

Numerical Solution of Plane Elasticity Problems with the Cell Method

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Abstract The aim of this paper is to present a methodology for solving the plane elasticity problem using the Cell Method. It is shown that with the use of a parabolic interpolation in a vectorial problem, a convergence rate of 3.5 is obtained. Such a convergence rate compares with, or is even better than, the one obtained with FEM with the same interpolation – depending on the integration technique used by the FEM application. The accuracy of the solution is also comparable or better.

keyword: Numerical method, Cell method, plane elasticity.

1 Introduction

The Cell Method (CM) has been recently introduced [Tonti (1993), Tonti (2001)]. It is currently being applied in several fields, such as thermal conduction, electromagnetism, mechanics of porous materials, and fracture mechanics [Tonti (to appear), Tonti (to appear-a), Cosmi and Di Marino (2000), Ferretti, Viola and Di Leo (2000), Marrone (2001), Nappi, Rajgelj and Zaccaria (2000)]. In all these cases, the CM results agree with those obtainable with other widely used numerical methods such as the Finite Element Method, Finite Difference Method and Finite Volume Method. Yet, the Cell Method concept is deeply different from that of the mentioned methods, and brings some advantages with it.

One of the major drawbacks of FEM and the other methods when dealing with elasticity problems is that “the description of a real elastic solid will be accurate only if the displacement field varies slowly over the size of the elements used...Fluctuations of elastic moduli from element to element should be small” [Roux (1990)]. This drawback directly derives from the use of a differential formulation of the physical laws involved: field equations in a differential formulation are subjected to restrictions imposed by derivability, restrictions that have nothing to do with the physical phenomenon investigated.

Actually, a real discrete formulation should not need to introduce relations among the differentials of the variables of the approximated field in order to write the balance equation.

In fact, CM uses constitutive equations at a local level – just like any other method – but writes the equilibrium equations directly for the discrete volume, not for a point, using global variables and not field variables, thus avoiding restrictions to guarantee their differentiability.

In this paper we will approximate the displacement field within the primal cell by an appropriate function. We will differentiate this function to express the strain components within each cell. We will then introduce the constitutive relation to write stress components and we will compute the forces acting through the dual complex sides. The point is that we are not going to use a differential apparatus to write the equilibrium equation: there is no need to write balance for a point – the node – introducing restrictions for differentiability, when equilibrium holds also for a whole region – a discrete region – namely the *influence region* of the node. In this sense we use a real discrete formulation: that the balance equation is expressed in finite terms.

As a consequence, CM is applicable whenever variables cannot be differentiated, for example when the displacement field undergoes large variations, i.e. when the size of the heterogeneities is the same scale of that of the discretization. In this sense, CM is deeply different from the other mentioned methods, in that CM uses global – integral – variables to derive directly a discrete formulation of the physical laws, it does not require neither energetic functionals nor their differentiation to find critical points.

Furthermore, using a parabolic interpolation in a vectorial problem such as plane elasticity, a convergence of order 3.5 is obtained with the Cell Method. This is greater than, or equal to, the one obtainable using FEM, with the same interpolation – depending on the integration technique used. As will be discussed, the Cell Method also improves accuracy of the solution with respect to ANSYS code.

2 The Cell Method grounds

We give here a description of the CM, for plane elasticity. Consider a plane elasticity problem, i.e. a body of constant thickness t loaded in the plane. First of all, we note that the

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geometry and kinematics of the body are described by the *configuration variables* – that is displacements, velocities, strain tensor and so on. No matter what the method employed for the discrete formulation, the points specifying the body configuration are called *nodes*. On the other hand, static and dynamic variables – such as forces, momenta, stress tensor and so on – describe the sources of the strain field and constitute the *source variables* for the problem. We shall recall that this classification is applicable to any field problem [Tonti (1972), Tonti (1972-a)]. For example, in a thermal conduction problem, temperatures and their gradients constitute the configuration variables, while heat fluxes and thermal sources are the source variables.

There is also a third class of variables, the *energy variables*, which result from the product of a configuration and a source variable. We are not going to use them in the following.

Given this classification, one of the fundamental aspects of the Cell Method is that dealing with two kinds of variables, two cell complexes will be needed in order to operate a correct discretization of the problem. Configuration variables and nodes will be associated with a primal complex of cells, while source variables will be linked to a dual, staggered, complex.

