Claw-free graphs as sets

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1 Basic laws on the union-set global operation

Def unionset: [Family of all members of members of a set] $\bigcup X \quad =_{\text{Def}} \quad \{u : v \in X, u \in v\}$

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THEOREM 2: [Union set as an upper bound] $(X \in S \to X \subseteq \bigcup S)$ & $(\langle \forall y \in S \mid y \subseteq X \rangle \to \bigcup S \subseteq X)$. PROOF:

Suppose_not(t, s) $\Rightarrow$ $(t \nsubseteq \bigcup s$ & $t \in s) \vee (\langle \forall y \in s \mid y \subseteq t \rangle$ & $\bigcup s \nsubseteq t)$

Use_def($\bigcup s$) $\Rightarrow$ AUTO

Suppose $\Rightarrow$ *Stat1*: $t \nsubseteq \bigcup s$ & $t \in s$

$\langle c \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ *Stat2*: $c \notin \{z : y \in s, z \in y\}$ & $c \in t$

$\langle t, c \rangle \hookrightarrow$ *Stat2* $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat3*: $\bigcup s \nsubseteq t$ & $\langle \forall y \in s \mid y \subseteq t \rangle$

$\langle d \rangle \hookrightarrow$ *Stat3* $\Rightarrow$ *Stat4*: $d \in \{z : y \in s, z \in y\}$ & $\langle \forall y \in s \mid y \subseteq t \rangle$ & $d \notin t$

$\langle b, a, b \rangle \hookrightarrow$ *Stat4* $\Rightarrow$ false; Discharge $\Rightarrow$ QED

THEORY imageOfDoubleton$(f(X), x_0, x_1)$
END imageOfDoubleton

ENTER_THEORY imageOfDoubleton

THEOREM imageOfDoubleton: [The image of a doubleton is either a doubleton or a singleton]
$\{f(v) : v \in \emptyset\} = \emptyset$ & $\{f(v) : v \in \{x_0\}\} = \{f(x_0)\}\, \{f(v) : v \in \{x_0, x_1\}\} = \{f(x_0), f(x_1)\}$. PROOF:

Suppose_not() $\Rightarrow$ AUTO

SIMPLF $\Rightarrow$ *Stat1*: $\{f(v) : v \in \{x_0, x_1\}\} \neq \{f(x_0), f(x_1)\}$

$\langle c \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ $c \in \{f(v) : v \in \{x_0, x_1\}\} \not\leftrightarrow c \in \{f(x_0), f(x_1)\}$

Suppose $\Rightarrow$ *Stat2*: $c \notin \{f(v) : v \in \{x_0, x_1\}\}$ & $c \notin \{f(v) : v \in \{x_0, x_1\}\}$

$\langle x_0, x_1 \rangle \hookrightarrow$ *Stat2* $\Rightarrow$ AUTO

Discharge $\Rightarrow$ *Stat3*: $c \in \{f(v) : v \in \{x_0, x_1\}\}$ & $c \notin \{f(x_0), f(x_1)\}$

$\langle x' \rangle \hookrightarrow$ *Stat3* $\Rightarrow$ $x' \in \{x_0, x_1\}$ & $f(x') \neq f(x_0)$ & $f(x') \neq f(x_1)$

Suppose $\Rightarrow$ $x' = x_0$

EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ $x' = x_1$

EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

ENTER_THEORY Set_theory
DISPLAY imageOfDoubleton

> THEORY imageOfDoubleton$(f(X), x_0, x_1)$
> $\{f(v) : v \in \emptyset\} = \emptyset$ & $\{f(v) : v \in \{x_0\}\} = \{f(x_0)\}$ & $\{f(v) : v \in \{x_0, x_1\}\} = \{f(x_0), f(x_1)\}$
> END imageOfDoubleton

THEOREM 2*a*: [Union of double-/single-tons] $Z = \{X, Y\} \to \bigcup Z = X \cup Y$. PROOF:

Suppose_not(z_0, x_0, y_0) $\Rightarrow$ AUTO

$\langle x_0, z_0 \rangle \hookrightarrow T2$ $\Rightarrow$ AUTO

$\langle y_0, z_0 \rangle \hookrightarrow T2$ $\Rightarrow$ AUTO

$\langle x_0 \cup y_0, z_0 \rangle \hookrightarrow T2 \Rightarrow$ $\quad$ *Stat1*: $\neg\langle \forall y \in z_0 \mid y \subseteq x_0 \cup y_0 \rangle$
$\langle v \rangle \hookrightarrow Stat1 \Rightarrow$ $\quad v \in \{x_0, y_0\}$ & $v \nsubseteq x_0 \cup y_0$
$(Stat1\star)$Discharge $\Rightarrow$ $\quad$ QED

THEOREM 2b: [Union of union] $\bigcup\bigcup X = \bigcup\{\bigcup y : y \in X\}$. PROOF:
Suppose_not$(x_0) \Rightarrow$ $\quad$ AUTO
Use_def$(\bigcup) \Rightarrow$ $\quad \{z : y \in \{u : v \in x_0, u \in v\}, z \in y\} \neq \{s : r \in \{\bigcup y : y \in x_0\}, s \in r\}$
SIMPLF $\Rightarrow$ $\quad$ *Stat1*: $\{z : v \in x_0, u \in v, z \in u\} \neq \{s : y \in x_0, s \in \bigcup y\}$
$\langle z_0 \rangle \hookrightarrow Stat1 \Rightarrow$ $\quad$ *Stat2*: $z_0 \in \{z : v \in x_0, u \in v, z \in u\} \not\leftrightarrow z_0 \in \{s : y \in x_0, s \in \bigcup y\}$
Suppose $\Rightarrow$ $\quad$ *Stat3*: $z_0 \in \{z : v \in x_0, u \in v, z \in u\}$ & $z_0 \notin \{s : y \in x_0, s \in \bigcup y\}$
Use_def$(\bigcup v_0) \Rightarrow$ $\quad$ AUTO
$\langle v_0, u_0, z, v_0, z_0 \rangle \hookrightarrow Stat3(Stat2\star) \Rightarrow$ $\quad$ *Stat4*: $z_0 \notin \{z : u \in v_0, z \in u\}$ & $v_0 \in x_0$ & $u_0 \in v_0$ & $z_0 \in u_0$
$\langle u_0, z_0 \rangle \hookrightarrow Stat4(Stat4\star) \Rightarrow$ $\quad$ false; $\quad$ Discharge $\Rightarrow$ $\quad$ *Stat5*: $z_0 \in \{s : y \in x_0, s \in \bigcup y\}$
Use_def$(\bigcup y_0) \Rightarrow$ $\quad$ AUTO
$\langle y_0, s_0 \rangle \hookrightarrow Stat5(Stat5\star) \Rightarrow$ $\quad$ *Stat6*: $z_0 \in \{s : u \in y_0, s \in u\}$ & $y_0 \in x_0$
$\langle u_1, s_1 \rangle \hookrightarrow Stat6(Stat5, Stat2\star) \Rightarrow$ $\quad$ *Stat7*: $z_0 \notin \{z : v \in x_0, u \in v, z \in u\}$ & $z_0 \in u_1$ & $u_1 \in y_0$
$\langle y_0, u_1, z_0 \rangle \hookrightarrow Stat7(Stat6\star) \Rightarrow$ $\quad$ false; $\quad$ Discharge $\Rightarrow$ $\quad$ QED

THEOREM 2c: [Additivity of monadic union] $\bigcup(X \cup Y) = \bigcup X \cup \bigcup Y$. PROOF:
Suppose_not$(x_0, y_0) \Rightarrow$ $\quad$ AUTO
$\langle \{x_0, y_0\} \rangle \hookrightarrow T2b \Rightarrow$ $\quad \bigcup\bigcup\{x_0, y_0\} = \bigcup\{\bigcup v : v \in \{x_0, y_0\}\}$
APPLY $\langle\,\rangle$ imageOfDoubleton$(f(X) \mapsto \bigcup X, x_0 \mapsto x_0, x_1 \mapsto y_0) \Rightarrow$ $\quad \{\bigcup v : v \in \{x_0, y_0\}\} = \{\bigcup x_0, \bigcup y_0\}$
$\langle \{x_0, y_0\}, x_0, y_0 \rangle \hookrightarrow T2a \Rightarrow$ $\quad \bigcup\{x_0, y_0\} = x_0 \cup y_0$
$\langle \{\bigcup x_0, \bigcup y_0\}, \bigcup x_0, \bigcup y_0 \rangle \hookrightarrow T2a \Rightarrow$ $\quad \bigcup\{\bigcup x_0, \bigcup y_0\} = \bigcup x_0 \cup \bigcup y_0$
EQUAL $\Rightarrow$ $\quad$ false; $\quad$ Discharge $\Rightarrow$ $\quad$ QED

THEOREM 2d: [Monotonicity of monadic union] $X \subseteq Y \rightarrow \bigcup X \subseteq \bigcup Y$. PROOF:
Suppose_not$(x_0, y_0) \Rightarrow$ $\quad y_0 = x_0 \cup y_0$ & $\bigcup x_0 \nsubseteq \bigcup y_0$
$\langle x_0, y_0 \rangle \hookrightarrow T2c \Rightarrow$ $\quad$ AUTO
EQUAL $\Rightarrow$ $\quad \bigcup y_0 = \bigcup x_0 \cup \bigcup y_0$
Discharge $\Rightarrow$ $\quad$ QED

THEOREM 2q: [Union of union, 2] $Y \in Z$ & $X \in \{Z, Z \setminus \{Y\}\} \rightarrow \bigcup Z = Y \cup \bigcup X$. PROOF:
Suppose_not$(y_0, z_0, x_0) \Rightarrow$ $\quad$ AUTO
ELEM $\Rightarrow$ $\quad z_0 = x_0 \cup \{y_0\}$
EQUAL $\Rightarrow$ $\quad \bigcup(x_0 \cup \{y_0\}) \neq y_0 \cup \bigcup x_0$
$\langle \{y_0\}, y_0, y_0 \rangle \hookrightarrow T2a \Rightarrow$ $\quad$ AUTO
$\langle x_0, \{y_0\} \rangle \hookrightarrow T2c \Rightarrow$ $\quad$ false; $\quad$ Discharge $\Rightarrow$ $\quad$ QED

THEOREM 2e: [Union of adjunction] $\bigcup(X \cup \{Y\}) = Y \cup \bigcup X$. PROOF:

Suppose_not$(x_0, y_0) \Rightarrow$ AUTO
$\langle y_0, x_0 \cup \{y_0\}, x_0 \rangle \hookrightarrow T2q(\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

2 Transitive sets

DEF transitivity: [Transitive, aka full, set] $\mathrm{Trans}(T) \leftrightarrow_{\mathrm{Def}} \{y \in T \mid y \not\subseteq T\} = \emptyset$

THEOREM 3a: [The unionset of a transitive set is included in it] $\mathrm{Trans}(T) \leftrightarrow T \supseteq \bigcup T$. PROOF:
Suppose_not$(t) \Rightarrow$ AUTO
Use_def$(\bigcup t) \Rightarrow$ AUTO
Use_def$(\mathrm{Trans}(t)) \Rightarrow$ AUTO
Suppose $\Rightarrow$ *Stat1*: $t \not\supseteq \bigcup t \;\&\; \mathrm{Trans}(t)$
$\langle c \rangle \hookrightarrow Stat1(\star) \Rightarrow$ *Stat2*: $c \in \{u : v \in t, u \in v\} \;\&\; \{y \in t \mid y \not\subseteq t\} = \emptyset \;\&\; c \notin t$
$\langle v, u, v \rangle \hookrightarrow Stat2(Stat2\star) \Rightarrow$ false; Discharge $\Rightarrow$ *Stat3*: $\{y \in t \mid y \not\subseteq t\} \neq \emptyset \;\&\; t \supseteq \{u : v \in t, u \in v\}$
Loc_def $\Rightarrow$ $a = \mathbf{arb}(d \backslash t)$
$\langle d \rangle \hookrightarrow Stat3(Stat3) \Rightarrow$ *Stat4*: $a \notin \{u : v \in t, u \in v\} \;\&\; d \in t \;\&\; a \in d \;\&\; a \notin t$
$\langle d, a \rangle \hookrightarrow Stat4(Stat4\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

THEOREM 3c: [For a transitive set, elements are also subsets] $\mathrm{Trans}(T) \;\&\; X \in T \rightarrow X \subseteq T$. PROOF:
Suppose_not$(t, x) \Rightarrow$ AUTO
$\langle t \rangle \hookrightarrow T3a \Rightarrow$ *Stat1*: $t = t \cup \{x\} \;\&\; \bigcup t \not\supseteq x \cup \bigcup t$
$\langle t, x \rangle \hookrightarrow T2e \Rightarrow$ AUTO
EQUAL(*Stat1*) $\Rightarrow$ false; Discharge $\Rightarrow$ QED

THEOREM 3d: [Trapping phenomenon for trivial sets] $\mathrm{Trans}(S) \;\&\; X, Z \in S \;\&\; X \notin Z \;\&\; Z \notin X \;\&\; S \backslash \{X, Z\} \subseteq \{\emptyset, \{\emptyset\}\} \rightarrow S \subseteq \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$. PROOF:
Suppose_not$(s, x, z) \Rightarrow$ AUTO
$\langle s, x \rangle \hookrightarrow T3c \Rightarrow$ AUTO
$\langle s, z \rangle \hookrightarrow T3c \Rightarrow$ AUTO
Discharge $\Rightarrow$ QED

THEOREM 4a: [Peddicord's lemma] $\mathrm{Trans}(T) \;\&\; Y \subseteq T \;\&\; Y \neq T \;\&\; A = \mathbf{arb}(T \backslash Y) \rightarrow A \subseteq Y \;\&\; A \in T \backslash Y$. PROOF:
Suppose_not$(t, y, a) \Rightarrow$ AUTO
$\langle t, a \rangle \hookrightarrow T3c \Rightarrow$ $a \subseteq t$
Discharge $\Rightarrow$ QED

THEOREM 4b: [$\emptyset$ belongs to every non-null transitive set t, $\{\emptyset\}$ also does if $t \not\subseteq \{\emptyset\}$, and so on]
$\mathrm{Trans}(T) \;\&\; N \in \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \;\&\; T \not\subseteq N \rightarrow N \subseteq T \;\&\; (N \in T \vee (N = \{\emptyset, \{\emptyset\}\} \;\&\; \{\{\emptyset\}\} \in T))$. PROOF:
Suppose_not$(t, n) \Rightarrow$ AUTO

$\langle \mathsf{t}, \emptyset, \mathbf{arb}(\mathsf{t}\backslash\emptyset) \rangle \hookrightarrow T4a(\star) \Rightarrow \quad \emptyset \in \mathsf{t}$
$\langle \mathsf{t}, \{\emptyset\}, \mathbf{arb}(\mathsf{t}\backslash \{\emptyset\}) \rangle \hookrightarrow T4a(\star) \Rightarrow \quad \{\emptyset\} \in \mathsf{t}$
$\langle \mathsf{t}, \{\emptyset, \{\emptyset\}\}, \mathbf{arb}(\mathsf{t}\backslash \{\emptyset, \{\emptyset\}\}) \rangle \hookrightarrow T4a(\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

THEOREM 4c: [Source removal does not disrupt transitivity] $\mathsf{Trans(S)}\ \&\ \mathsf{S} \supseteq \mathsf{T}\ \&\ (\mathsf{S}\backslash\mathsf{T}) \cap \bigcup \mathsf{S} = \emptyset \rightarrow \mathsf{Trans(T)}$. PROOF:
Suppose_not(s, t) $\Rightarrow$ AUTO
Use_def(Trans) $\Rightarrow$ $Stat1: \{\mathsf{y} \in \mathsf{t} \mid \mathsf{y} \not\subseteq \mathsf{t}\} \neq \emptyset\ \&\ \{\mathsf{y} \in \mathsf{s} \mid \mathsf{y} \not\subseteq \mathsf{s}\} = \emptyset$

$\langle \mathsf{y}, \mathsf{y} \rangle \hookrightarrow Stat1 \Rightarrow$ $Stat2: \mathsf{y} \not\subseteq \mathsf{t}\ \&\ \mathsf{y} \in \mathsf{s}\ \&\ \mathsf{y} \subseteq \mathsf{s}$
Use_def($\bigcup$s) $\Rightarrow$ AUTO
$\langle \mathsf{z} \rangle \hookrightarrow Stat2 \Rightarrow$ $Stat3: \mathsf{z} \notin \{\mathsf{u}: \mathsf{v} \in \mathsf{s}, \mathsf{u} \in \mathsf{v}\}\ \&\ \mathsf{z} \in \mathsf{y}$

$\langle \mathsf{y}, \mathsf{z} \rangle \hookrightarrow Stat3 \Rightarrow$ false; Discharge $\Rightarrow$ QED

3 Basic laws on the finiteness property

DEF $\mathcal{P}$: [Family of all subsets of a given set] $\mathcal{P}\mathsf{S} =_{\mathrm{Def}} \{\mathsf{x}: \mathsf{x} \subseteq \mathsf{S}\}$

DEF Fin: [Finiteness property] $\mathsf{Finite(F)} \leftrightarrow_{\mathrm{Def}} \langle \forall \mathsf{g} \in \mathcal{P}(\mathcal{P}\mathsf{F})\backslash \{\emptyset\}, \exists \mathsf{m} \mid \mathsf{g} \cap \mathcal{P}\mathsf{m} = \{\mathsf{m}\} \rangle$

