L&PS – Logic and Philosophy of Science
Vol. IX, No. 1, 2011, pp. 223-228

A note on symbols, syntax and sets

Luca Bellotti
Dipartimento di Filosofia
Università di Pisa
e-mail: lucabellotti@quipo.it

ABSTRACT. In this note I discuss some aspects of the role of symbols in mathematics. I concentrate on the mathematical treatment of mathematical symbols in the context of the syntax of formal systems, and I deal with the relationship of the concept of symbol with the notions of abstraction and set.

1. Introduction

Our subject in this note will be some philosophical problems arising when we consider the way in which *mathematical symbols* are subjected to a proper *mathematical treatment* in the usual syntax of formal systems. We shall not deal with the involvement of the syntax of formal systems with the concept of symbol *in general*. It is well-known that the philosophical richness of the concept of symbol, its wealth of meanings and aspects, and above all the tremendous difficulties surrounding its central function in the *subjective deduction* (in Kantian terms) of thought, make the apparently innocent question "what is a symbol?" one of the most fundamental and difficult problems in the whole philosophy of knowledge. We shall limit ourselves, here, to a necessarily very brief (and, for this reason, sometimes perhaps seemingly dogmatic) discussion of some aspects of the mathematical treatment of mathematical symbols.

2. Symbols and abstraction

Published by L&PS – Logic and Philosophy of Science
http://www2.units.it/ ~episteme — ISSN: 1826-1043

The first aspect is the manifold relationship of the concept of symbol with the notion of *abstraction*. It is very natural, and it usually passes unnoticed, to speak of the *representation* of symbols. This peculiar kind of representation can give rise to subtle (classic) problems in the philosophy of language, but we are not concerned with such problems here. In the representation of mathematical symbols one uses quotes, or (seldom) Quinean corners, or (almost always) autonymy. Syntactic *meta*variables are pervasively used in any logic textbook. Symbols can well represent themselves, even directly (in the case of their autonymous use). In fact, how could one hope to represent symbols *without symbols*? The point which concerns us here is the fact that in all cases of representation of symbols it is the symbol as an *abstract* object which is represented. The continuous, unobserved switch from symbols to representations of symbols and vice versa is in a sense unavoidable, because of the fundamental *ubiquity* of symbols between the world of the abstract and the world of the concrete. This is one of the most difficult puzzles surrounding the concept of symbol: symbols must be *concrete* if one wants them to have any role in the process of communication; but in another sense they must be *abstract*. Indeed, symbols can be the object of logical (and hence mathematical) study *only if* they are abstract objects. All our reflections are on *this* level, and all the remarks in this note make sense only if we consider the *abstract* side of the nature of symbols. The point is that symbols as abstract objects have very special features, the most important of which is the fact that they constitute the mathematical “essence” of symbols as *concrete* means of communication. They are abstract, but their realizations are just the ordinary symbols we use, and this gives them a singular position. For instance, a concrete object (an ink mark) can be a *concrete* symbol of an *abstract* symbol which is realized in its turn in many *concrete* instances of that symbol, each one symbolizing an *abstract* object.

A concrete symbol is an ink blot, or a sound, or a portion of a memory of a computer. The first problem regarding symbols as concrete objects is the question of their identification. But problems of ambiguity regard symbols *only* as concrete objects. There is no ambiguity at all as far as symbols as abstract objects are concerned. Similarly, although no use of symbols could be possible without certain gestaltic abilities of perceiving a definite *form*, this is a problem of cognitive psychology which should be kept distinct from the problems regarding the mathematical treatment of those peculiar abstract objects which are symbols.

But if symbols, as far as mathematics can have any interest in them, are abstract entities, we are in need of an *axiomatization*. In fact, an axiomatization is forced on us if we want to treat symbols mathematically, since axiomatization is *in general* the main way we have to isolate and constitute conceptually the relevant abstract features of the mathematical objects in which we are interested; otherwise, there could be no way to unify all the dispersed features of the various concrete realizations of those objects. Whatever one may think of this controversial thesis on the connection between abstraction and axiomatization, we assume here that it captures at least a decisive aspect of *mathematical* abstraction. One must take seriously the fact that mathematics is never the study of concrete objects. Even languages and their constituents, symbols, can be objects of mathematical investigation only if they are abstract. Not even the most elementary parts of recursion theory or proof theory study something concrete. They study *abstract* concepts which are the only way to understand *concrete* manipulations of concrete symbolic elements.