Using simplicial complexes – triangles – as primal cells, in a plane elasticity problem we may adopt:

- a triangular cell complex as primal complex
- a dual complex obtained connecting the barycentre of the primal cells and of their sides, as shown in Fig. 1.

Other choices for the dual complex are possible, as will be shown later.

Once the two cells complexes are established, we may think of the dual cell as an influence region for the inner node.

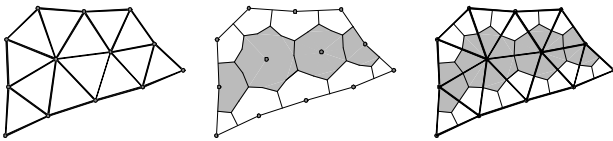


Figure 1 : Primal and dual complexes and their ensemble

It is then possible to write the equilibrium equations for each influence region, that is for each dual cell. We shall first examine the contributions of each portion of primal cell surrounding the node, and then collect the contributions from all the primal cells surrounding the node. In this way equilibrium relations are established over the entire influence region of each node: equilibrium equations are derived directly in a discrete form, using only global variables.

3 Plane elasticity with linear interpolation

Strain components are given by the symmetric part of displacement gradient and may be expressed as:

$$\{\mathbf{e}\}_c = [B]_c \{u\}_c \quad (1)$$

where $\{u\}_c$ collects the displacement components u_i and v_i . As in FEM,

$$\mathbf{e}_x = -\frac{1}{2A_c t} \sum_{i=1}^3 A_{ix} u_i, \quad \mathbf{e}_y = -\frac{1}{2A_c t} \sum_{i=1}^3 A_{iy} v_i,$$

$$\mathbf{g}_{yx} = -\frac{1}{2A_c t} \sum_{i=1}^3 (A_{ix} v_i + A_{iy} u_i),$$

where t is the thickness of the sample, A_c the area of the cell, and the meaning of A_{ij} is shown in Fig. 2.

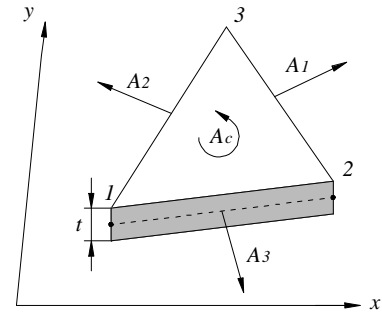


Figure 2 : Geometrical quantities

The constitutive equation may be written as usual in FEM:

$$\{\mathbf{s}\}_c = [D]_c \{\mathbf{e}\}_c = [D]_c [B]_c \{u\}_c \quad (2)$$

where $\{\mathbf{s}\}_c$ collects the stress components, and matrix $[D]_c$ represents Hooke law for the isotropic homogeneous material of the primal cell.

In order to write equilibrium equations it is necessary to express the forces acting through each side of the dual polyhedron surrounding the considered node. As stress components are uniform within each cell, the surface force \mathbf{T}_I (Fig. 3) will be given by

$$\begin{Bmatrix} T_{1x} \\ T_{1y} \end{Bmatrix}_c = \frac{1}{2} \begin{bmatrix} A_{1x} & 0 & A_{1y} \\ 0 & A_{1y} & A_{1x} \end{bmatrix}_c \begin{Bmatrix} \mathbf{s}_x \\ \mathbf{s}_y \\ \mathbf{t}_{xy} \end{Bmatrix}_c \quad (3)$$

and, for the three nodes of the cell,

$$\begin{Bmatrix} T_{1x} \\ T_{1y} \\ T_{2x} \\ T_{2y} \\ T_{3x} \\ T_{3y} \end{Bmatrix}_c = \frac{1}{2} \begin{bmatrix} A_{1x} & 0 & A_{1y} \\ 0 & A_{1y} & A_{1x} \\ A_{2x} & 0 & A_{2y} \\ 0 & A_{2y} & A_{2x} \\ A_{3x} & 0 & A_{3y} \\ 0 & A_{3y} & A_{3x} \end{bmatrix} \begin{Bmatrix} s_x \\ s_y \\ t_{xy} \end{Bmatrix}_c. \quad (4)$$