THEOREM 23: [Monotonicity of powerset] $\mathsf{S} \supseteq \mathsf{X} \rightarrow \mathcal{P}\mathsf{X} \cup \{\emptyset, \mathsf{X}\} \subseteq \mathcal{P}\mathsf{S}$. PROOF:
Suppose_not(s_0, x_0) $\Rightarrow$ AUTO
Set_monot $\Rightarrow$ $\{\mathsf{x}: \mathsf{x} \subseteq \mathsf{x}_0\} \subseteq \{\mathsf{x}: \mathsf{x} \subseteq \mathsf{s}_0\}$
Use_def($\mathcal{P}$) $\Rightarrow$ $Stat1: \emptyset \notin \{\mathsf{x}: \mathsf{x} \subseteq \mathsf{s}_0\} \vee \mathsf{x}_0 \notin \{\mathsf{x}: \mathsf{x} \subseteq \mathsf{s}_0\}$
$\langle \emptyset, \mathsf{x}_0 \rangle \hookrightarrow Stat1 \Rightarrow$ false; Discharge $\Rightarrow$ QED

THEOREM 24: [Monotonicity of finiteness] $\mathsf{Y} \supseteq \mathsf{X}\ \&\ \mathsf{Finite(Y)} \rightarrow \mathsf{Finite(X)}$. PROOF:
Suppose_not(y_0, x_0) $\Rightarrow$ AUTO
$\langle \mathsf{y}_0, \mathsf{x}_0 \rangle \hookrightarrow T23(\star) \Rightarrow$ $\mathcal{P}\mathsf{y}_0 \supseteq \mathcal{P}\mathsf{x}_0$
Use_def(Finite) $\Rightarrow$ $Stat1: \neg\langle \forall \mathsf{g} \in \mathcal{P}(\mathcal{P}\mathsf{x}_0)\backslash \{\emptyset\}, \exists \mathsf{m} \mid \mathsf{g} \cap \mathcal{P}\mathsf{m} = \{\mathsf{m}\} \rangle\ \&\ \langle \forall \mathsf{g}' \in \mathcal{P}(\mathcal{P}\mathsf{y}_0)\backslash \{\emptyset\}, \exists \mathsf{m} \mid \mathsf{g}' \cap \mathcal{P}\mathsf{m} = \{\mathsf{m}\} \rangle$
$\langle \mathcal{P}\mathsf{y}_0, \mathcal{P}\mathsf{x}_0 \rangle \hookrightarrow T23(\star) \Rightarrow$ $\mathcal{P}(\mathcal{P}\mathsf{y}_0) \supseteq \mathcal{P}(\mathcal{P}\mathsf{x}_0)$
$\langle \mathsf{g}_0, \mathsf{g}_0 \rangle \hookrightarrow Stat1(Stat1\star) \Rightarrow$ $\neg\langle \exists \mathsf{m} \mid \mathsf{g}_0 \cap \mathcal{P}\mathsf{m} = \{\mathsf{m}\} \rangle\ \&\ \langle \exists \mathsf{m} \mid \mathsf{g}_0 \cap \mathcal{P}\mathsf{m} = \{\mathsf{m}\} \rangle$
Discharge $\Rightarrow$ QED

THEORY finiteInduction(s_0, P(S))
Finite(s_0) & P(s_0)
END finiteInduction

ENTER_THEORY finiteInduction

THEOREM finiteInduction$_1$. $\langle \exists m \mid \{s \subseteq s_0 \mid P(s)\} \cap \mathcal{P}m = \{m\} \rangle$. PROOF:

Suppose_not() $\Rightarrow$ AUTO

Assump $\Rightarrow$ $\mathsf{Finite}(s_0) \,\&\, P(s_0)$

Use_def(Finite) $\Rightarrow$ *Stat1*: $\langle \forall g \in \mathcal{P}(\mathcal{P}s_0)\setminus\{\emptyset\}, \exists m \mid g \cap \mathcal{P}m = \{m\} \rangle$

$\langle \{s \subseteq s_0 \mid P(s)\} \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ $\{s \subseteq s_0 \mid P(s)\} \notin \mathcal{P}(\mathcal{P}s_0)\setminus\{\emptyset\}$

Suppose $\Rightarrow$ *Stat2*: $s_0 \notin \{s \subseteq s_0 \mid P(s)\}$

$\langle s_0 \rangle \hookrightarrow$ *Stat2* $\Rightarrow$ false; Discharge $\Rightarrow$ $\{s \subseteq s_0 \mid P(s)\} \notin \mathcal{P}(\mathcal{P}s_0)$

Use_def($\mathcal{P}$) $\Rightarrow$ *Stat3*: $\{s \subseteq s_0 \mid P(s)\} \notin \{y : y \subseteq \{z : z \subseteq s_0\}\}$

$\langle \{s \subseteq s_0 \mid P(s)\} \rangle \hookrightarrow$ *Stat3* $\Rightarrow$ *Stat4*: $\{s \subseteq s_0 \mid P(s)\} \not\subseteq \{z : z \subseteq s_0\}$

$\langle s_1 \rangle \hookrightarrow$ *Stat4* $\Rightarrow$ *Stat5*: $s_1 \in \{s : s \subseteq s_0 \mid P(s)\} \,\&\, s_1 \notin \{z : z \subseteq s_0\}$

$\langle s, s_1 \rangle \hookrightarrow$ *Stat5*(*Stat5*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ QED

APPLY $\langle v1_\Theta : \mathsf{fin}_\Theta \rangle$ Skolem$\Rightarrow$

THEOREM finiteInduction$_0$. $\{s \subseteq s_0 \mid P(s)\} \cap \mathcal{P}\mathsf{fin}_\Theta = \{\mathsf{fin}_\Theta\}$.

THEOREM finiteInduction$_2$: [Minimal finite set satisfying P] $S \subseteq \mathsf{fin}_\Theta \rightarrow \mathsf{Finite}(S) \,\&\, (P(S) \leftrightarrow S = \mathsf{fin}_\Theta)$. PROOF:

Suppose_not(s_1) $\Rightarrow$ AUTO

$\langle\,\rangle \hookrightarrow T$finiteInduction$_0$ $\Rightarrow$ $\{s \subseteq s_0 \mid P(s)\} \cap \mathcal{P}\mathsf{fin}_\Theta = \{\mathsf{fin}_\Theta\} \,\&\,$ *Stat1*: $\mathsf{fin}_\Theta \in \{s \subseteq s_0 \mid P(s)\}$

$\langle\,\rangle \hookrightarrow$ *Stat1* $\Rightarrow$ $\mathsf{fin}_\Theta \subseteq s_0 \,\&\, P(\mathsf{fin}_\Theta)$

Assump $\Rightarrow$ $\mathsf{Finite}(s_0)$

$\langle s_0, \mathsf{fin}_\Theta \rangle \hookrightarrow T24$ $\Rightarrow$ $\mathsf{Finite}(\mathsf{fin}_\Theta)$

$\langle \mathsf{fin}_\Theta, s_1 \rangle \hookrightarrow T24$ $\Rightarrow$ $P(s_1) \not\leftrightarrow s_1 = \mathsf{fin}_\Theta$

Suppose $\Rightarrow$ $s_1 = \mathsf{fin}_\Theta$

EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ $s_1 \notin \{s \subseteq s_0 \mid P(s)\} \cap \mathcal{P}\mathsf{fin}_\Theta \,\&\, P(s_1)$

Suppose $\Rightarrow$ $s_1 \notin \mathcal{P}\mathsf{fin}_\Theta$

Use_def($\mathcal{P}$) $\Rightarrow$ *Stat2*: $s_1 \notin \{y : y \subseteq \mathsf{fin}_\Theta\}$

$\langle s_1 \rangle \hookrightarrow$ *Stat2* $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat3*: $s_1 \notin \{s \subseteq s_0 \mid P(s)\}$

$\langle s_1 \rangle \hookrightarrow$ *Stat3* $\Rightarrow$ false; Discharge $\Rightarrow$ QED

ENTER_THEORY Set_theory

DISPLAY finiteInduction

```
THEORY finiteInduction(s0, P(S))
    Finite(s0) & P(s0)
⇒ (finΘ)
    ⟨∀S | S ⊆ finΘ → Finite(S) & (P(S) ↔ S = finΘ)⟩
END finiteInduction
```

4 Some combinatorics of the union-set operation

THEOREM 31*d*: [Unionset of $\emptyset$ and $\{\emptyset\}$] $Y \subseteq \{\emptyset\} \leftrightarrow \bigcup Y = \emptyset$. PROOF:

$\text{Suppose\_not}(x_0) \Rightarrow$ AUTO

$\text{Use\_def}(\bigcup x_0) \Rightarrow$ AUTO

$\text{Suppose} \Rightarrow$ *Stat1* : $\{z : y \in x_0, z \in y\} \neq \emptyset$

$\langle y_0, z_1\rangle \hookrightarrow$ *Stat1* $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat2* : $x_0 \nsubseteq \{\emptyset\}$ & $\{z : y \in x_0, z \in y\} = \emptyset$

$\langle y_1, y_1, \mathbf{arb}(y_1)\rangle \hookrightarrow$ *Stat2* $\Rightarrow$ false; Discharge $\Rightarrow$ QED

THEOREM 31*e*: [Unionset of a set obtained through single removal] $\bigcup(X \setminus \{Y\}) \supseteq \bigcup X \setminus Y$ & $\bigcup X \supseteq \bigcup(X \setminus \{Y\})$. PROOF:

$\text{Suppose\_not}(x, y) \Rightarrow$ AUTO

$\langle x \setminus \{y\}, x\rangle \hookrightarrow T2d(\star) \Rightarrow$ *Stat1* : $\bigcup(x \setminus \{y\}) \nsupseteq \bigcup x \setminus y$

$\langle c\rangle \hookrightarrow Stat1(Stat1\star) \Rightarrow$ *Stat2* : $c \in \bigcup x \setminus y$ & $c \notin \bigcup(x \setminus \{y\})$

$\text{Use\_def}(\bigcup) \Rightarrow$ *Stat3* : $c \in \{u : v \in x, u \in v\}$ & $c \notin \{u : v \in x \setminus \{y\}, u \in v\}$ & $c \notin y$

$\langle v_0, u_0, v_0, u_0\rangle \hookrightarrow Stat3(Stat3\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

THEOREM 31*f*: [Unionset, after a removal followed by two adjunctions] $\bigcup M \supseteq P$ & $Q \cup R = P \cup S \to \bigcup(M \setminus \{P\} \cup \{Q, R\}) = \bigcup M \cup S$. PROOF:

$\text{Suppose\_not}(m, p, q, r, s) \Rightarrow$ AUTO

$\text{TELEM} \Rightarrow$ $m \setminus \{p\} \cup \{q\} \cup \{r\} = m \setminus \{p\} \cup \{q, r\}$

$\text{EQUAL} \Rightarrow$ $\bigcup(m \setminus \{p\} \cup \{q\} \cup \{r\}) = \bigcup(m \setminus \{p\} \cup \{q, r\})$

$\langle m \setminus \{p\}, q\rangle \hookrightarrow T2e \Rightarrow$ AUTO

$\langle m \setminus \{p\} \cup \{q\}, r\rangle \hookrightarrow T2e(\star) \Rightarrow$ $\bigcup(m \setminus \{p\} \cup \{q, r\}) = \bigcup(m \setminus \{p\}) \cup (p \cup s)$

$\langle m, p\rangle \hookrightarrow T31e(\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

THEOREM 31*g*: [Incomparability of pre-pivotal elements] $Y \in X$ & $X \in Z$ & $X, Z \in S \to Y \in \bigcup(S \cap \bigcup S)$. PROOF:

$\text{Suppose\_not}(y, x, z, s) \Rightarrow$ $y \in x$ & $x \in z$ & $x, z \in s$ & $y \notin \bigcup(s \cap \bigcup s)$

$\text{Use\_def}(\bigcup) \Rightarrow$ *Stat1* : $y \notin \{v : u \in s \cap \bigcup s, v \in u\}$

$\text{Use\_def}(\bigcup s) \Rightarrow$ AUTO

$\langle x, y\rangle \hookrightarrow Stat1(\star) \Rightarrow$ *Stat2* : $x \notin \{t : w \in s, t \in w\}$

$\langle z, x\rangle \hookrightarrow Stat2(\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

THEOREM 31*h*: [Less-one lemma for unionset] $\bigcup M = T \setminus \{C\}$ & $S = T \cup X \cup \{V\}$ & $Y \in S \cap \{C, V\} \to \langle \exists d \mid \bigcup(M \cup \{X \cup \{Y\}\}) = S \setminus \{d\}\rangle$. PROOF:

$\text{Suppose\_not}(m, t, c, s, x, v, y) \Rightarrow$ *Stat0* : $\neg\langle \exists d \mid \bigcup(m \cup \{x \cup \{y\}\}) = s \setminus \{d\}\rangle$ & $\bigcup m = t \setminus \{c\}$ & $s = t \cup x \cup \{v\}$ & $(y = v \vee (c = y$ & $y \in s))$

$\langle s\rangle \hookrightarrow Stat0 \Rightarrow$ $\bigcup(m \cup \{x \cup \{y\}\}) \neq s$

$\langle c\rangle \hookrightarrow Stat0 \Rightarrow$ $\bigcup(m \cup \{x \cup \{y\}\}) \neq s \setminus \{c\}$

$\langle v\rangle \hookrightarrow Stat0 \Rightarrow$ $\bigcup(m \cup \{x \cup \{y\}\}) \neq s \setminus \{v\}$

$\langle m, x \cup \{y\}\rangle \hookrightarrow T2e \Rightarrow$ AUTO

$\text{EQUAL} \Rightarrow$ *Stat1* : $x \cup \{y\} \cup \bigcup m \neq s \setminus \{c\}$ & $x \cup \{y\} \cup \bigcup m \neq s \setminus \{v\}$ & $x \cup \{y\} \cup \bigcup m \neq s$

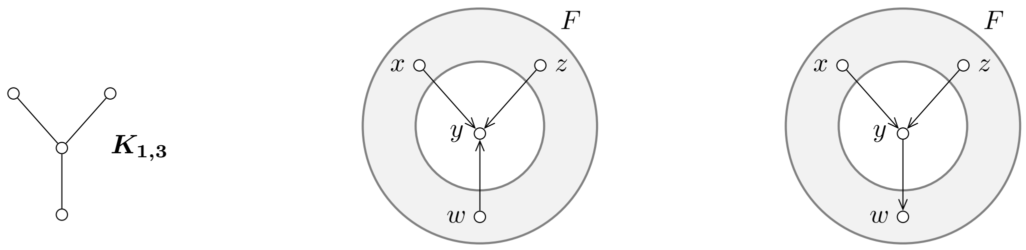

Figure 1: The forbidden orientations of a *claw* (as usually intended – see undirected graph on the left) inside a *claw-free set* (as formally defined in this scenario).

(*Stat0*, *Stat1*)Discharge $\Rightarrow$ QED

THEOREM 32: [Finite, non-null sets own sources] $\mathsf{Finite(F)} \;\&\; \mathsf{F} \neq \emptyset \rightarrow \mathsf{F}\backslash\bigcup\mathsf{F} \neq \emptyset$. PROOF:

Suppose_not(f_1) $\Rightarrow$ AUTO

APPLY $\langle \mathsf{fin}_\ominus : \mathsf{f}_0 \rangle$ finiteInduction$\left(\mathsf{s}_0 \mapsto \mathsf{f}_1, \mathsf{P(S)} \mapsto (\mathsf{S} \neq \emptyset \;\&\; \mathsf{S}\backslash\bigcup\mathsf{S} = \emptyset)\right) \Rightarrow$

Stat0 : $\langle \forall \mathsf{s} \,|\, \mathsf{s} \subseteq \mathsf{f}_0 \rightarrow \mathsf{Finite(s)} \;\&\; (\mathsf{s} \neq \emptyset \;\&\; \mathsf{s}\backslash\bigcup\mathsf{s} = \emptyset \leftrightarrow \mathsf{s} = \mathsf{f}_0)\rangle$

Loc_def $\Rightarrow$ $\mathsf{a} = \mathbf{arb}(\mathsf{f}_0)$

$\langle \mathsf{f}_0 \rangle \hookrightarrow$*Stat0* $\Rightarrow$ *Stat1* : $\mathsf{Finite}(\mathsf{f}_0) \;\&\; \mathsf{a} \in \mathsf{f}_0 \;\&\; \mathsf{f}_0\backslash\bigcup\mathsf{f}_0 = \emptyset$

Suppose $\Rightarrow$ $\mathsf{f}_0 = \{\mathsf{a}\} \;\&\; \bigcup\mathsf{f}_0 \not\subseteq \mathsf{a}$

EQUAL $\Rightarrow$ $\bigcup\{\mathsf{a}\} \not\subseteq \mathsf{a}$

Use_def($\bigcup$) $\Rightarrow$ $\{\mathsf{u} : \mathsf{v} \in \{\mathsf{a}\}, \mathsf{u} \in \mathsf{v}\} \not\subseteq \mathsf{a}$

SIMPLF $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO

$\langle \mathsf{f}_0\backslash\{\mathsf{a}\}, \mathsf{a} \rangle \hookrightarrow T2e(\star) \Rightarrow$ $\bigcup(\mathsf{f}_0\backslash\{\mathsf{a}\} \cup \{\mathsf{a}\}) = \bigcup(\mathsf{f}_0\backslash\{\mathsf{a}\}) \cup \mathsf{a} \;\&\; \mathsf{f}_0\backslash\{\mathsf{a}\} \cup \{\mathsf{a}\} = \mathsf{f}_0$