Let us suppose, as an example, that we want to express a property (e.g., transitivity) of a (binary) relation. The key point is that a relation *symbol*, by means of which no further hypotheses are assumed, has exactly *the same level of abstractness as the relation in general*. Just like the latter, it has no peculiar form of existence as an object, it does not subsist *an sich*, it is not some *thing* which exists separately as a substance. In a sense, "the relation in general" does not even *exist*, but not because of nominalistic reasons. This is not our point: on the contrary, we maintain that one can accept full generality and abstractness. Our point is that one should reinterpret this abstractness in the sense of a *transcendental* theory of the concept, in which the concept is a *rule* and a condition of constitution of the field of its objects, as opposed to the result of classic abstraction. *Any mathematical symbol is just the immediate expression of this "act" of unification and transcendental synthesis*: *this is the basis of the rather special "abstractness" of mathematical symbols*.

But then the function of mathematical symbols cannot be explained by taking them either simply as abstract entities (of a peculiar sort), or simply as concrete ink blots (let alone as a sort of abstract ink blots, as some formalist philosophies seem sometimes compelled to assume). We do not claim, of course, that this peculiar form of abstractness, in which abstractness *of the symbol* and abstractness *of the symbolized notion* essentially converge, is the usual way in which symbols perform their function in *any* process of abstraction; we do claim that this is an important aspect of their role as *mathematical* symbols.

More generally, we should remember that the abstract is not a sort of parody of the concrete. The abstract is not a peculiar genus of concrete objects which live in another (more or less parallel) world: mathematics is not at all the physics (let alone the zoology) of the invisible. This is perhaps the fundamental reason why any Platonism which "substantializes" abstract mathematical objects is, from the point of view adopted here, unacceptable. What by its own essence *holds*, can never be considered a *substance*. "The object does not obey here [in mathematics] other conditions beyond those of the very mathematical synthesis: it *is* and it *exists* as far as the mathematical synthesis *holds*. And on this holding no external or transcendent 'reality' of things decides, but only the immanent logic of the mathematical relations themselves" (Cassirer 1929, p. 462; our translation). It is true that the essential function of the abstract is to allow *the unification of the manifold*, by means of the recognition of *resemblances*. But the unified treatment of many concrete instances is not *based on* resemblance; on the contrary, the very possibility of singling out resemblances is a *result* of the fact that an abstract concept *holds* as a purely *functional* and not substantial unity.

3. Symbols and sets

For mathematics, at bottom, symbols are any multiplicity of items supplied with an operation which obeys the laws (expressed as axioms) of *concatenation*. This seems definitely too little; but, for *mathematics*, really *any* system of objects satisfying the natural constraints on an operation which allow to consider it a concatenation operation is a system of *symbols*. The realization of this fact leads us to the second aspect of the mathematical treatment of mathematical symbols which concerns us here: the relationship between the notion of symbol and the concept of *set*. We shall consider only some specific and peculiar aspects of *the role of sets for symbols* (not vice versa) here.

There is apparently a direct vicious circularity in the function of symbols in the formal expression of theories, a (philosophical) circularity which emerges as soon as one puts together three facts: the formal expression of a theory, the need of a theory axiomatizing the concept of symbol (in order to speak of formal theories in general), and *the possibly formal nature of the latter theory*. This possibility constitutes perhaps our main concern in this note. Of course, we do not ignore the fact that one could adopt a Hilbert-style view, leaving the theory of symbols at a (necessarily) informal level; but we want to see precisely what happens in the other option (also in view of the fact that the "informal" approaches have their own problems). The circularity which concerns us is very elementary and general: it regards the mere *formalization of syntax*, even in its simplest form (a pure theory of the concatenation of symbols) and for *any* formal theory, although the explicit involvement of set theory can make the features of the problematic “loop” much sharper. Let us recall, in this connection, the following remark of Benacerraf’s about Zermelo (Benacerraf 1985, p. 113): “formalization requires recursive definitions (of the syntactic concepts), and thus presupposes syntax, which itself presupposes the concept of ‘finite iteration’. Since in his view this concept was destined in turn to be reduced to set theory, he [Zermelo] felt that the presentation of set theory could not presuppose it”. While the stress is upon the relationship between finiteness and set theory in this passage, we now want to emphasize the *formal* nature of theories (both the object theory and the theory of symbols). But it is a difference in emphasis rather than in content.