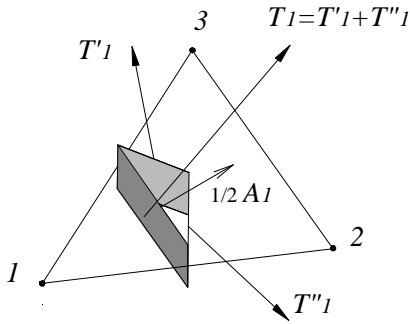


Figure 3 : Forces through the sides of the dual cell of node 1
Remembering (2), eq. (3) can be also written

$$\{T\}_c = -tA_c [B]_c^T [D]_c [B]_c \{u\}_c. \quad (5)$$

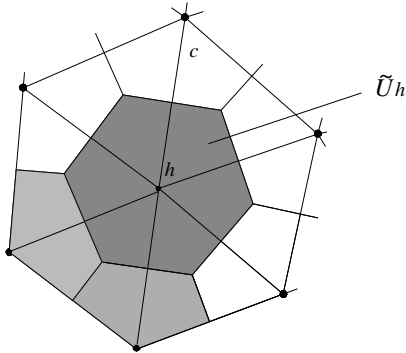


Figure 4 : Forces through the sides of the dual cell of node 1
It is now possible to write equilibrium condition for each dual cell. We make the following propositions (Fig. 4):

- U_h is the dual cell surrounding node h ;

- T_h^c is the resultant force that acts on the two sides of U_h belonging to cell c ;
- T_h is the total force acting on the boundary of U_h , due to all the cells that surround node h , $T_h = \sum_c T_h^c$,
- F_h^c is the volume force that acts on the part of cell c belonging to U_h ;
- F_h is the resultant volume force acting on U_h , $F_h = \sum_c F_h^c$.

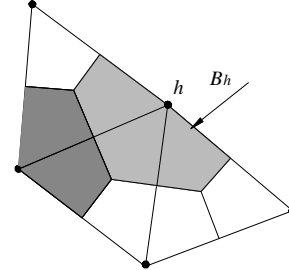


Figure 5 : Node h rests on the boundary of the sample

If cell c rests on the boundary of the region (Fig. 5),

- B_h^c is the external force that acts on the portion of cell c belonging to U_h ;
- B_h is the resultant external force acting on U_h through the boundary cells, $B_h = \sum_c B_h^c$.

Equilibrium can be then written for region U_h :

$$T_h + F_h + B_h = 0 \quad (5)$$

that is a set of $2n$ linear equations in the $2n$ unknowns u_i, v_i ($i=1, \dots, n$) which can be solved with the usual methods. It can be easily seen that with a linear interpolation of the displacement field, the stiffness matrix is the same as that of FEM, while the right-end side is different [Tonti (2001)].

4 Plane elasticity with quadratic interpolation

The Cell Method may bring to mind some aspects of the Finite Volumes Method and may also be regarded as similar to the so-called *direct* – or *physical* – approach that served as an introduction to Finite Elements Method and was used by Huebner (1975) to derive the equations governing the behavior of some simple elements. Yet, these two methods had an important limitation in the fact that the convergence order obtained has usually been limited to two.

Such a drawback does not hold for the Cell Method. In fact, the displacement field within the primal cell may be approximated by parabolic or higher order interpolation functions, which lead to higher convergence rates of the solution, as will be shown in the following.

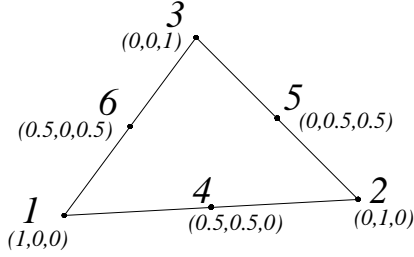


Figure 6 : Nodes and their triangular coordinates for quadratic interpolation.

For a quadratic interpolation function formulation we will use:

- a primal complex of six nodes triangular cells as shown in Fig. 6;
- a dual complex whose vertices are the two Gauss points of each side of the primal cell, as shown in Fig. 7.

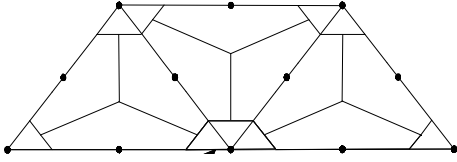


Figure 7 : Dual complex for quadratic interpolation.