$\langle \mathsf{f}_0\backslash\{\mathsf{a}\} \rangle \hookrightarrow$*Stat0*$(\star) \Rightarrow$ $\mathsf{f}_0\backslash\bigcup(\mathsf{f}_0\backslash\{\mathsf{a}\}) \neq \emptyset$

EQUAL $\Rightarrow$ $\mathsf{f}_0\backslash\left(\bigcup(\mathsf{f}_0\backslash\{\mathsf{a}\}) \cup \mathsf{a}\right) = \emptyset$

Discharge $\Rightarrow$ QED

5 Claw-free, transitive sets and their pivots

DEF claw: [Pair forming an $\in$-claw, possibly endowed with more than 3 el'ts] $\mathsf{MembClaw}(Y,F) \leftrightarrow_{\mathrm{Def}} F \cap \bigcup F = \emptyset \;\&\; \langle \exists x,z,w \mid F \supseteq \{x,z,w\} \;\&\; x \neq z \;\&\; w \notin \{x,z\} \;\&\; \{w\} \cap Y \supseteq \{v \in F \mid Y \notin v\} \rangle$

DEF clawFreeness: [Claw-freeness in a membership digraph] $\mathsf{ClawFree}(S) \leftrightarrow_{\mathrm{Def}} \langle \forall y \in S, e \subseteq S \mid \neg \mathsf{MembClaw}(y,e) \rangle$

THEOREM $\mathsf{clawFreeness}_a$: [Subsets of claw-free sets are claw-free] $\mathsf{ClawFree}(S) \;\&\; T \subseteq S \rightarrow \mathsf{ClawFree}(T)$. PROOF:
Suppose_not(s, t) $\Rightarrow$ AUTO
Use_def(ClawFree) $\Rightarrow$ *Stat1* : $\neg \langle \forall y \in t, e \subseteq t \mid \neg \mathsf{MembClaw}(y,e) \rangle \;\&\; \langle \forall y \in s, e \subseteq s \mid \neg \mathsf{MembClaw}(y,e) \rangle$
$\langle y,e,y,e \rangle \hookrightarrow$ *Stat1*($\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ QED

THEOREM $\mathsf{clawFreeness}_b$: [Any potential $\in$-claw must have a bypass] $\mathsf{ClawFree}(S) \;\&\; S \supseteq \{Y,X,Z,W\} \;\&\; Y \in X \cap Z \;\&\; W \in Y \;\&\; X \notin Z \cup \{Z\} \;\&\; Z \notin X \rightarrow W \in X \cup Z$. PROOF:
Suppose_not(s, y, x, z, w) $\Rightarrow$ AUTO
Use_def(ClawFree) $\Rightarrow$ *Stat0* : $\langle \forall y \in s, e \subseteq s \mid \neg \mathsf{MembClaw}(y,e) \rangle \;\&\; x \notin w \;\&\; z \notin w \;\&\; x \notin z \;\&\; w \notin x \;\&\; w \notin z \;\&\; z \notin x \;\&\; x \neq z \;\&\; w \in y \;\&\; y \in x \cap z$
Loc_def $\Rightarrow$ *Stat1* : $e = \{x,z,w\}$
Use_def(MembClaw(y, e)) $\Rightarrow$ AUTO
$\langle y,e \rangle \hookrightarrow$ *Stat0*(*Stat1*$\star$) $\Rightarrow$ $\neg (e \cap \bigcup e = \emptyset \;\&\; \langle \exists x,z,w \mid e \supseteq \{x,z,w\} \;\&\; x \neq z \;\&\; w \notin \{x,z\} \;\&\; \{w\} \cap y \supseteq \{v \in e \mid y \notin v\} \rangle)$
EQUAL $\Rightarrow$ $\bigcup e = \bigcup \{x,z,w\}$
Suppose $\Rightarrow$ *Stat2* : $e \cap \bigcup e \neq \emptyset$
Use_def($\bigcup e$) $\Rightarrow$ AUTO
$\langle c \rangle \hookrightarrow$ *Stat2*($\star$) $\Rightarrow$ *Stat3* : $c \in \{u : v \in e, u \in v\} \;\&\; c \in e$
$\langle v_0,u_0 \rangle \hookrightarrow$ *Stat3*(*Stat1*, *Stat1*$\star$) $\Rightarrow$ *Stat4* : $v_0, c \in \{x,z,w\} \;\&\; c \in v_0$
(*Stat0*, *Stat4*$\star$)Discharge $\Rightarrow$ *Stat5* : $\langle \exists x,z,w \mid e \supseteq \{x,z,w\} \;\&\; x \neq z \;\&\; w \notin \{x,z\} \;\&\; \{w\} \cap y \supseteq \{v \in e \mid y \notin v\} \rangle$
$\langle x,z,w \rangle \hookrightarrow$ *Stat5*(*Stat0*$\star$) $\Rightarrow$ *Stat6* : $\{w\} \cap y \not\supseteq \{v \in e \mid y \notin v\}$
$\langle d \rangle \hookrightarrow$ *Stat6*(*Stat6*$\star$) $\Rightarrow$ *Stat7* : $d \in \{v \in e \mid y \notin v\} \;\&\; d \notin \{w\} \cap y$
$\langle\, \rangle \hookrightarrow$ *Stat7*(*Stat1*, *Stat1*$\star$) $\Rightarrow$ *Stat8* : $d \in \{x,z,w\} \;\&\; y \notin d$
(*Stat0*, *Stat8*, *Stat7*$\star$)Discharge $\Rightarrow$ QED

THEORY $\mathsf{pivotsForClawFreeness}(s_0)$
$\mathsf{ClawFree}(s_0) \;\&\; \mathsf{Finite}(s_0) \;\&\; \mathsf{Trans}(s_0)$
$s_0 \not\subseteq \{\emptyset\}$
END pivotsForClawFreeness

ENTER_THEORY pivotsForClawFreeness

DEF frontier: [Frontier of a set] $\mathsf{front}(S) =_{\mathrm{Def}} \{x \in S \mid x \cap S \setminus \bigcup (S \cap \bigcup S) \neq \emptyset\}$

THEOREM frontier$_1$: [Non-trivial finite sets have a non-null frontier] $\mathsf{Finite}(S \cap \bigcup S)\ \&\ S \cap \bigcup S \neq \emptyset \rightarrow \mathsf{front}(S) \neq \emptyset$. PROOF:

Suppose_not(s) $\Rightarrow$ AUTO

$\langle s \cap \bigcup s\rangle \hookrightarrow T32 \Rightarrow$ $Stat1:\ s \cap \bigcup s \backslash \bigcup(s \cap \bigcup s) \neq \emptyset$

Use_def($\bigcup$s) $\Rightarrow$ AUTO

$\langle y\rangle \hookrightarrow Stat1 \Rightarrow$ $Stat2:\ y \in \{u:\ v \in s, u \in v\}\ \&\ y \in s\ \&\ y \notin \bigcup(s \cap \bigcup s)$

Use_def(front(s)) $\Rightarrow$ AUTO

$\langle x, u\rangle \hookrightarrow Stat2 \Rightarrow$ $Stat3:\ x \notin \{x_1 \in s \mid x_1 \cap s \backslash \bigcup(s \cap \bigcup s) \neq \emptyset\}\ \&\ x \in s\ \&\ y \in x$

$\langle x\rangle \hookrightarrow Stat3 \Rightarrow$ false; Discharge $\Rightarrow$ QED

THEOREM frontier$_2$: [Transitivity-preserving reduction of a transitive set]

$\mathsf{Trans}(S)\ \&\ X \in \mathsf{front}(S)\ \&\ Y \in X \backslash \bigcup\bigcup S\ \&\ T = \{z \in S \mid Y \notin z\} \rightarrow \mathsf{Trans}(T)\ \&\ T \subseteq S\ \&\ X \notin T\ \&\ Y \in T \backslash \bigcup T$. PROOF:

Suppose_not(s, x, y, t) $\Rightarrow$ AUTO

Set_monot $\Rightarrow$ $\{z \in s \mid y \notin z\} \subseteq \{z:\ z \in s\}$

Suppose $\Rightarrow$ $Stat0:\ x \in \{z \in s \mid y \notin z\}$

$\langle\ \rangle \hookrightarrow Stat0 \Rightarrow$ false; Discharge $\Rightarrow$ $x \notin t$

Use_def(front) $\Rightarrow$ $Stat1:\ x \in \{x' \in s \mid x' \cap s \backslash \bigcup(s \cap \bigcup s) \neq \emptyset\}$

$\langle\ \rangle \hookrightarrow Stat1 \Rightarrow$ $\mathsf{Trans}(s)\ \&\ x \in s\ \&\ x \cap s \backslash \bigcup(s \cap \bigcup s) \neq \emptyset\ \&\ y \in x \backslash \bigcup\bigcup s\ \&\ t = \{z \in s \mid y \notin z\}\ \&\ \mathsf{Trans}(s)\ \&\ \neg(\mathsf{Trans}(t)\ \&\ y \in t \backslash \bigcup t)$

Suppose $\Rightarrow$ $\neg\mathsf{Trans}(t)$

$\langle s, t\rangle \hookrightarrow T4c \Rightarrow$ $Stat2:\ (s \backslash t) \cap \bigcup s \neq \emptyset$

Use_def($\bigcup$s) $\Rightarrow$ AUTO

$\langle z\rangle \hookrightarrow Stat2 \Rightarrow$ $Stat3:\ z \in \{u':\ w' \in s, u' \in w'\}\ \&\ z \notin \{z' \in s \mid y \notin z'\}\ \&\ z \in s$

$\langle v, a, z\rangle \hookrightarrow Stat3(Stat3\star) \Rightarrow$ $y \in z\ \&\ z \in v\ \&\ v \in s$

Use_def($\bigcup\bigcup$s) $\Rightarrow$ AUTO

EQUAL($Stat1$) $\Rightarrow$ $y \notin \{u:\ w \in \{u':\ w' \in s, u' \in w'\}, u \in w\}$

SIMPLF $\Rightarrow$ $Stat4:\ y \notin \{u:\ w' \in s, w \in w', u \in w\}$

$\langle v, z, y\rangle \hookrightarrow Stat4(Stat1\star) \Rightarrow$ false; Discharge $\Rightarrow$ $y \in \bigcup t \vee y \notin t$

Use_def($\bigcup$t) $\Rightarrow$ AUTO

Use_def(Trans(s)) $\Rightarrow$ AUTO

EQUAL $\Rightarrow$ $Stat5:\ \{y \in s \mid y \not\subseteq s\} = \emptyset\ \&\ y \in \{u:\ v \in \{z \in s \mid y \notin z\}, u \in v\} \vee y \notin \{z \in s \mid y \notin z\}$

$\langle x\rangle \hookrightarrow Stat5(Stat1\star) \Rightarrow \quad y \in s$
SIMPLF $\Rightarrow \quad Stat6: \; y \in \{u: \; v \in s, u \in v \,|\, y \notin v\} \;\vee y \notin \{z \in s \,|\, y \notin z\}$
$\langle w, u, y\rangle \hookrightarrow Stat6(Stat5\star) \Rightarrow \quad$ false; Discharge $\Rightarrow$ QED

DEF clawFreeness$_{frontEl}$: [Frontier element of a claw-free, transitive non-trivial set] $x_\Theta =_{Def} \mathbf{arb}(front(s_0))$

DEF clawFreeness$_{pivotEl}$: [Pivotal element of a claw-free, transitive non-trivial set] $y_\Theta =_{Def} \mathbf{arb}(x_\Theta \setminus \bigcup\bigcup s_0)$

THEOREM clawFreeness$_c$: [x_Θ truly belongs to the frontier of s_0] $x_\Theta \in front(s_0) \;\&\; x_\Theta \setminus \bigcup\bigcup s_0 \neq \emptyset \;\&\; x_\Theta \in s_0$. PROOF:
Suppose_not() $\Rightarrow$ AUTO
Assump $\Rightarrow \quad Stat0: \; ClawFree(s_0) \;\&\; Finite(s_0 \cap \bigcup s_0) \;\&\; Trans(s_0) \;\&\; s_0 \not\subseteq \{\emptyset\}$
$\langle s_0\rangle \hookrightarrow T3a \Rightarrow \quad s_0 \cap \bigcup s_0 = \bigcup s_0$
$\langle s_0\rangle \hookrightarrow T31d \Rightarrow \quad s_0 \cap \bigcup s_0 \neq \emptyset$
$\langle s_0\rangle \hookrightarrow T frontier_1 \Rightarrow \quad Stat1: \; front(s_0) \neq \emptyset$
Use_def$(front(s_0)) \Rightarrow$ AUTO
Use_def$(x_\Theta) \Rightarrow \quad Stat2: \; x_\Theta \in \{x \in s_0 \,|\, x \cap s_0 \setminus \bigcup(s_0 \cap \bigcup s_0) \neq \emptyset\} \;\&\; x_\Theta \in front(s_0)$
$\langle\;\rangle \hookrightarrow Stat2(Stat2\star) \Rightarrow \quad x_\Theta \in s_0 \;\&\; x_\Theta \setminus \bigcup(s_0 \cap \bigcup s_0) \neq \emptyset$
EQUAL $\Rightarrow \quad$ false; Discharge $\Rightarrow$ QED

THEOREM clawFreeness$_0$: [Pivots own at most two predecessors] $Y \in X \setminus \bigcup\bigcup s_0 \;\&\; X \in s_0 \rightarrow \langle \exists z \,|\, \{v \in s_0 \,|\, Y \in v\} = \{X, z\}\rangle$. PROOF:
Suppose_not$(y, x) \Rightarrow \quad Stat1: \; \neg\langle \exists z \,|\, \{v \in s_0 \,|\, y \in v\} = \{x, z\}\rangle \;\&\; x \in s_0 \;\&\; y \in x \setminus \bigcup\bigcup s_0$

Suppose $\Rightarrow \quad Stat2: \; x \notin \{v \in s_0 \,|\, y \in v\}$
$\langle x\rangle \hookrightarrow Stat2(\star) \Rightarrow \quad$ false; Discharge $\Rightarrow$ AUTO
$\langle x\rangle \hookrightarrow Stat1(\star) \Rightarrow \quad Stat3: \; \{v \in s_0 \,|\, y \in v\} \neq \{x\}$
$\langle z\rangle \hookrightarrow Stat3(\star) \Rightarrow \quad Stat4: \; z \in \{v \in s_0 \,|\, y \in v\} \;\&\; x \neq z$
$\langle\;\rangle \hookrightarrow Stat4(Stat4\star) \Rightarrow \quad Stat5: \; z \in s_0 \;\&\; y \in z$
$\langle z\rangle \hookrightarrow Stat1(\star) \Rightarrow \quad Stat6: \; \{v \in s_0 \,|\, y \in v\} \neq \{z, x\}$
$\langle w\rangle \hookrightarrow Stat6(\star) \Rightarrow \quad Stat7: \; w \in \{v \in s_0 \,|\, y \in v\} \;\&\; w \notin \{x, z\}$
$\langle\;\rangle \hookrightarrow Stat7(Stat7\star) \Rightarrow \quad Stat8: \; w \in s_0 \;\&\; y \in w$
Loc_def $\Rightarrow \quad e = \{x, z, w\}$
Suppose $\Rightarrow \quad Stat9: \; \{v \in e \,|\, y \notin v\} \neq \emptyset$
$\langle v\rangle \hookrightarrow Stat9(\star) \Rightarrow \quad$ false; Discharge $\Rightarrow$ AUTO

Assump $\Rightarrow \quad ClawFree(s_0) \;\&\; Trans(s_0)$
Use_def(ClawFree) $\Rightarrow \quad Stat10: \; \langle \forall y \in s_0, e \subseteq s_0 \,|\, \neg MembClaw(y, e)\rangle$
$\langle s_0, x\rangle \hookrightarrow T3c(\star) \Rightarrow \quad y \in s_0$
$\langle y, e\rangle \hookrightarrow Stat10(\star) \Rightarrow \quad \neg MembClaw(y, e)$

Use_def(MembClaw) $\Rightarrow$ $\quad$ *Stat11* : $\neg\langle\exists x, z, w \mid e \supseteq \{x, z, w\}\ \&\ x \neq z\ \&\ w \notin \{x, z\}\ \&\ \{w\} \cap y \supseteq \{v \in e \mid y \notin v\}\rangle \vee e \cap \bigcup e \neq \emptyset$

$\langle x, z, w\rangle \hookrightarrow$ *Stat11*(*Stat4*$\star$) $\Rightarrow$ $\quad e \cap \bigcup e \neq \emptyset$

EQUAL(*Stat8*) $\Rightarrow$ $\quad$ *Stat12* : $\{x, z, w\} \cap \bigcup\{x, z, w\} \neq \emptyset$

$\langle a\rangle \hookrightarrow$ *Stat12*(*Stat12*$\star$) $\Rightarrow$ $\quad$ *Stat13* : $a \in \{x, z, w\}\ \&\ a \in \bigcup\{x, z, w\}$

$\langle \{x, z, w\}, s_0\rangle \hookrightarrow$ *T2d*(*Stat1*, *Stat5*, *Stat8*, *Stat13*$\star$) $\Rightarrow$ $\quad y \in a\ \&\ \bigcup s_0 \supseteq \bigcup\{x, z, w\}$

$\langle \bigcup\{x, z, w\}, \bigcup s_0\rangle \hookrightarrow$ *T2d*(*Stat13*$\star$) $\Rightarrow$ $\quad \bigcup\bigcup s_0 \supseteq \bigcup\bigcup\{x, z, w\}$

$\langle a, \bigcup\{x, z, w\}\rangle \hookrightarrow$ *T2*(*Stat13*$\star$) $\Rightarrow$ $\quad$ *Stat14* : $y \in \bigcup\bigcup s_0$

(*Stat1*, *Stat14*$\star$)Discharge $\Rightarrow$ $\quad$ QED

THEOREM clawFreeness$_d$: [Shape of the frontier at a pivotal pair] $\langle\exists z \mid \{v \in s_0 \mid y_\Theta \in v\} = \{x_\Theta, z\}\ \&\ y_\Theta \in z\rangle$. PROOF:

Suppose_not() $\Rightarrow$ $\quad$ AUTO

$\langle\,\rangle \hookrightarrow T$clawFreeness$_c$ $\Rightarrow$ $\quad$ *Stat1* : $x_\Theta \setminus \bigcup\bigcup s_0 \neq \emptyset\ \&\ x_\Theta \in s_0$

Use_def(y_Θ) $\Rightarrow$ $\quad y_\Theta \in x_\Theta \setminus \bigcup\bigcup s_0$

$\langle y_\Theta, x_\Theta\rangle \hookrightarrow T$clawFreeness$_0$ $\Rightarrow$ $\quad$ *Stat2* : $\langle\exists z \mid \{v \in s_0 \mid y_\Theta \in v\} = \{x_\Theta, z\}\rangle\ \&\ \neg\langle\exists z \mid \{v \in s_0 \mid y_\Theta \in v\} = \{x_\Theta, z\}\ \&\ y_\Theta \in z\rangle$

$\langle z_0, z_0\rangle \hookrightarrow$ *Stat2* $\Rightarrow$ $\quad$ *Stat3* : $z_0 \in \{v \in s_0 \mid y_\Theta \in v\}\ \&\ y_\Theta \notin z_0$

$\langle\,\rangle \hookrightarrow$ *Stat3* $\Rightarrow$ $\quad$ false; $\quad$ Discharge $\Rightarrow$ $\quad$ QED

APPLY $\langle v1_\Theta : z_\Theta\rangle$ Skolem$\Rightarrow$

THEOREM clawFreeness$_e$. $\{v \in s_0 \mid y_\Theta \in v\} = \{x_\Theta, z_\Theta\}\ \&\ y_\Theta \in z_\Theta$.