However, in mathematics the vicious circle was in a sense *broken*. “Another way was chosen to express the generality of what is common to all the possible concrete realizations of an abstract notion: the description of what is common is performed in an intuitive language based on the neutral and empty notion of ‘set’” (Lolli 1991, p. 26; our translation). But how is it possible to obtain by means of *sets* a description of what is common to all the realizations of an abstract mathematical notion? The very possibility of this is the result of the apparent conceptual *poverty* of the notion of set. Just the extreme “neutrality” of the latter notion, which has induced some mathematicians to refuse set theory as a basis for the whole of mathematics (usually in favor of category theory), is one of its strongest points, as far as one wants a mathematization of some basic and general logical relations relevant to mathematics (here, the relation between the abstract and the concrete).

A simple example is perhaps sufficient to suggest how the description of “what is common” can be performed: in order to “describe” something that (at the very least) is common to all binary relations, “instead of saying that a relation is something some pairs have and some other pairs have not, one says that a relation is a set of pairs” (ibid.). This reminds one of the unrestricted comprehension principle; but the great achievement of axiomatic set theories is the ability to save up the function of that principle in such contexts as the one envisaged here, without (as far as one knows nowadays) lapsing into paradox.

This example illuminates one of the deep reasons why all mathematical “entities” are set-theoretically definable: if the *abstract* objects which constitute the subject matter of mathematics have their source in the *formal* nature of mathematics, it is not surprising that they can be always characterized as what is common to all the instances of a certain notion.

In any case, what we should keep in mind is that symbols, as the subject of a proper *mathematical* study (basically, in the syntax of formal systems), ultimately *are sets*, and the operation of concatenation ultimately is an operation by means of which certain *structures of sets* are built up. The straight opposition of this set-theoretic definition of symbols to any formalist or even Hilbert-style foundation of mathematics should be clear. Unless one has an independent, non-mathematical notion of symbol, there is no way to extract a formalist foundation of mathematics from the already *abstract* notion of concatenation as an abstract operation on sets. Of course, if one does not adopt an overall set-theoretic ontological reduction for mathematical entities, symbols might well be something else, e.g., numbers; but this makes no substantial difference with respect to their abstractness, which is the crucial feature in which we are interested here.

In sum, one could say that "(naïve) set theory is a sort of intermediate filter between the interpretations of any theory and its formal version; it is not incompatible with the formal, axiomatic conception of mathematical theories. If the framework theory of sets is not left at its naïve level, but it is in its turn formalized (and this is possible), the phenomenon of self-reference, as one can easily imagine, rises to pyrotechnic levels" (Lolli 1991, p. 27).

We only recall, in conclusion, that the formalizability of set theory is not a sheer possibility which can be ignored at will, but can be considered an essential consequence of the mathematization (axiomatization) of the otherwise possibly confused and paradoxical notion of set.[1]

REFERENCES

BENACERRAF, P. (1985): "Skolem and the Skeptic", *Proceedings of the Aristotelian Society, Supplementary Volume*, 59, pp. 85-115.

CASSIRER, E. (1929): *Philosophie der symbolischen Formen*, III: *Phänomenologie der Erkenntnis*, Berlin: Bruno Cassirer.

LOLLI, G. (1991): *Introduzione alla logica formale*, Bologna: Il Mulino.

[1] I wish to thank Gabriele Lolli (whose writings first inspired this note) and Enrico Moriconi.