Displacements of $P(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ within the primal cell can be approximated by functions of the nodal displacements and strain components within a primal cell may be expressed as:

$$\{\mathbf{e}\}_c = \frac{1}{2A_c t} [G]_c \{u\}_c \quad (6)$$

where $[G]_c$ is a 3×12 matrix.

The expression for $[G]_c$ is

$$[G]_c = \begin{bmatrix} g_{11x} & 0 & g_{22x} & 0 & g_{33x} & 0 & g_{12x} & 0 & g_{23x} & 0 & g_{31x} & 0 \\ 0 & g_{11y} & 0 & g_{22y} & 0 & g_{33y} & 0 & g_{12y} & 0 & g_{23y} & 0 & g_{31y} \\ g_{11y} & g_{11x} & g_{22y} & g_{22x} & g_{33y} & g_{33x} & g_{12y} & g_{12x} & g_{23y} & g_{23x} & g_{31y} & g_{31x} \end{bmatrix}$$

where

$$g_{ix} = \frac{A_{ix}}{t} (4b_i - 1), \quad g_{iy} = \frac{A_{iy}}{t} (4b_i - 1),$$

$$g_{ijx} = \frac{4}{t} (b_i A_{jx} + b_j A_{ix}), \quad g_{ijy} = \frac{4}{t} (b_i A_{jy} + b_j A_{iy}),$$

$(i, j=1,2,3)$.

With the constitutive matrix, we obtain

$$\{\mathbf{s}\}_c = [D]_c [G]_c \{u\}_c \quad (7)$$

that shows stress components in a point within the primal cell as a function of the nodal displacements.

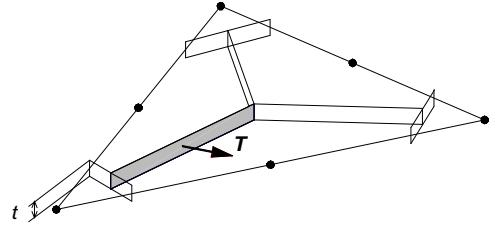


Figure 8 : Force on the dual region side.

In order to write equilibrium equations, once again we will find the expression of force T acting through the dual polyhedron sides.

Let us note that stress components are expressed by a linear function when parabolic interpolation of the displacement field is assumed. As a consequence, force T (Fig. 8) will be given by the product of the value of stress component computed in M (middle point of the dual region side) times the area of the side itself (Fig. 9).

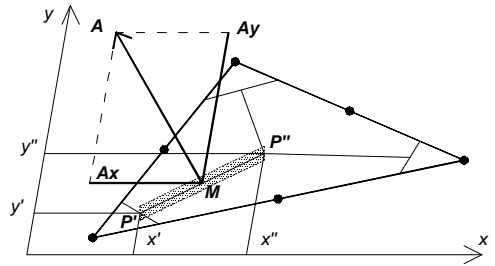


Figure 9 : Dual cell side area

If $P'(\mathbf{b}', \mathbf{b}_2', \mathbf{b}_3')$ and $P''(\mathbf{b}'', \mathbf{b}_2'', \mathbf{b}_3'')$ are the side ends, area components will be given by

$$A_x = -t(y''-y') = -t[y_1(\mathbf{b}_1''-\mathbf{b}_1') + y_2(\mathbf{b}_2''-\mathbf{b}_2') + y_3(\mathbf{b}_3''-\mathbf{b}_3')] \\ A_y = t(x''-x') = t[x_1(\mathbf{b}_1''-\mathbf{b}_1') + x_2(\mathbf{b}_2''-\mathbf{b}_2') + x_3(\mathbf{b}_3''-\mathbf{b}_3')]$$

and force \mathbf{T} will be

$$\begin{Bmatrix} T_x \\ T_y \end{Bmatrix} = \begin{bmatrix} A_x & 0 & A_y \\ 0 & A_y & A_x \end{bmatrix}_c \begin{Bmatrix} \mathbf{s}_x \\ \mathbf{s}_y \\ \mathbf{t}_{xy} \end{Bmatrix}_M \quad (8)$$

where the stress components are computed in M .
Substituting (7) we obtain

$$\begin{Bmatrix} T_x \\ T_y \end{Bmatrix} = \frac{1}{2A_c} [A]_c [D]_c [G]_c \{u\}_c \quad (9)$$

In order to compute $[A]_c$, the coordinates of the dual regions side ends, shown in Fig. 10, are given in Tab. 1.