THEOREM clawFreeness$_f$: [Tripleton pivot in a claw-free transitive non-trivial set]
$\{v \in s_0 \mid y_\Theta \in v\} = \{x_\Theta, z_\Theta\}\ \&\ \{x_\Theta, y_\Theta, z_\Theta\} \subseteq s_0\ \&\ y_\Theta \in x_\Theta \cap z_\Theta \setminus \bigcup\bigcup s_0\ \&\ x_\Theta \notin z_\Theta\ \&\ z_\Theta \notin x_\Theta$. PROOF:

Suppose_not() $\Rightarrow$ $\quad$ AUTO

$\langle\,\rangle \hookrightarrow T$clawFreeness$_c$ $\Rightarrow$ $\quad$ *Stat3* : $x_\Theta \setminus \bigcup\bigcup s_0 \neq \emptyset$

Use_def(y_Θ) $\Rightarrow$ $\quad y_\Theta \notin \bigcup\bigcup s_0\ \&\ y_\Theta \in x_\Theta$

$\langle\,\rangle \hookrightarrow T$clawFreeness$_e$ $\Rightarrow$ $\quad$ *Stat1* : $x_\Theta \in \{v : v \in s_0 \mid y_\Theta \in v\}\ \&\ z_\Theta \in \{v \in s_0 \mid y_\Theta \in v\}\ \&\ \{v : v \in s_0 \mid y_\Theta \in v\} = \{x_\Theta, z_\Theta\}\ \&\ y_\Theta \in z_\Theta$

$\langle v_0, v_1\rangle \hookrightarrow$ *Stat1* $\Rightarrow$ $\quad x_\Theta \in s_0\ \&\ z_\Theta \in s_0$

Assump $\Rightarrow$ $\quad$ Trans(s_0)

$\langle s_0, z_\Theta\rangle \hookrightarrow$ *T3c* $\Rightarrow$ $\quad y_\Theta \in s_0$

$\langle s_0\rangle \hookrightarrow$ *T3a* $\Rightarrow$ $\quad s_0 \cap \bigcup s_0 = \bigcup s_0$

EQUAL $\Rightarrow$ $\quad y_\Theta \notin \bigcup(s_0 \cap \bigcup s_0)$

$\langle y_\Theta, x_\Theta, z_\Theta, s_0\rangle \hookrightarrow$ *T31g* $\Rightarrow$ $\quad x_\Theta \notin z_\Theta$

$\langle y_\Theta, z_\Theta, x_\Theta, s_0\rangle \hookrightarrow$ *T31g* $\Rightarrow$ $\quad$ false; $\quad$ Discharge $\Rightarrow$ $\quad$ QED

DEF clawFreeness$_{remove}$: [Removal of elements above pivot from a transitive claw-free set] $\quad t_\Theta =_{\mathrm{Def}} \{v \in s_0 \mid y_\Theta \notin v\}$

THEOREM clawFreeness$_g$: [Removing elements above a pivot preserves transitivity]
Trans(t_Θ) & ClawFree(t_Θ) $\&\ t_\Theta \subseteq s_0\ \&\ x_\Theta \notin t_\Theta\ \&\ y_\Theta \in t_\Theta \setminus \bigcup t_\Theta\ \&\ t_\Theta = s_0 \setminus \{x_\Theta, z_\Theta\}$. PROOF:

Suppose_not() ⇒ AUTO
Use_def(t_Θ) ⇒ *Stat1*: $t_\Theta = \{v \in s_0 \mid y_\Theta \notin v\}$
Set_monot ⇒ $\{v \in s_0 \mid y_\Theta \notin v\} \subseteq \{v : v \in s_0\}$
Assump ⇒ Trans(s_0) & ClawFree(s_0)
$\langle s_0, t_\Theta \rangle \hookrightarrow T$clawFreeness$_a$(*Stat1*⋆) ⇒ ClawFree($t_\Theta$)
$\langle\ \rangle \hookrightarrow T$clawFreeness$_c$ ⇒ $x_\Theta \in$ front(s_0)
$\langle\ \rangle \hookrightarrow T$clawFreeness$_f$ ⇒ *Stat2*: $\{v \in s_0 \mid y_\Theta \in v\} = \{x_\Theta, z_\Theta\}$ & $y_\Theta \in x_\Theta \setminus \bigcup\bigcup s_0$ & $y_\Theta \notin \bigcup\bigcup s_0$
$\langle s_0, x_\Theta, y_\Theta, t_\Theta \rangle \hookrightarrow T$frontier$_2$(⋆) ⇒ *Stat3*: $t_\Theta \neq s_0 \setminus \{x_\Theta, z_\Theta\}$ & Trans(t_Θ) & $t_\Theta \subseteq s_0$ & $x_\Theta \notin t_\Theta$ & $y_\Theta \in t_\Theta \setminus \bigcup t_\Theta$
$\langle e \rangle \hookrightarrow$ *Stat3*(*Stat3*⋆) ⇒ $e \in t_\Theta \not\leftrightarrow e \in s_0 \setminus \{x_\Theta, z_\Theta\}$
Suppose ⇒ *Stat4*: $e \in \{v \in s_0 \mid y_\Theta \notin v\}$ & $e \notin s_0 \setminus \{x_\Theta, z_\Theta\}$
$\langle\ \rangle \hookrightarrow$ *Stat4*(*Stat2*⋆) ⇒ *Stat5*: $e \in \{v \in s_0 \mid y_\Theta \in v\}$ & $e \in s_0$ & $y_\Theta \notin e$
$\langle\ \rangle \hookrightarrow$ *Stat5*(*Stat5*⋆) ⇒ false; Discharge ⇒ *Stat6*: $e \notin \{v \in s_0 \mid y_\Theta \in v\}$ & $e \notin \{v \in s_0 \mid y_\Theta \notin v\}$ & $e \in s_0$
$\langle e, e \rangle \hookrightarrow$ *Stat6*(*Stat6*⋆) ⇒ false; Discharge ⇒ QED

ENTER_THEORY Set_theory
DISPLAY pivotsForClawFreeness

> THEORY pivotsForClawFreeness(s_0)
> ClawFree(s_0) & Finite(s_0) & Trans(s_0)
> $s_0 \not\subseteq \{\emptyset\}$
> ⇒ $(x_\Theta, y_\Theta, z_\Theta, t_\Theta)$
> $\langle \forall y, x \mid y \in x \setminus \bigcup\bigcup s_0$ & $x \in s_0 \rightarrow \langle \exists z \mid \{v \in s_0 \mid y \in v\} = \{x, z\} \rangle \rangle$
> $\{v \in s_0 \mid y_\Theta \in v\} = \{x_\Theta, z_\Theta\}$ & $\{x_\Theta, y_\Theta, z_\Theta\} \subseteq s_0$ & $y_\Theta \in x_\Theta \cap z_\Theta \setminus \bigcup\bigcup s_0$ & $x_\Theta \notin z_\Theta$ & $z_\Theta \notin x_\Theta$
> $t_\Theta = \{v \in s_0 \mid y_\Theta \notin v\}$
> ClawFree(t_Θ) & Trans(t_Θ) & $t_\Theta \subseteq s_0$ & $x_\Theta \notin t_\Theta$ & $y_\Theta \in t_\Theta \setminus \bigcup t_\Theta$ & $t_\Theta = s_0 \setminus \{x_\Theta, z_\Theta\}$
> END pivotsForClawFreeness

6 Hanks, cycles, and Hamiltonian cycles

DEF cycle$_0$: [Collection of edges whose endpoints have degree greater than 1] Hank(H) $\leftrightarrow_{\text{Def}}$ $\emptyset \notin H$ & $\langle \forall e \in H \mid e \subseteq \bigcup(H \setminus \{e\}) \rangle$

DEF cycle$_1$: [Cycle (unless null)] Cycle(C) $\leftrightarrow_{\text{Def}}$ Hank(C) & $\langle \forall d \subseteq C \mid$ Hank(d) & $d \neq \emptyset \rightarrow d = C \rangle$

THEOREM hank$_0$: [Alternative characterization of a hank] Hank(H) $\leftrightarrow (\emptyset \notin H$ & $\langle \forall e \in H, x \in e, \exists q \in H \mid q \neq e$ & $x \in q \rangle)$. PROOF:
Suppose_not(h) ⇒ AUTO
Suppose ⇒ $\neg\langle \forall e \in h, x \in e, \exists q \in h \mid q \neq e$ & $x \in q \rangle$ & $\langle \forall e \in h \mid e \subseteq \bigcup(h \setminus \{e\}) \rangle$
Use_def($\bigcup$) ⇒ *Stat1*: $\neg\langle \forall e \in h, x \in e, \exists q \in h \mid q \neq e$ & $x \in q \rangle$ & $\langle \forall e \in h \mid e \subseteq \{v : u \in h \setminus \{e_0\}, v \in u\} \rangle$
$\langle e_0, x_0, e_0 \rangle \hookrightarrow$ *Stat1* ⇒ *Stat2*: $x_0 \in \{v : u \in h \setminus \{e_0\}, v \in u\}$ & $\neg\langle \exists q \in h \mid q \neq e_0$ & $x_0 \in q \rangle$ & $e_0 \in h$ & $x_0 \in e_0$
$\langle q_0, v_0, q_0 \rangle \hookrightarrow$ *Stat2* ⇒ false; Discharge ⇒ AUTO
Use_def(Hank) ⇒ *Stat3*: $\neg\langle \forall e \in h \mid e \subseteq \bigcup(h \setminus \{e\}) \rangle$ & $\langle \forall e \in h, x \in e, \exists q \in h \mid q \neq e$ & $x \in q \rangle$

$\langle e_1\rangle\hookrightarrow Stat3 \Rightarrow$ $Stat4:\ e_1 \not\subseteq \bigcup(h\setminus\{e_1\})\ \&\ \langle\forall e\in h, x\in e, \exists q\in h\,|\, q\neq e\ \&\ x\in q\rangle\ \&\ e_1\in h$
Use_def$(\bigcup(h\setminus\{e_1\}))\Rightarrow$ AUTO
$\langle x_1,e_1,x_1\rangle\hookrightarrow Stat4 \Rightarrow$ $Stat5:\ \langle\exists q\in h\,|\,q\neq e_1\ \&\ x_1\in q\rangle\ \&\ x_1\notin\{v:\ u\in h\setminus\{e_1\}, v\in u\}\ \&\ x_1\in e_1$
$\langle q_1,q_1,x_1\rangle\hookrightarrow Stat5 \Rightarrow$ false; Discharge $\Rightarrow$ QED

THEOREM hank$_1$: [No singleton or doubleton set is a cycle] $H\subseteq\{X,U\}\ \&\ \mathrm{Hank}(H)\rightarrow H=\emptyset$. PROOF:
Suppose_not$(h_0,x_0,u_0)\Rightarrow$ $Stat0:\ h_0\neq\emptyset\ \&\ h_0\subseteq\{x_0,u_0\}\ \&\ \mathrm{Hank}(h_0)$

$\langle a\rangle\hookrightarrow Stat0(Stat0\star)\Rightarrow$ $Stat1:\ a\in h_0$
Use_def(Hank) $\Rightarrow$ $Stat2:\ \langle\forall e\in h_0\,|\,e\subseteq\bigcup(h_0\setminus\{e\})\rangle\ \&\ \emptyset\notin h_0$
$\langle a\rangle\hookrightarrow Stat2\Rightarrow$ $a\subseteq\bigcup(h_0\setminus\{a\})$
$\langle h_0\setminus\{a\}\rangle\hookrightarrow T31d\Rightarrow$ $Stat3:\ h_0\neq\{a\}$
$\langle b\rangle\hookrightarrow Stat3\Rightarrow$ $Stat4:\ b\in h_0\ \&\ b\neq a$
$\langle b\rangle\hookrightarrow Stat2\Rightarrow$ $b\subseteq\bigcup(h_0\setminus\{b\})$
$\langle\{a\},a,a\rangle\hookrightarrow T2a\Rightarrow$ $\bigcup\{a\}=a$
$\langle\{b\},b,b\rangle\hookrightarrow T2a\Rightarrow$ $\bigcup\{b\}=b$
$(Stat0, Stat1, Stat4\star)$ELEM $\Rightarrow$ $h_0\setminus\{a\}=\{b\}\ \&\ h_0\setminus\{b\}=\{a\}$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

THEOREM hank$_2$: [A membership chain and an extra edge form a hank (basic case)] $X\in Y\ \&\ Y\in Z\rightarrow\mathrm{Hank}(\{\{X,Y\},\{Y,Z\},\{Z,X\}\})$. PROOF:
Suppose_not$(x_0,y_0,z_0)\Rightarrow$ AUTO
Use_def(Hank) $\Rightarrow$ $Stat0:\ \neg\langle\forall e\in\{\{x_0,y_0\},\{y_0,z_0\},\{z_0,x_0\}\}\,|\,e\subseteq\bigcup(\{\{x_0,y_0\},\{y_0,z_0\},\{z_0,x_0\}\}\setminus\{e\})\rangle\ \&\ x_0\in y_0\ \&\ y_0\in z_0$
$\langle e_0\rangle\hookrightarrow Stat0\Rightarrow$ $e_0\in\{\{x_0,y_0\},\{y_0,z_0\},\{z_0,x_0\}\}\ \&\ e_0\not\subseteq\bigcup(\{\{x_0,y_0\},\{y_0,z_0\},\{z_0,x_0\}\}\setminus\{e_0\})$
Suppose $\Rightarrow$ $e_0=\{x_0,y_0\}\ \&\ \{\{x_0,y_0\},\{y_0,z_0\},\{z_0,x_0\}\}\setminus\{e_0\}=\{\{y_0,z_0\},\{z_0,x_0\}\}$
$\langle\{\{y_0,z_0\},\{z_0,x_0\}\},\{y_0,z_0\},\{z_0,x_0\}\rangle\hookrightarrow T2a\Rightarrow$ $\bigcup\{\{y_0,z_0\},\{z_0,x_0\}\}=\{y_0,z_0,x_0\}$
EQUAL $\Rightarrow$ $\{x_0,y_0\}\not\subseteq\{y_0,z_0,x_0\}$
Discharge $\Rightarrow$ AUTO
Suppose $\Rightarrow$ $e_0=\{y_0,z_0\}\ \&\ \{\{x_0,y_0\},\{y_0,z_0\},\{z_0,x_0\}\}\setminus\{e_0\}=\{\{x_0,y_0\},\{z_0,x_0\}\}$
$\langle\{\{x_0,y_0\},\{z_0,x_0\}\},\{x_0,y_0\},\{z_0,x_0\}\rangle\hookrightarrow T2a\Rightarrow$ $\bigcup\{\{x_0,y_0\},\{z_0,x_0\}\}=\{x_0,y_0,z_0\}$
EQUAL $\Rightarrow$ $\{y_0,z_0\}\not\subseteq\{x_0,y_0,z_0\}$
Discharge $\Rightarrow$ AUTO
$\langle\{\{x_0,y_0\},\{y_0,z_0\}\},\{x_0,y_0\},\{y_0,z_0\}\rangle\hookrightarrow T2a\Rightarrow$ $\bigcup\{\{x_0,y_0\},\{y_0,z_0\}\}=\{x_0,y_0,z_0\}\ \&\ e_0=\{z_0,x_0\}\ \&$
$\{\{x_0,y_0\},\{y_0,z_0\},\{z_0,x_0\}\}\setminus\{e_0\}=\{\{x_0,y_0\},\{y_0,z_0\}\}$
EQUAL $\Rightarrow$ $\{z_0,x_0\}\not\subseteq\{x_0,y_0,z_0\}$
Discharge $\Rightarrow$ QED

