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$

7 $\langle \mathsf{s}_0 \rangle \hookrightarrow$ *Thamiltonian*$_4$ $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO

8 $\langle \mathsf{t} \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ *Stat9* : $\langle \exists \mathsf{h} \mid \mathsf{SqHamiltonian(h, t)} \rangle$

9 $\langle \mathsf{h}_0 \rangle \hookrightarrow$ *Stat9* $\Rightarrow$ $\mathsf{SqHamiltonian(h_0, t)}$

10 $\mathsf{Use\_def}(\mathsf{Hamiltonian}(\mathsf{h}_0, \mathsf{t}, \mathsf{sqEdges(t)})) \Rightarrow$ AUTO

11 $\mathsf{Use\_def(SqHamiltonian)} \Rightarrow$ *Stat11* : $\langle \forall \mathsf{x} \in \mathsf{t} \setminus \bigcup \mathsf{t}, \exists \mathsf{y} \in \mathsf{x} \mid \{\mathsf{x}, \mathsf{y}\} \in \mathsf{h}_0 \rangle \;\&$
$\mathsf{Cycle(h_0)} \;\&\; \bigcup \mathsf{h}_0 = \mathsf{t} \;\&\; \mathsf{h}_0 \subseteq \mathsf{sqEdges(t)}$

12 $\langle \mathsf{y}, \mathsf{w} \rangle \hookrightarrow$ *Stat11* $\Rightarrow$ $\mathsf{w} \in \mathsf{y} \;\&\; \{\mathsf{w}, \mathsf{y}\} \in \mathsf{h}_0$

13 Suppose $\Rightarrow$ $\mathsf{x} = \mathsf{z}$

14 $\langle \mathsf{s}_0, \mathsf{t}, \mathsf{x}, \mathsf{y}, \mathsf{h}_0, \mathsf{w}, \emptyset \rangle \hookrightarrow$ *Thamiltonian*$_1$ $\Rightarrow$
$\mathsf{SqHamiltonian}(\mathsf{h}_0 \setminus \{\{\mathsf{w}, \mathsf{y}\}\} \cup \{\{\mathsf{w}, \mathsf{x}\}, \{\mathsf{x}, \mathsf{y}\}\}, \mathsf{s}_0)$

15 $\langle \mathsf{h}_0 \setminus \{\{\mathsf{w}, \mathsf{y}\}\} \cup \{\{\mathsf{w}, \mathsf{x}\}, \{\mathsf{x}, \mathsf{y}\}\} \rangle \hookrightarrow$ *Stat2* $\Rightarrow$ false; Discharge $\Rightarrow$ $\mathsf{x} \neq \mathsf{z}$

16 $\langle \mathsf{s}_0, \mathsf{y} \rangle \hookrightarrow$ *T3c* $\Rightarrow$ $\mathsf{w} \in \mathsf{s}_0$

17 $\langle \mathsf{s}_0, \mathsf{y}, \mathsf{x}, \mathsf{z}, \mathsf{w} \rangle \hookrightarrow$ *TclawFreeness*$_b$ $\Rightarrow$ $\mathsf{w} \in \mathsf{x} \cup \mathsf{z}$

18 Suppose $\Rightarrow$ $\mathsf{w} \in \mathsf{x}$

19 $\langle \mathsf{s}_0, \mathsf{t}, \mathsf{x}, \mathsf{z}, \mathsf{y}, \mathsf{h}_0, \mathsf{w} \rangle \hookrightarrow$ *Thamiltonian*$_2$ $\Rightarrow$
$\mathsf{SqHamiltonian}(\mathsf{h}_0 \setminus \{\{\mathsf{w}, \mathsf{y}\}\} \cup \{\{\mathsf{w}, \mathsf{x}\}, \{\mathsf{x}, \mathsf{z}\}, \{\mathsf{z}, \mathsf{y}\}\}, \mathsf{s}_0)$

20 $\langle \mathsf{h}_0 \setminus \{\{\mathsf{w}, \mathsf{y}\}\} \cup \{\{\mathsf{w}, \mathsf{x}\}, \{\mathsf{x}, \mathsf{z}\}, \{\mathsf{z}, \mathsf{y}\}\} \rangle \hookrightarrow$ *Stat2* $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO

21 ELEM $\Rightarrow$ $\mathsf{w} \in \mathsf{z}$

22 $\langle \mathsf{s}_0, \mathsf{t}, \mathsf{z}, \mathsf{x}, \mathsf{y}, \mathsf{h}_0, \mathsf{w} \rangle \hookrightarrow$ *Thamiltonian*$_2$ $\Rightarrow$
$\mathsf{SqHamiltonian}(\mathsf{h}_0 \setminus \{\{\mathsf{w}, \mathsf{y}\}\} \cup \{\{\mathsf{w}, \mathsf{z}\}, \{\mathsf{z}, \mathsf{x}\}, \{\mathsf{x}, \mathsf{y}\}\}, \mathsf{s}_0)$

23 $\langle \mathsf{h}_0 \setminus \{\{\mathsf{w}, \mathsf{y}\}\} \cup \{\{\mathsf{w}, \mathsf{z}\}, \{\mathsf{z}, \mathsf{x}\}, \{\mathsf{x}, \mathsf{y}\}\} \rangle \hookrightarrow$ *Stat2* $\Rightarrow$ false; Discharge $\Rightarrow$ QED

inclusion-minimal finite transitive non-trivial claw-free set s_0 likewise lacking such a cycle (steps **2**, **3**).

The THEORY pivotsForClawFreeness can be applied to s_0 (step **4**): we thereby pick an element x from the frontier of s_0, and an element y of x which is pivotal relative to s_0. This y will have at most two in-neighbors (one of the two being x) in s_0. We denote by z an in-neighbor of y in s_0, such that z differs from x, if possible. Observe, among others, that neither one of x, z can belong to the other.

If the removal of x, z from s_0 leads to a set t included in $\{\emptyset, \{\emptyset\}\}$ (step **5**), then by Thm $3d$ we get $\mathsf{s}_0 \subseteq \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$. This leads us to a contradiction, in light of Thm $\mathsf{hamiltonian}_4$ (step **7**). Therefore, t is not trivial and the inductive hypothesis applies to it (step **8**): thanks to that hypothesis, we can find a Hamiltonian cycle h_0 for t (step **9**).

Recalling the definitions of Hamiltonian and sqHamiltonian (steps **10**, **11**), it follows from y being a source of $\mathsf{t} = \bigcup \mathsf{h}_0$ that there is an edge $\{\mathsf{y}, \mathsf{w}\}$ in h_0, with $\mathsf{w} \in \mathsf{y}$ (step **12**).

If $\mathsf{x} = \mathsf{z}$, the set $\mathsf{h}_1 = \mathsf{h}_0 \setminus \{\{\mathsf{y}, \mathsf{w}\}\} \cup \{\{\mathsf{x}, \mathsf{y}\}, \{\mathsf{x}, \mathsf{w}\}\}$ is a Hamiltonian cycle for s_0, by Thm $\mathsf{hamiltonian}_1$ (step **14**). This conflicts with the minimality of s_0 (step **15**): in fact $\{\mathsf{x}, \mathsf{w}\}$ is a square edge, since $\mathsf{w} \in \mathsf{y}$ and $\mathsf{y} \in \mathsf{x}$ both hold.

On the other hand, if $\mathsf{x} \neq \mathsf{z}$, claw-freeness implies, via Thm $\mathsf{clawFreeness}_\mathsf{b}$, that either $\mathsf{w} \in \mathsf{x}$ or $\mathsf{w} \in \mathsf{z}$ must hold (step **17**). Assume the former (step **18**), and put $\mathsf{h}_2 = \mathsf{h}_0 \setminus \{\{\mathsf{y}, \mathsf{w}\}\} \cup \{\{\mathsf{y}, \mathsf{z}\}, \{\mathsf{z}, \mathsf{x}\}, \{\mathsf{x}, \mathsf{w}\}\}$, where $\{\mathsf{x}, \mathsf{z}\}$ is a square edge and $\{\mathsf{x}, \mathsf{w}\}$ and $\{\mathsf{y}, \mathsf{z}\}$ are genuine edges incident in the sources x, z. By Thm $\mathsf{hamiltonian}_2$, h_2 is a Hamiltonian cycle for s_0 (step **19**), and we are again facing a contradiction (step **20**). The case $\mathsf{w} \in \mathsf{z}$ is entirely symmetric (steps **21**, **22**, **23**), which proves the initial claim.