Table 1 : Triangular coordinates of Gauss points -

$$g = \frac{(1-\sqrt{3})}{2}$$

	\mathbf{b}_1	\mathbf{b}_2	\mathbf{b}_3
$P1$	$1-g$	$g/2$	$g/2$
$P2$	$g/2$	$1-g$	$g/2$
$P3$	$g/2$	$g/2$	$1-g$
$P4$	$1-g$	0	g
$P5$	$1-g$	g	0
$P6$	g	$1-g$	0
$P7$	0	$1-g$	g
$P8$	g	0	$1-g$
$P9$	0	g	$1-g$
$P10$	$1/3$	$1/3$	$1/3$

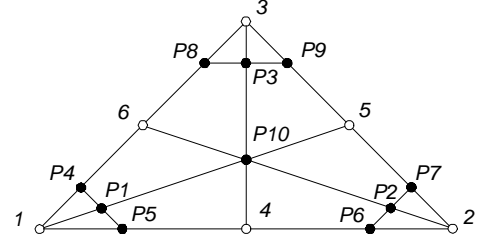


Figure 10 : Gauss points for quadratic interpolation.

Let us consider node 1 of cell c (see Fig. 11). The forces acting on the two sides, a and b , of the portion of the dual region of node 1 that belong to the cell are T_a and T_b .

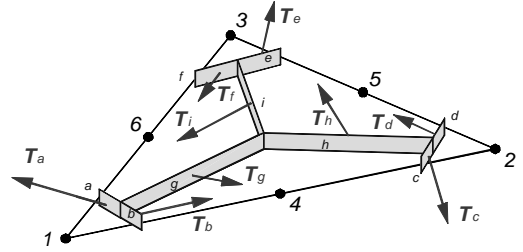


Figure 11 : Forces on the dual cells

In the same way we may express the surface forces for each of the six portions of dual regions surrounding each node of the cell:

$$\begin{aligned} T_1^c &= -T_a + T_b \\ T_2^c &= -T_c + T_d \\ T_3^c &= -T_e + T_f \\ T_4^c &= -T_b - T_g + T_h + T_c \\ T_5^c &= -T_d - T_h + T_i + T_e \\ T_6^c &= -T_f - T_i + T_g + T_a \end{aligned} \quad (10)$$

It is now possible to write equilibrium for each dual cell. With the same propositions as in the linear interpolation, equilibrium can be then written for region U_h (see Fig. 12):

$$T_h + F_h + B_h = 0 \quad (11)$$

which again is a set of $2n$ equations in the $2n$ unknown nodal displacement components and can be solved with the usual methods.

It should be noted that the stiffness matrix hereby obtained does not coincide with that of FEM for triangular elements with six nodes, as shown by Tonti (2001).

5 Result discussion.

In order to test convergence and accuracy of the proposed method, a problem has been considered for which the exact solution is known in literature.

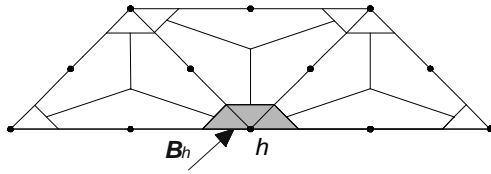


Figure 12: Influence region for a boundary node.

Consider the well-known problem of bending a cantilever loaded at the end (Fig. 13), as developed by Timoshenko and Goodier (1970).

The upper and lower edges carry no load, and shearing forces of resultant P are distributed along the end $x=0$. If I is the moment of inertia, the solution

$$s_x = \frac{Pxy}{I}, \quad s_y = 0, \quad t_{xy} = -\frac{P}{2I}(c^2 - y^2) \quad (12)$$

represents an exact solution if the shearing forces on the end are distributed with the same parabolic law as t_{xy} and intensity of normal forces at the built-in end is proportional to y .

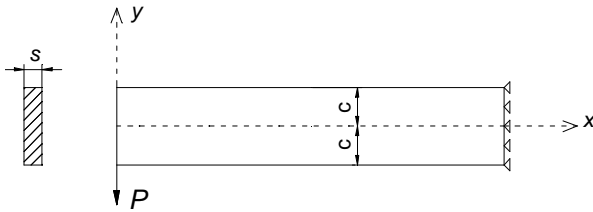


Figure 13 : Cantilever loaded at the end.