THEOREM hank$_3$: [Replacing an edge by a 2-path with the same endpoints does not disrupt a hank]
$\mathrm{Hank}(H)\ \&\ \{W,Y\}\in H\ \&\ W\neq Y\ \&\ X\notin\bigcup H\ \&\ H'=H\setminus\{\{W,Y\}\}\cup\{\{W,X\},\{X,Y\}\}\rightarrow\mathrm{Hank}(H')$. PROOF:
Suppose_not$(h_0,w_0,y_0,x_1,h_1)\Rightarrow$ $Stat0:\ (\mathrm{Hank}(h_0)\ \&\ \{w_0,y_0\}\in h_0\ \&\ w_0\neq y_0\ \&\ x_1\notin\bigcup h_0\ \&\ h_1=h_0\setminus\{\{w_0,y_0\}\}\cup\{\{w_0,x_1\},\{x_1,y_0\}\})\ \&\ \neg\mathrm{Hank}(h_1)$

Use_def(Hank) $\Rightarrow$ *Stat1*: $\neg\langle\forall e \in h_1 \mid e \subseteq \bigcup(h_1 \setminus \{e\})\rangle$ & *Stat2*: $\langle\forall e \in h_0 \mid e \subseteq \bigcup(h_0 \setminus \{e\})\rangle$
$\langle e_1, e_1\rangle \hookrightarrow$ *Stat1* $\Rightarrow$ *Stat3*: $e_1 \not\subseteq \bigcup(h_1 \setminus \{e_1\})$ & $e_1 \in h_1$ & $(e_1 \in h_0 \rightarrow e_1 \subseteq \bigcup(h_0 \setminus \{e_1\}))$
Use_def$(\bigcup(h_1 \setminus \{e_1\}))$ $\Rightarrow$ AUTO
$\langle z_1\rangle \hookrightarrow$ *Stat3*(*Stat3*$\star$) $\Rightarrow$ *Stat4*: $z_1 \notin \{v : u \in h_1 \setminus \{e_1\}, v \in u\}$ & $z_1 \in e_1$

Suppose $\Rightarrow$ $z_1 = x_1$
$\langle \{w_0, x_1\}, x_1\rangle \hookrightarrow$ *Stat4* $\Rightarrow$ $\{w_0, x_1\} = e_1$
$\langle \{x_1, y_0\}, x_1\rangle \hookrightarrow$ *Stat4* $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO

Suppose $\Rightarrow$ $e_1 \in \{\{w_0, x_1\}, \{x_1, y_0\}\}$
Use_def$(\bigcup(h_0 \setminus \{\{w_0, y_0\}\}))$ $\Rightarrow$ AUTO
$\langle \{w_0, y_0\}\rangle \hookrightarrow$ *Stat2* $\Rightarrow$ *Stat5*: $z_1 \in \{v : u \in h_0 \setminus \{\{w_0, y_0\}\}, v \in u\}$
$\langle e', z'\rangle \hookrightarrow$ *Stat5* $\Rightarrow$ $e' \in h_0 \setminus \{\{w_0, y_0\}\}$ & $z_1 \in e'$
Use_def$(\bigcup h_0)$ $\Rightarrow$ AUTO
$\langle e', z_1\rangle \hookrightarrow$ *Stat4* $\Rightarrow$ $x_1 \in e'$ & *Stat6*: $x_1 \notin \{v : u \in h_0, v \in u\}$
$\langle e', x_1\rangle \hookrightarrow$ *Stat6* $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO
Use_def$(\bigcup(h_0 \setminus \{e_1\}))$ $\Rightarrow$ AUTO

(*Stat0*$\star$)ELEM $\Rightarrow$ $e_1 \in h_0$ & *Stat7*: $z_1 \in \{v : u \in h_0 \setminus \{e_1\}, v \in u\}$ & $z_1 \notin \{v : u \in h_1 \setminus \{e_1\}, v \in u\}$
$\langle e_0, z_0, e_0, z_1\rangle \hookrightarrow$ *Stat7* $\Rightarrow$ *Stat8*: $z_1 \in \{w_0, y_0\}$
$\langle \{w_0, x_1\}, w_0\rangle \hookrightarrow$ *Stat4* $\Rightarrow$ $z_1 \neq w_0$
$\langle \{x_1, y_0\}, y_0\rangle \hookrightarrow$ *Stat4* $\Rightarrow$ $z_1 \neq y_0$
(*Stat8*$\star$)Discharge $\Rightarrow$ QED

THEOREM cycle_0: [A membership 2-chain and an extra edge make a cycle] $X \in Y$ & $Y \in Z \rightarrow \text{Cycle}(\{\{X, Y\}, \{Y, Z\}, \{Z, X\}\})$. PROOF:
Suppose_not(x_0, y_0, z_0) $\Rightarrow$ AUTO
Use_def$(\text{Cycle}(\{\{x_0, y_0\}, \{y_0, z_0\}, \{z_0, x_0\}\}))$ $\Rightarrow$ AUTO
$\langle x_0, y_0, z_0\rangle \hookrightarrow$ *Thank*$_2(\star)$ $\Rightarrow$ *Stat0*: $\neg\langle\forall d \subseteq \{\{x_0, y_0\}, \{y_0, z_0\}, \{z_0, x_0\}\} \mid \text{Hank}(d)$ & $d \neq \emptyset \rightarrow d = \{\{x_0, y_0\}, \{y_0, z_0\}, \{z_0, x_0\}\}\rangle$
$\langle d\rangle \hookrightarrow$ *Stat0* $\Rightarrow$ *Stat2*: $\text{Hank}(d)$ & $d \neq \emptyset$ & $d \neq \{\{x_0, y_0\}, \{y_0, z_0\}, \{z_0, x_0\}\}$ & $d \subseteq \{\{x_0, y_0\}, \{y_0, z_0\}, \{z_0, x_0\}\}$
$\langle d, \{y_0, z_0\}, \{z_0, x_0\}\rangle \hookrightarrow$ Thank$_1$(*Stat2*$\star$) $\Rightarrow$ $d \not\subseteq \{\{y_0, z_0\}, \{z_0, x_0\}\}$
$\langle d, \{x_0, y_0\}, \{z_0, x_0\}\rangle \hookrightarrow$ Thank$_1$(*Stat2*, *Stat2*$\star$) $\Rightarrow$ $d \not\subseteq \{\{x_0, y_0\}, \{z_0, x_0\}\}$
$\langle d, \{x_0, y_0\}, \{y_0, z_0\}\rangle \hookrightarrow$ Thank$_1$(*Stat2*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ QED

THEOREM cycle_1: [The replacement of an edge by a 2-path with the same endpoints does not disrupt a cycle]
$\text{Cycle}(C)$ & $\{W, Y\} \in C$ & $W \neq Y$ & $X \notin \bigcup C$ & $C' = C \setminus \{\{W, Y\}\} \cup \{\{W, X\}, \{X, Y\}\} \rightarrow \text{Cycle}(C')$. PROOF:
Suppose_not$(h_0, w_0, y_0, x_1, h_1)$ $\Rightarrow$ AUTO

$\langle h_0, w_0, y_0, x_1, h_1\rangle \hookrightarrow$ *Thank*$_3$ $\Rightarrow$ AUTO
Use_def(Cycle) $\Rightarrow$ *Stat0*: $\{w_0, y_0\} \in h_0$ & $w_0 \neq y_0$ & $x_1 \notin \bigcup h_0$ & $h_1 = h_0 \setminus \{\{w_0, y_0\}\} \cup \{\{w_0, x_1\}, \{x_1, y_0\}\}$ &
Stat1: $\neg\langle\forall d \subseteq h_1 \mid \text{Hank}(d)$ & $d \neq \emptyset \rightarrow d = h_1\rangle$ & *Stat2*: $\langle\forall d \subseteq h_0 \mid \text{Hank}(d)$ & $d \neq \emptyset \rightarrow d = h_0\rangle$ & $\text{Hank}(h_0)$ & $\text{Hank}(h_1)$

$\langle d_1, d_1\rangle \hookrightarrow Stat1(Stat0\star) \Rightarrow \quad Stat3: \; d_1 \subseteq h_1 \;\&\; d_1 \neq h_1 \;\&\; d_1 \neq \emptyset \;\&\; \mathsf{Hank}(d_1) \;\&\; \neg(\{w_0, x_1\} \notin d_1 \;\&\; \{x_1, y_0\} \notin d_1)$
$\mathsf{Use\_def}(\bigcup h_0) \Rightarrow \quad$ AUTO
$\langle d_1\rangle \hookrightarrow T\mathsf{hank}_0 \Rightarrow \quad Stat4: \; \langle \forall e \in d_1, x \in e, \exists q \in d_1 \mid q \neq e \;\&\; x \in q\rangle \;\&\; Stat4a: \; x_1 \notin \{v: \; u \in h_0, v \in u\} \;\&\; \emptyset \notin d_1$

$\langle \{w_0, x_1\}, x_1, q_0, q_0, x_1\rangle \hookrightarrow Stat4(Stat0\star) \Rightarrow \quad \neg(\{w_0, x_1\} \in d_1 \;\&\; \{x_1, y_0\} \notin d_1)$
$\langle \{x_1, y_0\}, x_1, q_1, q_1, x_1\rangle \hookrightarrow Stat4(Stat0\star) \Rightarrow \quad Stat5: \; \{w_0, x_1\}, \{x_1, y_0\} \in d_1$

$\langle \{w_0, y_0\}, x_1\rangle \hookrightarrow Stat4a(Stat0\star) \Rightarrow \quad Stat6: \; x_1 \neq w_0 \;\&\; x_1 \neq y_0 \;\&\; w_0 \neq y_0 \;\&\; \{w_0, y_0\} \notin d_1$
$\mathsf{Loc\_def} \Rightarrow \quad Stat7: \; d_0 = d_1 \cup \{\{w_0, y_0\}\} \setminus \{\{w_0, x_1\}, \{x_1, y_0\}\}$
$(Stat5, Stat7, Stat6, Stat3, Stat0, Stat4\star)\mathsf{ELEM} \Rightarrow \quad d_0 \subseteq h_0 \;\&\; d_0 \neq \emptyset \;\&\; d_0 \neq h_0 \;\&\; \emptyset \notin d_0$
$\mathsf{Use\_def}(\bigcup d_0) \Rightarrow \quad$ AUTO
$\langle d_0, h_0\rangle \hookrightarrow T2d\;(Stat0\star) \Rightarrow \quad Stat8: \; x_1 \notin \{u: \; v \in d_0, u \in v\}$

$\langle d_0\rangle \hookrightarrow T\mathsf{hank}_0 \Rightarrow \quad$ AUTO
$\langle d_0\rangle \hookrightarrow Stat2(Stat7\star) \Rightarrow \quad Stat9: \; \neg\langle \forall e \in d_0, x \in e, \exists q \in d_0 \mid q \neq e \;\&\; x \in q\rangle$
$\langle e_0, x_0\rangle \hookrightarrow Stat9 \Rightarrow \quad Stat10: \; \neg\langle \exists q \in d_0 \mid q \neq e_0 \;\&\; x_0 \in q\rangle \;\&\; e_0 \in d_0 \;\&\; x_0 \in e_0$

$\mathsf{Suppose} \Rightarrow \quad Stat11: \; e_0 \neq \{w_0, y_0\}$

$\langle e_0, x_0\rangle \hookrightarrow Stat4(Stat7, Stat10, Stat11\star) \Rightarrow \quad Stat12: \; \langle \exists q \in d_1 \mid q \neq e_0 \;\&\; x_0 \in q\rangle$
$\langle q_2\rangle \hookrightarrow Stat12(Stat12\star) \Rightarrow \quad Stat13: \; x_0 \in q_2 \;\&\; q_2 \neq e_0 \;\&\; q_2 \in d_1$

$\mathsf{Suppose} \Rightarrow \quad Stat14: \; q_2 = \{w_0, x_1\} \vee q_2 = \{x_1, y_0\}$

$\langle \{w_0, y_0\}\rangle \hookrightarrow Stat10(Stat7, Stat6, Stat11\star) \Rightarrow \quad Stat15: \; x_0 \notin \{w_0, y_0\}$
$\langle e_0, x_0\rangle \hookrightarrow Stat8 \Rightarrow \quad$ AUTO
$(Stat10\star)\mathsf{Discharge} \Rightarrow \quad Stat16: \; \neg(q_2 = \{w_0, x_1\} \vee q_2 = \{x_1, y_0\})$

$\langle q_2\rangle \hookrightarrow Stat10(Stat7, Stat16, Stat13\star) \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad Stat17: \; e_0 = \{w_0, y_0\}$

$\langle \{w_0, x_1\}, w_0, q_4\rangle \hookrightarrow Stat4(Stat5, Stat7, Stat6, Stat17\star) \Rightarrow \quad Stat19: \; q_4 \neq e_0 \;\&\; q_4 \in d_0 \;\&\; w_0 \in q_4$
$\langle \{x_1, y_0\}, y_0, q_5\rangle \hookrightarrow Stat4(Stat5, Stat5\star) \Rightarrow \quad Stat20: \; y_0 \in q_5 \;\&\; q_5 \neq \{x_1, y_0\} \;\&\; q_5 \in d_1$
$(Stat20, Stat7, Stat6, Stat17\star)\mathsf{ELEM} \Rightarrow \quad Stat21: \; q_5 \neq e_0 \;\&\; q_5, e_0 \in d_0$
$\langle q_5\rangle \hookrightarrow Stat10(Stat21, Stat17, Stat20\star) \Rightarrow \quad Stat22: \; x_0 \notin q_5 \;\&\; x_0 = w_0$
$\langle q_4\rangle \hookrightarrow Stat10(Stat19, Stat22\star) \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad$ QED

DEF hamiltonian$_1$: [Hamiltonian cycle, in graph devoid of isolated vertices] $\quad \mathsf{Hamiltonian}(H, S, E) \quad \leftrightarrow_{\mathrm{Def}} \quad \mathsf{Cycle}(H) \;\&\; \bigcup H = S \;\&\; H \subseteq E$

DEF hamiltonian$_2$: [Edges in squared membership] $\quad \mathsf{sqEdges}(S) \quad =_{\mathrm{Def}} \quad \{\{x, y\}: \; x \in S, y \in S \setminus \{x\}, z \in S \cap x \mid y = z \vee y \in z \vee z \in y\}$

DEF hamiltonian$_3$: [Restraining condition for Hamiltonian cycles] $\mathsf{SqHamiltonian}(H,S) \leftrightarrow_{Def} \mathsf{Hamiltonian}(H,S,\mathsf{sqEdges}(S)) \;\&\; \langle \forall x \in S \setminus \bigcup S, \exists y \in x \mid \{x,y\} \in H \rangle$

THEOREM hamiltonian$_1$: [Enriched Hamiltonian cycles]

$S = T \cup \{X\} \;\&\; X \notin T \;\&\; Y \in X \;\&\; \mathsf{SqHamiltonian}(H,T) \;\&\; \{W,Y\} \in H \;\&\; (W \in Y \vee (Y \in W \;\&\; K \neq Y \;\&\; \{W,K\} \in H \;\&\; K \in W)) \rightarrow$
$\mathsf{SqHamiltonian}(H \setminus \{\{W,Y\}\} \cup \{\{W,X\},\{X,Y\}\},S)$. PROOF:

Suppose_not$(s_0,t_0,x_1,y_0,h_0,w_0,k_0) \Rightarrow$ AUTO

Use_def(SqHamiltonian) $\Rightarrow$ *Stat0*: $\langle \forall x \in t_0 \setminus \bigcup t_0, \exists y \in x \mid \{x,y\} \in h_0 \rangle \;\&\; \mathsf{Hamiltonian}(h_0,t_0,\mathsf{sqEdges}(t_0)) \;\&$
$\neg\Big(\mathsf{Hamiltonian}(h_0 \setminus \{\{w_0,y_0\}\} \cup \{\{w_0,x_1\},\{x_1,y_0\}\},s_0,\mathsf{sqEdges}(s_0)) \;\&\; \langle \forall x \in s_0 \setminus \bigcup s_0, \exists y \in x \mid \{x,y\} \in h_0 \setminus \{\{w_0,y_0\}\} \cup \{\{w_0,x_1\},\{x_1,y_0\}\} \rangle\Big)$

Suppose $\Rightarrow$ *Stat1*: $\neg\langle \forall x \in s_0 \setminus \bigcup s_0, \exists y \in x \mid \{x,y\} \in h_0 \setminus \{\{w_0,y_0\}\} \cup \{\{w_0,x_1\},\{x_1,y_0\}\} \rangle$

$\langle x' \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ *Stat2*: $\neg\langle \exists k \in x' \mid \{x',k\} \in h_0 \setminus \{\{w_0,y_0\}\} \cup \{\{w_0,x_1\},\{x_1,y_0\}\} \rangle \;\&\; x' \in s_0 \setminus \bigcup s_0$

Suppose $\Rightarrow$ $x' \notin t_0 \setminus \bigcup t_0$

$\langle y_0 \rangle \hookrightarrow$ *Stat2*$(\star)$ $\Rightarrow$ $x' \in t_0 \;\&\; s_0 = t_0 \cup \{x_1\} \;\&\; x_1 \notin t_0 \;\&\; y_0 \in x_1$