The result on the existence of a perfect matching is usually referred to graphs whose set of vertices has an even cardinality, as we have done in our Proposition 2; but here, since numbers pop in only in this place, we omit the evenness constraint: transitive, claw-free sets admit a 'near-perfect matching' (see [15]), that is to say, a perfect matching which does not cover at most one of its elements. In Ref:

Thm $\mathsf{clawFreeness}_2$: [Every claw-free transitive set has a near-perfect matching]
$\mathsf{Finite}(\mathsf{S})\ \&\ \mathsf{Trans}(\mathsf{S})\ \&\ \mathsf{ClawFree}(\mathsf{S}) \rightarrow \langle \exists \mathsf{m}, \mathsf{y} \mid \mathsf{perfectMatching}(\mathsf{m})\ \&\ \mathsf{S} \setminus \{\mathsf{y}\} = \bigcup \mathsf{m} \rangle$

As one sees from our previous treatment of Proposition 2, this theorem's proof bears a close resemblance with the proof about Hamiltonicity just detailed; hence it seems pointless to supply again here many formal details.

6 Conclusions

This paper continues a series investigating transfers of techniques and results across the areas of graphs and sets. In such transfers, claw-free graphs turned out to occupy a preeminent place; e.g., the set-theoretic interpretation led to simpler proofs of two classical results about connected claw-free graphs. We have taken here a natural step, by formalizing these two set-inspired proofs in the proof-checker Referee, which ultimately represents every entity in the user's domain of discourse as a set.

To take advantage of the set-theoretic foundation of Referee, we exploited set equivalents of the graph-theoretic notions involved in our experiment: edge, source, square, etc. To ease some proofs, we have often resorted to weak counterparts of well-established notions such as cycle, claw-freeness, longest directed path, etc.

- In the above, we could have defined a transitive set to be *claw-free* if none of the four non-isomorphic membership renderings of a claw are induced by any quadruple of its elements; but actually, it sufficed to forbid two out of these four to get the desired proofs. This explains why our results are easier to achieve but under some respects more general. To see the difference, observe that the graph in Fig. 11, once suitably oriented, can be handled by our theorems, whereas its Hamiltonicity and its perfect matchings are not seen either by the traditional results [23,26,17], or by subsequent generalizations regarding *quasi claw-free graphs* [2], *almost claw-free graphs* [21], and $S(K_{1,3})$-*free graphs* [13].
- The graph 'squares' about which our Hamiltonicity proof speaks are actually poorer in edges than the standard ones, since we allow only three out of the four membership alignments, cf. Fig. 10.
- We have addressed issues regarding graphs, which we see as pre-algorithmic and, as such, application-oriented. Nonetheless, our results are so close to the foundations of mathematics that we found no reason to introduce numbers, and we were able to avoid recursion even in the determination of the pivots, as explained in Section 4.3. For the time being, we succeeded even in doing without basic conceptual tools which, as we expect, will enter into play in continuations of this work; for example, the notion of spanning tree.
- The graphs that can be represented by sets form a broad class of graphs, which includes, in partial overlap with connected claw-free graphs, all graphs endowed with a Hamiltonian path [18]. By allowing the presence of 'atoms' in our sets, as will emerge from the next section, we can actually represent all graphs.

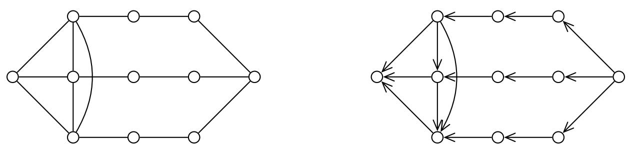


Fig. 11 A graph endowed with a claw, but admitting an orientation compatible with our set-theoretic definition of claw-freeness (cf. Fig. 7).

6.1 An outward look

To end, let us now place the results presented so far under the more general perspective motivating this work. We display in this section the interfaces of two representation THEORYs (not developed formally with Referee, as of today), and of a THEORY auxiliary to one of these two, explaining why we can work with membership as a convenient surrogate for the edge relationship of general graphs.

One of these, THEORY finGraphRepr, will implement the proof given in [18] that any finite graph $(\mathsf{v}_0, \mathsf{e}_0)$ is 'isomorphic', via a suitable orientation of its edges and an injection f of v_0 onto a set ν, to a digraph $(\nu, \{(x, y) : x \in \nu, y \in x \cap \nu\})$ enjoying the *weak extensionality* property: "distinct non-sink vertices have different out-neighborhoods". Sinks can, at taste, be seen as pairwise distinct *atoms* (or 'urelements' [14]) entering in the formation of the sets assigned to the internal vertices, or as sets whose internal structure is immaterial.