Under these conditions, displacement field is given by

$$u = \frac{Px^2y}{2EI} + \frac{nPy^3}{6EI} - \frac{Py^3}{6GI} - \left(\frac{Pl^2}{2EI} - \frac{Pc^2}{2GI} \right) y$$

$$v = -\frac{nPxy^2}{2EI} - \frac{Px^3}{6EI} + \frac{Pl^2x}{2EI} - \frac{Pl^3}{3EI}. \quad (13)$$

This solution has been imposed on the ends of a bar with $P = 100$ N/mm, $E = 100$ GPa, $n = 0.3$, $l = 100$ mm, $c = 10$ mm, $s = 1$ mm.

In order to compare approximate and exact values, the root-means-square values (rms) of the errors at nodes have been computed. Cell Method results have also been compared with those obtained with the same six nodes triangles mesh using two commercial FEM codes: ANSYS 5.5 and ABAQUS 5.8. We first considered a structured primal cell complex with two horizontal divisions and successively halved the discretization length as shown in Fig. 14.

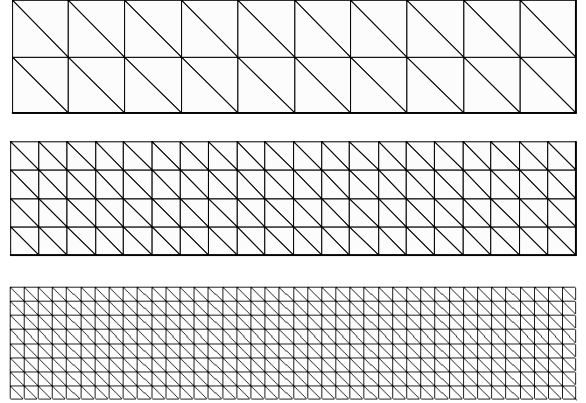


Fig.14 : Cell complexes.

Results are shown in Fig. 15 and Tab. 2. It can be seen that the convergence rate obtainable with both the Cell Method and ABAQUS is 3.5, while that obtained with ANSYS is 2.9. Only the two coarser meshes have been tested with ANSYS, while all three have been tested with both Cell Method and ABAQUS, obtaining coinciding results.

Table 2 : Results

rms ANSYS	rms ABAQUS	rms Cell Method
0,0000159700	0,0000160704	0,0000160791
0,0000021743	0,0000014759	0,0000014636
	0,0000001263	0,0000001272

It should be noted that the rate of convergence based solely on polynomial approximation theory is 3 for six nodes triangular finite elements, see Burnett (1988). It is also known that FEM results are computed more accurately at the integration – superconvergent – points. The location of these data is then moved to nodal locations by extrapolation in ANSYS (see Theory Reference Manual) and ABAQUS (see Theory Manual).

We may also note that accuracy with the Cell method tends to increase more than it does with ANSYS when the mesh size is decreased.

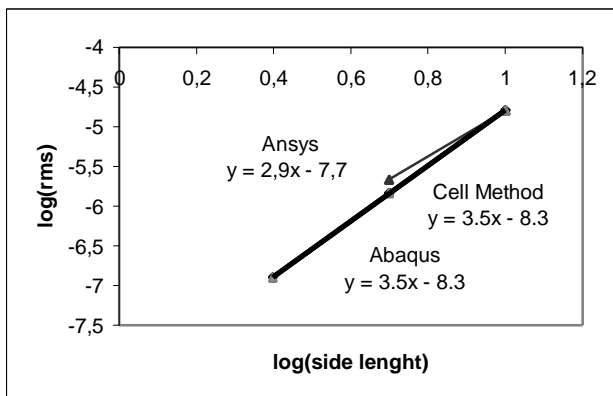


Fig. 16 : Results

6 Conclusions

Plane elasticity with the Cell Method has been presented, and it has been shown that using a parabolic interpolation in a vectorial problem, a convergence rate 3.5 is obtained, that compares with, or is greater than, the one obtained with FEM with the same interpolation, depending on the integration scheme employed.

In conclusion, the Cell Method has proved to be very promising, being able to overcome some of the restrictions of FEM while providing a comparable or better convergence and accuracy.

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