$\langle t_0, s_0 \rangle \hookrightarrow$ *T2d* (*Stat2*$\star$) $\Rightarrow$ $\bigcup s_0 \supseteq \bigcup t_0$

(*Stat2*$\star$)Discharge $\Rightarrow$ AUTO

$\langle x', y_1 \rangle \hookrightarrow$ *Stat0*$(\star)$ $\Rightarrow$ *Stat3*: $y_1 \in x' \;\&\; \{x',y_1\} \in h_0 \;\&\; x_1 \in s_0 \;\&\; y_0 \in x_1$

Use_def$(\bigcup s_0)$ $\Rightarrow$ AUTO

$\langle y_1 \rangle \hookrightarrow$ *Stat2*(*Stat3*$\star$) $\Rightarrow$ *Stat4*: $x' \notin \{u : v \in s_0, u \in v\} \;\&\; \{x',y_1\} = \{w_0,y_0\}$

$\langle x_1, y_0 \rangle \hookrightarrow$ *Stat4*(*Stat2*$\star$) $\Rightarrow$ $x' = w_0$

$\langle x_1, y_0 \rangle \hookrightarrow$ *Stat4*$(\star)$ $\Rightarrow$ $x' \notin y_0 \;\&\; \{x',k_0\} \in h_0 \;\&\; k_0 \in x' \;\&\; k_0 \neq y_0$

$\langle k_0 \rangle \hookrightarrow$ *Stat2*(*Stat4*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat5*: $\mathsf{Hamiltonian}(h_0,t_0,\mathsf{sqEdges}(t_0)) \;\&\; \neg\mathsf{Hamiltonian}(h_0 \setminus \{\{w_0,y_0\}\} \cup \{\{w_0,x_1\},\{x_1,y_0\}\},s_0,\mathsf{sqEdges}(s_0))$

Use_def$\Big(\mathsf{Hamiltonian}(h_0,t_0,\mathsf{sqEdges}(t_0))\Big)$ $\Rightarrow$ AUTO

ELEM $\Rightarrow$ *Stat6*: $s_0 = t_0 \cup \{x_1\} \;\&\; x_1 \notin t_0 \;\&\; y_0 \in x_1 \;\&\; \{w_0,y_0\} \in h_0 \;\&\; (w_0 \in y_0 \vee (y_0 \in w_0 \;\&\; k_0 \neq y_0 \;\&\; \{w_0,k_0\} \in h_0 \;\&\; k_0 \in w_0))$

Loc_def $\Rightarrow$ *Stat7*: $h_1 = h_0 \setminus \{\{w_0,y_0\}\} \cup \{\{w_0,x_1\},\{x_1,y_0\}\}$

Use_def$\Big(\mathsf{Hamiltonian}(h_1,s_0,\mathsf{sqEdges}(s_0))\Big)$ $\Rightarrow$ AUTO

EQUAL(*Stat5*) $\Rightarrow$ *Stat8*: $(\mathsf{Cycle}(h_0) \;\&\; \bigcup h_0 = t_0 \;\&\; h_0 \subseteq \mathsf{sqEdges}(t_0)) \;\&\; \neg(\mathsf{Cycle}(h_1) \;\&\; \bigcup h_1 = s_0 \;\&\; h_1 \subseteq \mathsf{sqEdges}(s_0))$

$\langle h_0, w_0, y_0, x_1, h_1 \rangle \hookrightarrow$ *Tcycle*$_1$ (*Stat6*$\star$) $\Rightarrow$ *Stat9*: $(\mathsf{Cycle}(h_0) \;\&\; \bigcup h_0 = t_0 \;\&\; h_0 \subseteq \mathsf{sqEdges}(t_0)) \;\&\; \neg(\bigcup h_1 = s_0 \;\&\; h_1 \subseteq \mathsf{sqEdges}(s_0))$

Suppose $\Rightarrow$ *Stat10*: $\{w_0,y_0\} \not\subseteq \bigcup h_0$

Use_def$(\bigcup h_0)$ $\Rightarrow$ AUTO

$\langle b \rangle \hookrightarrow$ *Stat10*(*Stat10*$\star$) $\Rightarrow$ *Stat11*: $b \notin \{u : v \in h_0, u \in v\} \;\&\; b \in \{w_0,y_0\}$

$\langle \{w_0,y_0\}, b \rangle \hookrightarrow$ *Stat11*(*Stat11*, *Stat6*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat12*: $w_0, y_0 \in t_0$

Suppose $\Rightarrow$ *Stat13*: $h_1 \not\subseteq \mathsf{sqEdges}(s_0)$

Use_def$(\mathsf{sqEdges}(s_0))$ $\Rightarrow$ AUTO

$\langle e\rangle \hookrightarrow Stat13(Stat7\star) \Rightarrow$ $(e \in h_0 \vee e = \{w_0, x_1\} \vee e = \{x_1, y_0\})$ & $Stat14:\ e \notin \{\{x,y\} : x \in s_0, y \in s_0 \setminus \{x\}, z \in s_0 \cap x \mid y = z \vee y \in z \vee z \in y\}$
$(Stat6, Stat12\star)$ELEM $\Rightarrow$ $Stat15:\ x_1, y_0, w_0 \in s_0$ & $y_0 \in x_1$ & $(w_0 \in y_0 \vee y_0 \in w_0)$ & $x_1 \neq w_0$
$\langle x_1, y_0, y_0\rangle \hookrightarrow Stat14(Stat15\star) \Rightarrow$ $e \neq \{x_1, y_0\}$
$\langle x_1, w_0, y_0\rangle \hookrightarrow Stat14(Stat15\star) \Rightarrow$ $e \neq \{w_0, x_1\}$
Use_def$(\text{sqEdges}(t_0)) \Rightarrow$ AUTO
$(Stat8\star)$ELEM $\Rightarrow$ $Stat16:\ e \in \{\{x,y\} : x \in t_0, y \in t_0 \setminus \{x\}, z \in t_0 \cap x \mid y = z \vee y \in z \vee z \in y\}$
$\langle x_2, y_2, z_2\rangle \hookrightarrow Stat16(Stat16\star) \Rightarrow$ $Stat17:\ e = \{x_2, y_2\}$ & $x_2, y_2, z_2 \in t_0$ & $x_2 \neq y_2$ & $z_2 \in x_2$ & $(y_2 = z_2 \vee y_2 \in z_2 \vee z_2 \in y_2)$
$(Stat6\star)$ELEM $\Rightarrow$ $s_0 \supseteq t_0$
$\langle x_2, y_2, z_2\rangle \hookrightarrow Stat14(Stat17\star) \Rightarrow$ false; Discharge $\Rightarrow$ $Stat18:\ \bigcup h_1 \neq s_0$

$\langle \{\{w_0, x_1\}, \{x_1, y_0\}\}, \{w_0, x_1\}, \{x_1, y_0\}\rangle \hookrightarrow T2a\ (Stat18\star) \Rightarrow$ $\bigcup\{\{w_0, x_1\}, \{x_1, y_0\}\} = \{w_0, x_1, y_0\}$
$\langle h_0 \setminus \{\{w_0, y_0\}\}, \{\{w_0, x_1\}, \{x_1, y_0\}\}\rangle \hookrightarrow T2c\ (Stat18\star) \Rightarrow$ $\bigcup(h_0 \setminus \{\{w_0, y_0\}\} \cup \{\{w_0, x_1\}, \{x_1, y_0\}\}) = \bigcup(h_0 \setminus \{\{w_0, y_0\}\}) \cup \bigcup\{\{w_0, x_1\}, \{x_1, y_0\}\}$
$\langle h_0 \setminus \{\{w_0, y_0\}\}, h_0\rangle \hookrightarrow T2d\ (Stat8\star) \Rightarrow$ $\bigcup h_0 \supseteq \bigcup(h_0 \setminus \{\{w_0, y_0\}\})$ & $\{w_0, y_0\} \subseteq \bigcup h_0$
EQUAL$(Stat7) \Rightarrow$ $\bigcup h_1 = \bigcup(h_0 \setminus \{\{w_0, y_0\}\}) \cup \{w_0, x_1, y_0\}$
$(Stat6\star)$ELEM $\Rightarrow$ $\bigcup h_1 \subseteq s_0$

Use_def$(\bigcup h_1) \Rightarrow$ AUTO
$\langle a\rangle \hookrightarrow Stat18(Stat18\star) \Rightarrow$ $Stat23:\ a \notin \{u : v \in h_1, u \in v\}$ & $a \in s_0 \setminus \{x_1\}$

Use_def$(\bigcup) \Rightarrow$ $Stat24:\ a \in \{u : v \in h_0, u \in v\}$
$\langle e', u'\rangle \hookrightarrow Stat24(Stat25\star) \Rightarrow$ $Stat25:\ e' \in h_0$ & $a \in e'$

$\langle e', a\rangle \hookrightarrow Stat23(Stat7, Stat25\star) \Rightarrow$ $e' = \{w_0, y_0\}$
$\langle \{w_0, x_1\}, w_0\rangle \hookrightarrow Stat23(Stat7\star) \Rightarrow$ $a = y_0$
$\langle \{x_1, y_0\}, y_0\rangle \hookrightarrow Stat23(Stat7\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

THEOREM hamiltonian$_2$: [Doubly enriched Hamiltonian cycles]
$S = T \cup \{X, Z\}$ & $\{X, Z\} \cap T = \emptyset$ & $X \neq Z$ & $Y \in X \cap Z$ & SqHamiltonian(H, T) & $\{W, Y\} \in H$ & $W \in Y \cap X \rightarrow$
SqHamiltonian$(H \setminus \{\{W, Y\}\} \cup \{\{W, X\}, \{X, Z\}, \{Z, Y\}\}, S)$. PROOF:
Suppose_not$(s_0, t_0, x_0, z_0, y_0, h_0, w_0) \Rightarrow$ AUTO
$\langle t_0 \cup \{x_0\}, t_0, x_0, y_0, h_0, w_0, \emptyset\rangle \hookrightarrow Thamiltonian_1 \Rightarrow$ SqHamiltonian$(h_0 \setminus \{\{w_0, y_0\}\} \cup \{\{w_0, x_0\}, \{x_0, y_0\}\}, t_0 \cup \{x_0\})$
Loc_def $\Rightarrow$ $Stat1:\ h_1 = h_0 \setminus \{\{w_0, y_0\}\} \cup \{\{w_0, x_0\}, \{x_0, y_0\}\}$ & $t_1 = t_0 \cup \{x_0\}$
ELEM $\Rightarrow$ $Stat2:\ s_0 = t_1 \cup \{z_0\}$ & $x_0 \notin t_0$
EQUAL $\Rightarrow$ SqHamiltonian(h_1, t_1)
Suppose $\Rightarrow$ $\{x_0, y_0\} \in h_0$

Use_def$(\text{SqHamiltonian}(h_0, t_0)) \Rightarrow$ AUTO
Use_def$(\text{Hamiltonian}(h_0, t_0, \text{sqEdges}(t_0))) \Rightarrow$ AUTO
Use_def$(\bigcup) \Rightarrow$ $Stat3:\ x_0 \notin \{u : v \in h_0, u \in v\}$
$\langle \{x_0, y_0\}, x_0\rangle \hookrightarrow Stat3(Stat2\star) \Rightarrow$ false; Discharge $\Rightarrow$ $\{x_0, y_0\} \neq \{w_0, x_0\}$ & $z_0 \notin t_1$ & $y_0 \in z_0$ & $y_0 \in x_0$ & $w_0 \in y_0$ & $w_0 \in x_0$

$\langle t_1 \cup \{z_0\}, t_1, z_0, y_0, h_1, x_0, w_0\rangle \hookrightarrow T\mathsf{hamiltonian}_1\ (\mathit{Stat1}\star) \Rightarrow\quad \mathsf{SqHamiltonian}(h_1 \setminus \{\{x_0, y_0\}\} \cup \{\{x_0, z_0\}, \{z_0, y_0\}\}, t_1 \cup \{z_0\})$
$(\mathit{Stat1}\star)\mathsf{ELEM} \Rightarrow\quad h_1 \setminus \{\{x_0, y_0\}\} = h_0 \setminus \{\{w_0, y_0\}\} \cup \{\{w_0, x_0\}\}$
$(\mathit{Stat1}\star)\mathsf{ELEM} \Rightarrow\quad h_1 \setminus \{\{x_0, y_0\}\} \cup \{\{x_0, z_0\}, \{z_0, y_0\}\} = h_0 \setminus \{\{w_0, y_0\}\} \cup \{\{w_0, x_0\}, \{x_0, z_0\}, \{z_0, y_0\}\}$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

THEOREM hamiltonian$_3$: [Trivial Hamiltonian cycles] $S = \{X, Y, Z\}\ \&\ X \in Y\ \&\ Y \in Z \rightarrow \mathsf{SqHamiltonian}(\{\{X, Y\}, \{Y, Z\}, \{Z, X\}\}, S)$. PROOF:
Suppose_not$(s, x_0, y_0, z_0) \Rightarrow$ AUTO

Use_def$(\mathsf{SqHamiltonian}(\{\{x_0, y_0\}, \{y_0, z_0\}, \{z_0, x_0\}\}, s)) \Rightarrow$ AUTO
Use_def$\Big(\mathsf{Hamiltonian}(\{\{x_0, y_0\}, \{y_0, z_0\}, \{z_0, x_0\}\}, s, \mathsf{sqEdges}(s))\Big) \Rightarrow$ AUTO
$\langle x_0, y_0, z_0\rangle \hookrightarrow T\mathsf{cycle}_0 \Rightarrow$ AUTO
ELEM $\Rightarrow$ $\mathit{Stat1}:\ s = \{x_0, y_0, z_0\}\ \&\ x_0 \in y_0\ \&\ y_0 \in z_0$
Suppose $\Rightarrow$ $\mathit{Stat8}:\ \{\{x_0, y_0\}, \{y_0, z_0\}, \{z_0, x_0\}\} \not\subseteq \mathsf{sqEdges}(s)$
Use_def$(\mathsf{sqEdges}(s)) \Rightarrow$ AUTO
$\langle e_0\rangle \hookrightarrow \mathit{Stat8}(\mathit{Stat8}\star) \Rightarrow$ $\mathit{Stat9}:\ e_0 \notin \{\{x, y\} : x \in s, y \in s \setminus \{x\}, z \in s \cap x \mid y = z \vee y \in z \vee z \in y\}\ \&\ e_0 \in \{\{x_0, y_0\}, \{y_0, z_0\}, \{z_0, x_0\}\}$
$\langle z_0, y_0, y_0\rangle \hookrightarrow \mathit{Stat9}(\mathit{Stat1}, \mathit{Stat1}\star) \Rightarrow$ $e_0 \neq \{y_0, z_0\}$
$\langle z_0, x_0, y_0\rangle \hookrightarrow \mathit{Stat9}(\mathit{Stat1}, \mathit{Stat9}\star) \Rightarrow$ $e_0 \neq \{z_0, x_0\}$
$\langle y_0, x_0, x_0\rangle \hookrightarrow \mathit{Stat9}(\mathit{Stat1}, \mathit{Stat1}\star) \Rightarrow$ $e_0 \neq \{x_0, y_0\}$
$(\mathit{Stat9}\star)$Discharge $\Rightarrow$ AUTO
Suppose $\Rightarrow$ $\mathit{Stat4}:\ \bigcup\{\{x_0, y_0\}, \{y_0, z_0\}, \{z_0, x_0\}\} \neq s$
$(\mathit{Stat1}, \mathit{Stat1}\star)$ELEM $\Rightarrow$ $s = \{x_0, y_0, z_0\}\ \&\ \{\{x_0, y_0\}, \{y_0, z_0\}, \{z_0, x_0\}\} = \{\{x_0, y_0\}, \{y_0, z_0\}\} \cup \{\{z_0, x_0\}\}\ \&\ \{x_0, y_0, z_0\} \cup \{z_0, x_0\} = \{x_0, y_0, z_0\}$
$\Big\langle \{\{x_0, y_0\}, \{y_0, z_0\}\}, \{\{z_0, x_0\}\}\Big\rangle \hookrightarrow T2c\ (\mathit{Stat5}\star) \Rightarrow$ $\mathit{Stat5}:\ \bigcup(\{\{x_0, y_0\}, \{y_0, z_0\}\} \cup \{\{z_0, x_0\}\}) = \bigcup\{\{x_0, y_0\}, \{y_0, z_0\}\} \cup \bigcup\{\{z_0, x_0\}\}$
$\Big\langle \{\{x_0, y_0\}, \{y_0, z_0\}\}, \{x_0, y_0\}, \{y_0, z_0\}\Big\rangle \hookrightarrow T2a\ (\mathit{Stat6}\star) \Rightarrow$ $\mathit{Stat6}:\ \bigcup\{\{x_0, y_0\}, \{y_0, z_0\}\} = \{x_0, y_0, z_0\}$
$\Big\langle \{\{z_0, x_0\}, \{z_0, x_0\}\}, \{z_0, x_0\}, \{z_0, x_0\}\Big\rangle \hookrightarrow T2a\ (\mathit{Stat7}\star) \Rightarrow$ $\mathit{Stat7}:\ \bigcup\{\{z_0, x_0\}, \{z_0, x_0\}\} = \{z_0, x_0\}\ \&\ \{\{z_0, x_0\}, \{z_0, x_0\}\} = \{\{z_0, x_0\}\}$
EQUAL$(\mathit{Stat4}) \Rightarrow$ false; Discharge $\Rightarrow$ $\mathit{Stat10}:\ \neg\langle \forall z \in s \setminus \bigcup s, \exists y \in z \mid \{z, y\} \in \{\{x_0, y_0\}, \{y_0, z_0\}, \{z_0, x_0\}\}\rangle$