$$
\begin{array}{l}
\text{THEORY finGraphRepr}(\mathsf{v}_0, \mathsf{e}_0)\\
\quad \mathsf{Finite}(\mathsf{v}_0) \;\&\; \mathsf{e}_0 \subseteq \{\{\mathsf{x},\mathsf{y}\} : \mathsf{x},\mathsf{y} \in \mathsf{v}_0 \mid \mathsf{x} \neq \mathsf{y}\}\\
\Rightarrow (f_\Theta, \nu_\Theta)\\
\quad 1\text{--}1(f_\Theta) \;\&\; \mathbf{domain}(f_\Theta) = \mathsf{v}_0 \;\&\; \mathbf{range}(f_\Theta) = \nu_\Theta\\
\quad \langle \forall \mathsf{x} \in \mathsf{v}_0, \mathsf{y} \in \mathsf{v}_0 \mid \{\mathsf{x},\mathsf{y}\} \in \mathsf{e}_0 \leftrightarrow f_\Theta\,\mathsf{x} \in f_\Theta\,\mathsf{y} \vee f_\Theta\,\mathsf{y} \in f_\Theta\,\mathsf{x} \rangle\\
\quad \{\mathsf{x} \in \nu_\Theta \mid \mathsf{x} \cap \nu_\Theta \neq \emptyset\} \subseteq \mathscr{P}(\nu_\Theta)\\
\text{END finGraphRepr}
\end{array}
$$

Although accessory, the weak extensionality condition (last claim in the THEORY's interface just displayed) is the clue for getting the desired f; in fact, for any weakly extensional digraph, acyclicity always ensures that a variant of Mostowski's collapse is well-defined: in order to get it, one starts by assigning a distinct set Mt to each sink t and then proceeds by putting recursively

$$Mw = \{\, Mu \mid (w, u) \text{ is an arc} \,\}$$

for all non-sink nodes w; plainly, injectivity of the function $u \mapsto Mu$ can be ensured globally by a suitable choice of the images Mt of the sinks t. The said variant Mostowski's collapse for a well-founded weakly extensional digraph (even an infinite one) can be specified in Ref as a THEORY whose interface reads as follows:

$$
\begin{array}{l}
\text{THEORY mostowskiCollapse}(\mathsf{v}_0, \mathsf{a}_0)\\
\quad \langle \forall \mathsf{t} \subseteq \mathsf{v}_0, \exists \mathsf{m}, \forall \mathsf{x} \in \mathsf{t} \mid \mathsf{m} \in \mathsf{t} \;\&\; (\mathsf{m},\mathsf{x}) \notin \mathsf{a}_0 \rangle\\
\quad \langle \forall \mathsf{w} \in \mathsf{v}_0, \mathsf{w}' \in \mathsf{v}_0, \mathsf{u} \in \mathsf{v}_0 \mid\\
\qquad (\mathsf{w},\mathsf{u}) \in \mathsf{a}_0 \;\&\; \{\mathsf{x} \in \mathsf{v}_0 \mid (\mathsf{w},\mathsf{x}) \in \mathsf{a}_0\} = \{\mathsf{x} \in \mathsf{v}_0 \mid (\mathsf{w}',\mathsf{x}) \in \mathsf{a}_0\} \rightarrow \mathsf{w} = \mathsf{w}' \rangle\\
\Rightarrow (\mathsf{M}_\Theta)\\
\quad 1\text{--}1(\mathsf{M}_\Theta) \;\&\; \mathbf{domain}(\mathsf{M}_\Theta) = \mathsf{v}_0\\
\quad \langle \forall \mathsf{w} \in \mathsf{v}_0, \mathsf{u} \in \mathsf{v}_0 \mid (\mathsf{w},\mathsf{u}) \in \mathsf{a}_0 \rightarrow \mathsf{M}_\Theta \mathsf{w} = \{\mathsf{M}_\Theta \mathsf{u} : \mathsf{u} \in \mathsf{v}_0 \mid (\mathsf{w},\mathsf{u}) \in \mathsf{a}_0\} \rangle\\
\text{END mostowskiCollapse}
\end{array}
$$

Our second representation THEORY, cfGraphRepr, will specialize finGraphRepr to the case of a *connected, claw-free* (undirected, finite) graph—connectedness and claw-freeness appear, respectively, as the second and the third assumption of this THEORY. For these graphs, we can insist that the orientation be so imposed as to ensure *extensionality* in full: "distinct vertices have different out-neighborhoods".