$\langle z'\rangle \hookrightarrow \mathit{Stat10}(\mathit{Stat10}\star) \Rightarrow$ $\mathit{Stat11}:\ \neg\langle \exists y \in z' \mid \{z', y\} \in \{\{x_0, y_0\}, \{y_0, z_0\}, \{z_0, x_0\}\}\rangle\ \&\ z' \in s \setminus \bigcup s$
$\langle y_0\rangle \hookrightarrow \mathit{Stat11}(\mathit{Stat11}\star) \Rightarrow$ $\mathit{Stat12}:\ y_0 \notin z' \vee \{z', y_0\} \notin \{\{x_0, y_0\}, \{y_0, z_0\}, \{z_0, x_0\}\}$
Use_def$(\bigcup) \Rightarrow$ $\mathit{Stat13}:\ z' \notin \{u : v \in s, u \in v\}\ \&\ z' \in s$
$\langle z_0, y_0\rangle \hookrightarrow \mathit{Stat13}(\mathit{Stat12}, \mathit{Stat1}\star) \Rightarrow$ $\mathit{Stat14}:\ z' = x_0$
$\langle y_0, x_0\rangle \hookrightarrow \mathit{Stat13}(\mathit{Stat14}, \mathit{Stat13}, \mathit{Stat12}, \mathit{Stat1}\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

THEOREM hamiltonian$_4$: [Potential revealers of non-Hamiltonicity]
$\mathsf{Trans}(S)\ \&\ S \not\subseteq \{\emptyset, \{\emptyset\}\}\ \&\ \neg\langle \exists h \mid \mathsf{SqHamiltonian}(h, S)\rangle \rightarrow$
$S \neq \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}\ \&\ S \neq \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}\ \&\ S \neq \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}\ \&\ S \supseteq \{\emptyset, \{\emptyset\}\}\ \&\ (\{\{\emptyset\}\} \in S \vee \{\emptyset, \{\emptyset\}\} \in S)$. PROOF:
Suppose_not$(t) \Rightarrow$ AUTO

$\langle t, \{\emptyset, \{\emptyset\}\}\rangle \hookrightarrow T4b \Rightarrow$ $\mathit{Stat1}:\ \neg\langle \exists h \mid \mathsf{SqHamiltonian}(h, t)\rangle\ \&\ (t \supseteq \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\} \vee t \supseteq \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\})$

$\langle t, \emptyset, \{\emptyset\}, \{\{\emptyset\}\} \rangle \hookrightarrow Thamiltonian_3 \Rightarrow$ AUTO
$\langle \{\{\emptyset, \{\emptyset\}\}, \{\{\emptyset\}, \{\{\emptyset\}\}\}, \{\{\{\emptyset\}\}, \emptyset\}\} \rangle \hookrightarrow Stat1(\star) \Rightarrow$ $t \neq \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$
$\langle t, \emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\} \rangle \hookrightarrow Thamiltonian_3 \Rightarrow$ AUTO
$\langle \{\{\emptyset, \{\emptyset\}\}, \{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \{\{\emptyset, \{\emptyset\}\}, \emptyset\}\} \rangle \hookrightarrow Stat1(\star) \Rightarrow$ $Stat2: \ t = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$

$\langle \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}, \emptyset, \{\emptyset\}, \{\{\emptyset\}\} \rangle \hookrightarrow Thamiltonian_3(Stat2\star) \Rightarrow$ $\mathsf{SqHamiltonian}(\{\{\emptyset, \{\emptyset\}\}, \{\{\emptyset\}, \{\{\emptyset\}\}\}, \{\{\{\emptyset\}\}, \emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\})$
$\langle \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}, \emptyset, \{\emptyset\}, \{\{\emptyset\}\} \rangle \hookrightarrow Thamiltonian_3 \Rightarrow$ $\mathsf{SqHamiltonian}(\{\{\emptyset, \{\emptyset\}\}, \{\{\emptyset\}, \{\{\emptyset\}\}\}, \{\{\{\emptyset\}\}, \emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\})$
$\langle \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}, \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset\}, \{\{\emptyset, \{\emptyset\}\}, \{\{\emptyset\}, \{\{\emptyset\}\}\}, \{\{\{\emptyset\}\}, \emptyset\}\}, \emptyset \rangle \hookrightarrow Thamiltonian_1(Stat2\star) \Rightarrow$
$\mathsf{SqHamiltonian}(\{\{\emptyset, \{\emptyset\}\}, \{\{\emptyset\}, \{\{\emptyset\}\}\}, \{\{\{\emptyset\}\}, \emptyset\}\} \setminus \{\{\emptyset, \{\emptyset\}\}\} \cup \{\{\emptyset, \{\emptyset, \{\emptyset\}\}\}, \{\{\emptyset, \{\emptyset\}\}, \{\emptyset\}\}\}, \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\})$
EQUAL$(Stat2) \Rightarrow$ $\mathsf{SqHamiltonian}(\{\{\emptyset, \{\emptyset\}\}, \{\{\emptyset\}, \{\{\emptyset\}\}\}, \{\{\{\emptyset\}\}, \emptyset\}\} \setminus \{\{\emptyset, \{\emptyset\}\}\} \cup \{\{\emptyset, \{\emptyset, \{\emptyset\}\}\}, \{\{\emptyset, \{\emptyset\}\}, \{\emptyset\}\}\}, t)$
$\langle \{\{\emptyset, \{\emptyset\}\}, \{\{\emptyset\}, \{\{\emptyset\}\}\}, \{\{\{\emptyset\}\}, \emptyset\}\} \setminus \{\{\emptyset, \{\emptyset\}\}\} \cup \{\{\emptyset, \{\emptyset, \{\emptyset\}\}\}, \{\{\emptyset, \{\emptyset\}\}, \{\emptyset\}\}\} \rangle \hookrightarrow Stat1(Stat2\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

7 Hamiltonicity of squared claw-free sets

THEOREM clawFreeness$_1$: [Hamiltonicity of non-trivial, claw-free sets]
$\mathsf{Finite}(S) \ \& \ \mathsf{Trans}(S) \ \& \ \mathsf{ClawFree}(S) \ \& \ S \not\subseteq \{\emptyset, \{\emptyset\}\} \rightarrow \langle \exists h \mid \mathsf{SqHamiltonian}(h, S) \rangle$. PROOF:
Suppose_not$(s_1) \Rightarrow$ AUTO

APPLY $\langle \mathsf{fin}_\Theta : s_0 \rangle$ finiteInduction$\big(s_0 \mapsto s_1, P(S) \mapsto (\mathsf{Trans}(S) \ \& \ \mathsf{ClawFree}(S) \ \& \ S \not\subseteq \{\emptyset, \{\emptyset\}\} \ \& \ \neg\langle \exists h \mid \mathsf{SqHamiltonian}(h, S) \rangle)\big) \Rightarrow$
$Stat1: \ \langle \forall s \mid s \subseteq s_0 \rightarrow \mathsf{Finite}(s) \ \& \ (\mathsf{Trans}(s) \ \& \ \mathsf{ClawFree}(s) \ \& \ s \not\subseteq \{\emptyset, \{\emptyset\}\} \ \& \ \neg\langle \exists h \mid \mathsf{SqHamiltonian}(h, s) \rangle \leftrightarrow s = s_0) \rangle$
$\langle s_0 \rangle \hookrightarrow Stat1(Stat1\star) \Rightarrow$ $Stat2: \ \neg\langle \exists h \mid \mathsf{SqHamiltonian}(h, s_0) \rangle \ \& \ \mathsf{Finite}(s_0) \ \& \ \mathsf{Trans}(s_0) \ \& \ \mathsf{ClawFree}(s_0) \ \& \ s_0 \not\subseteq \{\emptyset, \{\emptyset\}\}$

APPLY $\langle x_\Theta : x, y_\Theta : y, z_\Theta : z, t_\Theta : t \rangle$ pivotsForClawFreeness$(s_0 \mapsto s_0) \Rightarrow$
$Stat3: \ \{v \in s_0 \mid y \in v\} = \{x, z\} \ \& \ z \in s_0 \ \& \ y \in z \ \& \ y \in x \ \& \ y, x \in s_0 \ \& \ y \notin \bigcup\bigcup s_0 \ \& \ t = \{u \in s_0 \mid y \notin u\} \ \&$
$\mathsf{Trans}(t) \ \& \ \mathsf{ClawFree}(t) \ \& \ x \notin t \ \& \ y \in t \setminus \bigcup t \ \& \ t = s_0 \setminus \{x, z\} \ \& \ x \notin z \ \& \ z \notin x$

Suppose $\Rightarrow$ $t \subseteq \{\emptyset, \{\emptyset\}\}$

$\langle s_0, x, z \rangle \hookrightarrow T3d(Stat2\star) \Rightarrow$ $Stat7: \ s_0 \subseteq \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$
$\langle s_0 \rangle \hookrightarrow Thamiltonian_4(Stat2, Stat7\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO
$\langle t \rangle \hookrightarrow Stat1(Stat3\star) \Rightarrow$ $Stat9: \ \langle \exists h \mid \mathsf{SqHamiltonian}(h, t) \rangle \ \& \ t \not\subseteq \{\emptyset, \{\emptyset\}\}$
$\langle h_0 \rangle \hookrightarrow Stat9(Stat9\star) \Rightarrow$ $Stat10: \ \mathsf{SqHamiltonian}(h_0, t)$
Use_def$\big(\mathsf{Hamiltonian}(h_0, t, \mathsf{sqEdges}(t))\big) \Rightarrow$ AUTO
Use_def$(\mathsf{SqHamiltonian}) \Rightarrow$ $Stat11: \ \langle \forall x \in t \setminus \bigcup t, \exists y \in x \mid \{x, y\} \in h_0 \rangle \ \& \ \mathsf{Cycle}(h_0) \ \& \ \bigcup h_0 = t \ \& \ h_0 \subseteq \mathsf{sqEdges}(t)$

$\langle y, w \rangle \hookrightarrow Stat11(Stat3\star) \Rightarrow$ $Stat12: \ w \in y \ \& \ \{w, y\} \in h_0$
Suppose $\Rightarrow$ $x = z$
$\langle s_0, t, x, y, h_0, w, \emptyset \rangle \hookrightarrow Thamiltonian_1(Stat3\star) \Rightarrow$ $\mathsf{SqHamiltonian}(h_0 \setminus \{\{w, y\}\} \cup \{\{w, x\}, \{x, y\}\}, s_0)$

$\langle h_0 \setminus \{\{w,y\}\} \cup \{\{w,x\},\{x,y\}\}\rangle \hookrightarrow Stat2(Stat2\star) \Rightarrow$ false; Discharge $\Rightarrow$ $Stat13: x \neq z$
$\langle s_0, y\rangle \hookrightarrow T3c(Stat2\star) \Rightarrow$ $Stat14: w \in s_0$
$\langle s_0, y, x, z, w\rangle \hookrightarrow T\mathsf{clawFreeness_b}(Stat2, Stat3, Stat12, Stat13, Stat14\star) \Rightarrow$ $w \in x \cup z$
Suppose $\Rightarrow$ $Stat15: w \in x$
$\langle s_0, t, x, z, y, h_0, w\rangle \hookrightarrow T\mathsf{hamiltonian_2}(Stat3, Stat13, Stat10, Stat12, Stat15\star) \Rightarrow$ $\mathsf{SqHamiltonian}(h_0 \setminus \{\{w,y\}\} \cup \{\{w,x\},\{x,z\},\{z,y\}\}, s_0)$
$\langle h_0 \setminus \{\{w,y\}\} \cup \{\{w,x\},\{x,z\},\{z,y\}\}\rangle \hookrightarrow Stat2(Stat2\star) \Rightarrow$ false; Discharge $\Rightarrow$ $Stat16: w \in z$
$\langle s_0, t, z, x, y, h_0, w\rangle \hookrightarrow T\mathsf{hamiltonian_2}(Stat3, Stat13, Stat10, Stat12, Stat16\star) \Rightarrow$ $\mathsf{SqHamiltonian}(h_0 \setminus \{\{w,y\}\} \cup \{\{w,z\},\{z,x\},\{x,y\}\}, s_0)$
$\langle h_0 \setminus \{\{w,y\}\} \cup \{\{w,z\},\{z,x\},\{x,y\}\}\rangle \hookrightarrow Stat2(Stat2\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

8 Matchings

DEF matching: [Set of disjoint membership pairs] $\mathsf{Matching}(M) \leftrightarrow_{Def} \langle \forall p \in M, \exists x \in p, y \in x, \forall q \in M \mid x \in q \vee y \in q \rightarrow \{x,y\} = q\rangle$

THEOREM $\mathsf{matching_0}$: [The null set is a matching] $\mathsf{Matching}(\emptyset)$. PROOF:
Suppose_not() $\Rightarrow$ AUTO
Use_def(Matching) $\Rightarrow$ $Stat0: \neg\langle \forall p \in \emptyset, \exists x \in p, y \in x, \forall q \in \emptyset \mid x \in q \vee y \in q \rightarrow \{x,y\} = q\rangle$
$\langle p_1\rangle \hookrightarrow Stat0 \Rightarrow$ false; Discharge $\Rightarrow$ QED

THEOREM $\mathsf{matching_1}$: [Matchings consist of doubletons proper] $\mathsf{Matching}(M)\ \&\ P \in M \rightarrow P \notin \{\emptyset, \{X\}\}$. PROOF:
Suppose_not$(m, p_0, x_0) \Rightarrow$ AUTO
Use_def(Matching) $\Rightarrow$ $Stat1: \langle \forall p \in m, \exists x \in p, y \in x, \forall q \in m \mid x \in q \vee y \in q \rightarrow \{x,y\} = q\rangle$
$\langle p_0, x, y, p_0\rangle \hookrightarrow Stat1(\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

THEOREM $\mathsf{matching_2}$: [All subsets of a matching are matchings] $\mathsf{Matching}(M)\ \&\ M \supseteq N \rightarrow \mathsf{Matching}(N)$. PROOF:
Suppose_not$(m, n) \Rightarrow$ AUTO
Set_monot $\Rightarrow$
$\langle \forall p \in m, \exists x \in p, y \in x, \forall q \in m \mid x \in q \vee y \in q \rightarrow \{x,y\} = q\rangle \rightarrow$
$\langle \forall p \in n, \exists x \in p, y \in x, \forall q \in n \mid x \in q \vee y \in q \rightarrow \{x,y\} = q\rangle$
Use_def(Matching) $\Rightarrow$ false; Discharge $\Rightarrow$ QED

THEOREM $\mathsf{matching_3}$: [Bottom-up assembly of a finite matching] $\mathsf{Matching}(M)\ \&\ X \notin \bigcup M\ \&\ Y \notin \bigcup M\ \&\ Y \in X \rightarrow \mathsf{Matching}(M \cup \{\{X,Y\}\})$. PROOF:
Suppose_not$(m, x_0, y_0) \Rightarrow$ $Stat2: \mathsf{Matching}(m)\ \&\ x_0 \notin \bigcup m\ \&\ y_0 \notin \bigcup m\ \&\ y_0 \in x_0\ \&\ \neg\mathsf{Matching}(m \cup \{\{x_0,y_0\}\})$
Suppose $\Rightarrow$ $Stat3: \neg\langle \forall q \in m \mid x_0 \notin q\ \&\ y_0 \notin q\rangle$
Use_def$(\bigcup) \Rightarrow$ $Stat4: x_0 \notin \{v : u \in m, v \in u\}\ \&\ y_0 \notin \{v : u \in m, v \in u\}$
$\langle q_2\rangle \hookrightarrow Stat3(Stat3\star) \Rightarrow$ $q_2 \in m\ \&\ x_0 \in q_2 \vee y_0 \in q_2$
$\langle q_2, x_0, q_2, y_0\rangle \hookrightarrow Stat4(Stat4\star) \Rightarrow$ false; Discharge $\Rightarrow$ $Stat5: \langle \forall q \in m \mid x_0 \notin q\ \&\ y_0 \notin q\rangle$
Use_def(Matching) $\Rightarrow$ $Stat6: \neg\langle \forall p \in m \cup \{\{x_0,y_0\}\}, \exists x \in p, y \in x, \forall q \in m \cup \{\{x_0,y_0\}\} \mid x \in q \vee y \in q \rightarrow \{x,y\} = q\rangle\ \&$
$Stat7: \langle \forall p \in m, \exists x \in p, y \in x, \forall q \in m \mid x \in q \vee y \in q \rightarrow \{x,y\} = q\rangle$