DEF connectedness: [Connectedness of a graph]
$\mathsf{Connected}(V,E) \leftrightarrow_{\mathrm{Def}}$
$$\big\langle \forall x \in V, y \in V \mid x \neq y \;\&\; \{x,y\} \notin E \rightarrow \big\langle \exists p \subseteq E \mid \mathsf{Cycle}(p \cup \{\{y,x\}\})\big\rangle\big\rangle$$

DEF clawFreeGraph: [Claw-freeness of a graph]
$\mathsf{ClawFreeG}(V,E) \leftrightarrow_{\mathrm{Def}} \big\langle \forall w \in V, x \in V, y \in V, z \in V \mid \{w,y\}, \{y,x\}, \{y,z\} \in E \rightarrow$
$x = z \vee w \in \{z,x\} \vee \{x,z\} \in E \vee \{z,w\} \in E \vee \{w,x\} \in E \big\rangle$

THEORY cfGraphRepr(v_0, e_0)
$\mathsf{Finite}(v_0) \;\&\; e_0 \subseteq \{\{x,y\} : x, y \in v_0 \mid x \neq y\}$
$\mathsf{Connected}(v_0, e_0)$
$\mathsf{ClawFreeG}(v_0, e_0)$
$\Rightarrow (f_\Theta, \nu_\Theta)$
$1\text{--}1(f_\Theta) \;\&\; \mathbf{domain}(f_\Theta) = v_0 \;\&\; \mathbf{range}(f_\Theta) = \nu_\Theta$
$\big\langle \forall x \in v_0, y \in v_0 \mid \{x,y\} \in e_0 \leftrightarrow f_\Theta\, x \in f_\Theta\, y \vee f_\Theta\, y \in f_\Theta\, x \big\rangle$
$\mathsf{Trans}(\nu_\Theta) \;\&\; \mathsf{ClawFree}(\nu_\Theta)$
END cfGraphRepr

Consequently, the following will hold:

– there is a unique sink, $\emptyset$; moreover,
– the set ν underlying the image digraph is transitive. Also, rather trivially,
– ν is a claw-free set, in an even stronger sense than the definition with which we have been working throughout this paper.

Via the THEORY cfGraphRepr, the above-proved existence results about perfect matchings and Hamiltonian cycles can be transferred from the realm of membership digraphs to the *a priori* more general realm of connected claw-free graphs.

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A Proof on the representation of connected claw-free graphs

We say that a vertex x of a connected graph G is a *cut vertex* if the removal of x from G (together with all of its incident edges) disconnects the graph.

Proposition 4 *If G is a connected claw-free graph and $x \in V(G)$ is not a cut vertex of G, then G admits an extensional acyclic orientation whose (unique) sink is x.*

Proof We reason by induction on the number of vertices. Let G be a connected claw-free graph and let $x \in V(G)$ that is not a cut vertex of G. If there exists a vertex $y \in N(x)$ which is not a cut vertex of $G - \{x\}$, then from the inductive hypothesis $G - \{x\}$ admits an orientation D having y as sink. Extending the orientation D by orienting the edges incident to x as out-going towards x produces the desired e.a.o. of G.

Observe that if y is a cut vertex of a connected claw-free graph G, then $G - \{y\}$ has exactly two components. Suppose now that every neighbor of x is a cut vertex for $G - \{x\}$. Consider a minimum connected subgraph of $G - \{x\}$ which contains all neighbors of x, which must be a tree having as leaves neighbors of x. Let y be such a leaf, and denote by C_1 and C_2 the components of $G - \{x, y\}$, where x has no neighbors in C_2. Observe that y is not a cut vertex for $G - C_i - \{x\}$, $i = 1, 2$. Applying the inductive hypothesis to $G - C_i - \{x\}$, we can obtain the e.a.o. D_i whose sink is y, $i = 1, 2$. Let s_i be the vertex of C_i having y as unique out-neighbor in D_i. From the choice of y, the fact that G is claw-free, and $s_1 s_2 \notin E(G)$, the edge $s_1 x$ must be present in G, while $s_2 x \notin E(G)$. Then, obtain the e.a.o. D of G by extending D_1 and D_2 and orienting the edges incident to x as out-going towards x. □

It is easy to see that any nonempty graph has a vertex which is not a cut vertex; therefore we have:

Proposition 5 *If G is a connected claw-free graph, then there exists a finite full set ν_G and a bijection $f : V(G) \to \nu_G$ so that $xy \in E(G)$ if and only if either $fx \in fy$ or $fy \in fx$.*

Proof Apply Proposition 4 to obtain an extensional acyclic orientation D of G. Bijection f is Mostowski's collapse on D, where ν_G is taken to be $\{fx \mid x \in V(D)\}$. An example of a claw-free graph is given in Figure 3. □