$\langle p_0\rangle \hookrightarrow Stat6(Stat6\star) \Rightarrow$ $Stat8:\ \neg\langle \exists x \in p_0, y \in x, \forall q \in m \cup \{\{x_0, y_0\}\} \mid x \in q \vee y \in q \rightarrow \{x, y\} = q\rangle \ \&\ p_0 \in m \cup \{\{x_0, y_0\}\}$
Suppose $\Rightarrow$ $Stat9:\ p_0 = \{x_0, y_0\}$
$\langle x_0, y_0\rangle \hookrightarrow Stat8(Stat2, Stat9\star) \Rightarrow$ $Stat10:\ \neg\langle \forall q \in m \cup \{\{x_0, y_0\}\} \mid x_0 \in q \vee y_0 \in q \rightarrow \{x_0, y_0\} = q\rangle$
$\langle q_1\rangle \hookrightarrow Stat9(Stat9\star) \Rightarrow$ $q_1 \in m \ \&\ x_0 \in q_1 \vee y_0 \in q_1$
$\langle q_1\rangle \hookrightarrow Stat5(Stat10\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO
$\langle p_0, x_1, y_1\rangle \hookrightarrow Stat7 \Rightarrow$ AUTO
$\langle x_1, y_1\rangle \hookrightarrow Stat8(Stat8\star) \Rightarrow$ $Stat13:\ \neg\langle \forall q \in m \cup \{\{x_0, y_0\}\} \mid x_1 \in q \vee y_1 \in q \rightarrow \{x_1, y_1\} = q\rangle \ \&$
$Stat12:\ \langle \forall q \in m \mid x_1 \in q \vee y_1 \in q \rightarrow \{x_1, y_1\} = q\rangle \ \&\ p_0 \in m \ \&\ x_1 \in p_0 \ \&\ y_1 \in x_1$
$\langle q_0, q_0\rangle \hookrightarrow Stat13(Stat13\star) \Rightarrow$ $Stat14:\ q_0 = \{x_0, y_0\} \ \&\ x_1 \in q_0 \vee y_1 \in q_0$
$\langle p_0\rangle \hookrightarrow Stat5(\star) \Rightarrow$ $x_0 \notin p_0 \ \&\ y_0 \notin p_0$
$\langle p_0\rangle \hookrightarrow Stat12(Stat13, Stat13\star) \Rightarrow$ $\{x_1, y_1\} = p_0$
$(Stat14\star)$Discharge $\Rightarrow$ QED

THEOREM matching$_4$: [Deviated matching] $\text{Matching}(M) \ \&\ \{Y, W\} \in M \ \&\ X \notin \bigcup M \ \&\ Z \notin \bigcup M \ \&\ Y \in Z \ \&\ Y \neq X \ \&\ X \neq Z \ \&\ W \in X \rightarrow$
$\text{Matching}(M \setminus \{\{Y, W\}\} \cup \{\{Y, Z\}, \{X, W\}\})$. PROOF:
Suppose_not$(m, y_0, w_0, x_0, z_0) \Rightarrow$ AUTO

Suppose $\Rightarrow$ $Stat1:\ \{y_0, w_0\} \cap \bigcup(m \setminus \{\{y_0, w_0\}\}) \neq \emptyset$
Use_def(Matching) $\Rightarrow$ $Stat2:\ \langle \forall p \in m, \exists x \in p, y \in x, \forall q \in m \mid x \in q \vee y \in q \rightarrow \{x, y\} = q\rangle$
Use_def$(\bigcup(m \setminus \{\{y_0, w_0\}\})) \Rightarrow$ AUTO
$\langle w_1\rangle \hookrightarrow Stat1 \Rightarrow$ $Stat3:\ w_1 \in \{u :\ v \in m \setminus \{\{y_0, w_0\}\}, u \in v\} \ \&\ w_1 \in \{y_0, w_0\}$
$\langle p_0, w_2\rangle \hookrightarrow Stat3 \Rightarrow$ $p_0 \in m \setminus \{\{y_0, w_0\}\} \ \&\ w_1 \in p_0$
$\langle p_0, x_2, y_2\rangle \hookrightarrow Stat2 \Rightarrow$ $Stat4:\ \langle \forall q \in m \mid x_2 \in q \vee y_2 \in q \rightarrow \{x_2, y_2\} = q\rangle \ \&\ x_2 \in p_0 \ \&\ y_2 \in x_2$
$\langle p_0\rangle \hookrightarrow Stat4 \Rightarrow$ $p_0 = \{x_2, y_2\}$
$\langle \{y_0, w_0\}\rangle \hookrightarrow Stat4 \Rightarrow$ false; Discharge $\Rightarrow$ AUTO
$\langle m, m \setminus \{\{y_0, w_0\}\}\rangle \hookrightarrow T\text{matching}_2 \Rightarrow$ $\text{Matching}(m \setminus \{\{y_0, w_0\}\})$

$\langle m \setminus \{\{y_0, w_0\}\}, m\rangle \hookrightarrow T2d \Rightarrow$ $x_0 \notin \bigcup(m \setminus \{\{y_0, w_0\}\})$
$\langle m \setminus \{\{y_0, w_0\}\}, x_0, w_0\rangle \hookrightarrow T\text{matching}_3 \Rightarrow$ $\text{Matching}(m \setminus \{\{y_0, w_0\}\} \cup \{\{x_0, w_0\}\})$

Suppose $\Rightarrow$ $x_0 = w_0 \vee z_0 = w_0$
Use_def$(\bigcup) \Rightarrow$ $Stat5:\ z_0 \notin \{u :\ v \in m, u \in v\} \ \&\ x_0 \notin \{u :\ v \in m, u \in v\}$
$\langle \{y_0, w_0\}, w_0, \{y_0, w_0\}, y_0\rangle \hookrightarrow Stat5 \Rightarrow$ false; Discharge $\Rightarrow$ AUTO
Suppose $\Rightarrow$ $z_0 \in \bigcup(m \setminus \{\{y_0, w_0\}\} \cup \{\{x_0, w_0\}\})$
$\langle m \setminus \{\{y_0, w_0\}\}, m\rangle \hookrightarrow T2d \Rightarrow$ $z_0 \notin \bigcup(m \setminus \{\{y_0, w_0\}\})$
$\langle m \setminus \{\{y_0, w_0\}\}, \{x_0, w_0\}\rangle \hookrightarrow T2e \Rightarrow$ false; Discharge $\Rightarrow$ AUTO

$\langle m, \{y_0, w_0\}, w_0\rangle \hookrightarrow T\text{matching}_1 \Rightarrow$ $y_0 \neq w_0$
$\langle m \setminus \{\{y_0, w_0\}\}, \{x_0, w_0\}\rangle \hookrightarrow T2e \Rightarrow$ $y_0 \notin \bigcup(m \setminus \{\{y_0, w_0\}\} \cup \{\{x_0, w_0\}\})$

$\langle m\setminus\{\{y_0,w_0\}\}\cup\{\{x_0,w_0\}\},z_0,y_0\rangle\hookrightarrow T\mathsf{matching}_3 \Rightarrow$ $\mathsf{Matching}(m\setminus\{\{y_0,w_0\}\}\cup\{\{x_0,w_0\}\}\cup\{\{y_0,z_0\}\})$ & $m\setminus\{\{y_0,w_0\}\}\cup\{\{x_0,w_0\}\}\cup\{\{y_0,z_0\}\} = m\setminus\{\{y_0,w_0\}\}\cup\{\{y_0,z_0\},\{x_0,w_0\}\}$

EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

9 Every claw-free set admits a near-perfect matching

THEOREM clawFreeness$_2$: [Claw-free sets admit near-perfect matchings] $\mathsf{Finite}(S)$ & $\mathsf{Trans}(S)$ & $\mathsf{ClawFree}(S) \to$ $\langle \exists m, y \mid \mathsf{Matching}(m)$ & $S\setminus\{y\} = \bigcup m\rangle$. PROOF:

Suppose_not$(s_1) \Rightarrow$ AUTO

APPLY $\langle \mathsf{fin}_\Theta : s_0\rangle$ finiteInduction$\Big(s_0 \mapsto s_1, P(S) \mapsto \big(\mathsf{Trans}(S)$ & $\mathsf{ClawFree}(S)$ & $\neg\langle\exists m, y \mid \mathsf{Matching}(m)$ & $S\setminus\{y\} = \bigcup m\rangle\big)\Big) \Rightarrow$

Stat1 : $\langle \forall s \mid s \subseteq s_0 \to \mathsf{Finite}(s)$ & $\big(\mathsf{Trans}(s)$ & $\mathsf{ClawFree}(s)$ & $\neg\langle\exists m, y \mid \mathsf{Matching}(m)$ & $s\setminus\{y\} = \bigcup m\rangle \leftrightarrow s = s_0\big)\rangle$

$\langle s_0\rangle\hookrightarrow$ *Stat1(Stat1⋆)* $\Rightarrow$ *Stat2* : $\neg\langle\exists m, y \mid \mathsf{Matching}(m)$ & $s_0\setminus\{y\} = \bigcup m\rangle$ & $\mathsf{Trans}(s_0)$ & $\mathsf{ClawFree}(s_0)$ & $\mathsf{Finite}(s_0)$

$\langle\emptyset\rangle\hookrightarrow T31d \Rightarrow$ AUTO; $\langle\,\rangle\hookrightarrow T\mathsf{matching}_0 \Rightarrow$ AUTO; $\langle\emptyset,\emptyset\rangle\hookrightarrow$ *Stat2* $\Rightarrow$ $s_0 \not\subseteq \{\emptyset\}$

APPLY $\langle x_\Theta : x, y_\Theta : y, z_\Theta : z, t_\Theta : t'\rangle$ pivotsForClawFreeness$(s_0 \mapsto s_0) \Rightarrow$

Stat3 : $\{v \in s_0 \mid y \in v\} = \{x, z\}$ & $z \in s_0$ & $y \in z$ & $y \in x$ & $y, x \in s_0$ & $y \notin \bigcup\bigcup s_0$ & $t' = \{z \in s_0 \mid y \notin z\}$ & $\mathsf{Trans}(t')$ & $\mathsf{ClawFree}(t')$ & $x \notin t'$ & $y \in t'\setminus\bigcup t'$ & $t' = s_0\setminus\{x, z\}$ & $x \notin z$ & $z \notin x$

Loc_def $\Rightarrow$ $t = \textbf{if } x = z \textbf{ then } t'\setminus\{y\} \textbf{ else } t' \textbf{ fi}$

$\langle t', t\rangle\hookrightarrow T4c \Rightarrow$ AUTO $\langle t', t\rangle\hookrightarrow T\mathsf{clawFreeness_a}$(*Stat3⋆*) $\Rightarrow$ *Stat4* : $\mathsf{Trans}(t)$ & $\mathsf{ClawFree}(t)$ & $x \notin t$

$\langle t\rangle\hookrightarrow$ *Stat1(Stat3⋆)* $\Rightarrow$ *Stat5* : $\langle\exists m, y \mid \mathsf{Matching}(m)$ & $t\setminus\{y\} = \bigcup m\rangle$

$\langle m_0, y_0\rangle\hookrightarrow$ *Stat5(Stat5)* $\Rightarrow$ *Stat6* : $\mathsf{Matching}(m_0)$ & $\bigcup m_0 = t\setminus\{y_0\}$

Use_def$(\bigcup(m_0)) \Rightarrow$ AUTO

Suppose $\Rightarrow$ *Stat7* : $y \notin \bigcup m_0$

$\langle m_0, x, y\rangle\hookrightarrow T\mathsf{matching}_3$(*Stat6, Stat4, Stat7, Stat3⋆*) $\Rightarrow$ $\mathsf{Matching}(m_0 \cup \{\{x, y\}\})$

$\langle m_0 \cup \{\{x, y\}\}, d_0\rangle\hookrightarrow$ *Stat2(Stat7)* $\Rightarrow$ *Stat8* : $\bigcup(m_0 \cup \{\{x, y\}\}) \neq s_0\setminus\{d_0\}$

Loc_def $\Rightarrow$ $v = \textbf{if } x = z \textbf{ then } y \textbf{ else } z \textbf{ fi}$

$\langle m_0, t, y_0, s_0, \{x\}, v, y\rangle\hookrightarrow T31h$(*Stat3⋆*) $\Rightarrow$ *Stat9* : $\langle\exists d \mid \bigcup(m_0 \cup \{\{x\} \cup \{y\}\}) = s_0\setminus\{d\}\rangle$

$\langle d_0\rangle\hookrightarrow$ *Stat9(Stat9)* $\Rightarrow$ $\bigcup(m_0 \cup \{\{x\} \cup \{y\}\}) = s_0\setminus\{d_0\}$ & $\{x\} \cup \{y\} = \{x, y\}$

EQUAL(*Stat8*) $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat11* : $y \in \{h : p \in m_0, h \in p\}$ & $y \in x \cap z$ & $x \neq z$ & $t = t'$ & $x \notin z$ & $z \notin x$ & $\bigcup m_0 = \{h : p \in m_0, h \in p\}$

Suppose $\Rightarrow$ *Stat12* : $\neg\langle\exists w \mid \{y, w\} \in m_0$ & $w \in y\rangle$

$\langle p_1, h_1\rangle\hookrightarrow$ *Stat11(Stat12⋆)* $\Rightarrow$ $p_1 \in m_0$ & $y \in p_1$

Use_def(Matching)(*Stat6, Stat6*) $\Rightarrow$ *Stat13* : $\langle\forall p \in m_0, \exists x \in p, y \in x, \forall q \in m_0 \mid x \in q \vee y \in q \to \{x, y\} = q\rangle$

$\langle p_1, x_2, w_2, p_1\rangle\hookrightarrow$ *Stat13(Stat12⋆)* $\Rightarrow$ $x_2 \in p_1$ & $w_2 \in x_2$ & $p_1 \in m_0$ & $\{x_2, w_2\} = p_1$

$\langle w_2 \rangle \hookrightarrow Stat12(Stat12\star) \Rightarrow \quad Stat14:\ y \in x_2\ \&\ \{y, x_2\} \in m_0$
Suppose $\Rightarrow \quad Stat15:\ x_2 \notin \{u:\ v \in m_0, u \in v\}$
$\langle\ \{y, x_2\}, x_2 \rangle \hookrightarrow Stat15(Stat14\star) \Rightarrow \quad$ false; Discharge $\Rightarrow \quad Stat16:\ x_2 \notin \{v \in s_0 \mid y \in v\}\ \&\ x_2 \in s_0\ \&\ \{v \in s_0 \mid y \in v\} = \{x, z\}\ \&\ x_2 \in \bigcup m_0$
$\langle x_2 \rangle \hookrightarrow Stat16(Stat14\star) \Rightarrow \quad$ false; Discharge $\Rightarrow \quad Stat17:\ \langle \exists w \mid\ \{y, w\} \in m_0\ \&\ w \in y \rangle$

$\langle w \rangle \hookrightarrow Stat17(Stat17\star) \Rightarrow \quad Stat18:\ w \in y\ \&\ \{y, w\} \in m_0$
$\langle s_0, y \rangle \hookrightarrow T3c(Stat2, Stat3, Stat18\star) \Rightarrow \quad Stat19:\ w, y, x, z \in s_0$
Loc_def $\Rightarrow \quad y_1 = \textbf{if}\ y_0 \in \{x, z\}\ \textbf{then}\ s_0\ \textbf{else}\ y_0\ \textbf{fi}$
$(Stat19\star)$ELEM $\Rightarrow \quad s_0 \setminus \{x, z, y_0\} \cup \{z, x\} = s_0 \setminus \{y_1\}$
$\langle s_0, y, x, z, w \rangle \hookrightarrow T$clawFreeness$_b(Stat2, Stat11, Stat18, Stat19\star) \Rightarrow \quad w \in x \cup z$

Suppose $\Rightarrow \quad Stat20:\ w \notin \{u:\ v \in m_0, u \in v\}$
$\langle\ \{y, w\}, w \rangle \hookrightarrow Stat20(Stat18, Stat18\star) \Rightarrow \quad$ false; Discharge $\Rightarrow$ AUTO
$(Stat6, Stat11, Stat4, Stat3\star)$ELEM $\Rightarrow \quad Stat21:\ x \notin \bigcup m_0\ \&\ z \notin \bigcup m_0\ \&\ \bigcup m_0 \cup \{z, x\} = s_0 \setminus \{x, z, y_0\} \cup \{z, x\}$
EQUAL$(Stat11) \Rightarrow \quad Stat22:\ \bigcup m_0 \cup \{z, x\} = s_0 \setminus \{y_1\}\ \&\ w \in \bigcup m_0$

Suppose $\Rightarrow \quad Stat23:\ w \in x$
$\langle m_0, y, w, x, z \rangle \hookrightarrow T$matching$_4(Stat6, Stat11, Stat18, Stat23, Stat21\star) \Rightarrow \quad Stat24:\ \mathsf{Matching}(m_0 \setminus \{\{y, w\}\} \cup \{\{y, z\}, \{x, w\}\})$
$\langle m_0, \{y, w\}, \{y, z\}, \{x, w\}, \{z, x\} \rangle \hookrightarrow T31f(Stat11, Stat22\star) \Rightarrow \quad \bigcup(m_0 \setminus \{\{y, w\}\} \cup \{\{y, z\}, \{x, w\}\}) = s_0 \setminus \{y_1\}$
$\langle m_0 \setminus \{\{y, w\}\} \cup \{\{y, z\}, \{x, w\}\}, y_1 \rangle \hookrightarrow Stat2(Stat24\star) \Rightarrow \quad$ false; Discharge $\Rightarrow \quad Stat25:\ w \in z$
$\langle m_0, y, w, z, x \rangle \hookrightarrow T$matching$_4(Stat6, Stat11, Stat18, Stat25, Stat21\star) \Rightarrow \quad Stat26:\ \mathsf{Matching}(m_0 \setminus \{\{y, w\}\} \cup \{\{y, x\}, \{z, w\}\})$
$\langle m_0, \{y, w\}, \{y, x\}, \{z, w\}, \{z, x\} \rangle \hookrightarrow T31f(Stat11, Stat22\star) \Rightarrow \quad \bigcup(m_0 \setminus \{\{y, w\}\} \cup \{\{y, x\}, \{z, w\}\}) = s_0 \setminus \{y_1\}$
$\langle m_0 \setminus \{\{y, w\}\} \cup \{\{y, x\}, \{z, w\}\}, y_1 \rangle \hookrightarrow Stat2(Stat26\star) \Rightarrow \quad$ false; Discharge $\Rightarrow$ QED