Curry's Paradox. A new Argument for *Trivialism*

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ABSTRACT. In dialetheism some sentences, called *dialetheias*, are both true and false. A crucial problem of this logical theory is that of avoiding *trivialism*, i.e. the consequence that all sentences are dialetheias. Priest tries to avoid trivialism by rejecting some principles of classical logic. In the present paper we formulate a new version of *Curry's paradox* and argue that trivialism follows even from logical principles that are dialetheistically correct. To the purpose we will use a notion of *naïve deducibility*, defended by Priest in his discussion of Gödel's theorem.

1. Introduction

As is well known, dialetheism maintains the thesis that there are true contradictions, i.e. true sentences of form (A∧¬A), called *dialetheias*. More generally, we will call "dialetheia" any sentence that is both true and false. In a reach series of papers and books (see, for example, Priest 1979, 2001, 2002,

2006a, 2006), Priest claims that dialetheism supplies the best solution to the the *strengthen liar paradox*, a paradox originated from the sentence:

(a): (a) is not true

by holding that (a) is both true and not true. More generally, he holds that the paradoxical sentences obtained from self-reference are *dialetheiae*.

Priest's dialetheism has been extensively criticized in the literature (for an overview of criticism see Berto 2007, part IV). In this paper we will not discuss the crucial problem concerning the acceptance of a dialetheia. Rather, we will focus on the following claims by Priest:

(i) The presence of dialetheiae does not entail trivialism: a) a contradictory theory may not be exploding, i.e. it does not necessarily prove all sentences; b) in a model with a dialetheia not all sentences are necessarily true.
(ii) The meaning of logical constants is the same in the object language and in the metalanguage.

Of course, claim (i) is of a vital importance for dialetheism: if a dialetheia implied everything, dialetheic theories would be of no interest.

Claim (ii) should guarantee that dialetheiae are not produced by altering ad hoc the intended meaning of logical constants. Using Priest's words: "the distinction between a theory (…) and its metatheory makes perfectly good sense to a dialetheist. But there is no reason to insist that the metatheory must be stronger than, and therefore different from, the theory." "The same logic must be used in both 'object language' and 'metatheory' " (Priest 2006b, 98).

In classical logic, explosion is produced using the rule *ex contradictione quodlibet* (ECQ). The classical justification for ECQ rests on the alleged evidence that no contradiction can be true, which is obviously rejected by dialetheists. In standard natural deduction ECQ can be derived using *reductio ad absurdum* (RAA) and other apparently non-problematic rules. RAA is immediately rejected by a dialetheist. However, the banish of RAA is insufficient to avoid trivialism: Curry's paradox, from which trivialism follows, can be generated without using RAA, but just with MPP and the derived rule of *Absorption* (ABS: $(A \rightarrow (A \rightarrow B)) \mathrel{|\!\!-} (A \rightarrow B)$). In order to save dialetheism from trivialism, Priest adopts in LP the material conditional, for which he rejects the general validity of (MPP).

The crucial problem we intend to face in this paper is whether trivialism can follow even from logical principles that are dialetheistically correct.

2. Curry's paradox and its arithmetical formalization

Curry's paradox belongs to the family of so-called paradoxes of self-reference (or paradoxes of circularity). Shortly, the paradox is derived in natural language from sentences like (a):

(a) If sentence (a) is true, then Santa Claus exists.

Suppose that the antecedent of the conditional in (a) is true, i.e. that sentence (a) is true. Then, by MPP Santa Claus exists. In this way the consequent of (a) is proved under the assumption of its antecedent. In other words, we have proved (a). Finally, by MPP, Santa Claus exists.

Of course, we could substitute any arbitrary sentence for "Santa Claus exists": that means that every sentence can be proved and *trivialism* follows. In Priest (1979, IV.5) Priest observes that, in a semantically closed theory, using (MPP) and absorption, a version of Curry's paradox is derivable.
We reconstruct his argument in the language of first order arithmetic with a truth predicate.

Let L be the language of first order arithmetic and N be its standard model. Extend L to the language L^* by introducing a new predicate T. With reference to a codification of the syntax of L^* by natural numbers, extend N to a model N^* of L^* by interpreting T as the truth predicate of L^*, so that, for all $n \in N$, $T(n)$ is true *iff* n is the code of a true sentence A of L^*, in symbols $n = \ulcorner A \urcorner$.

Of course, classically, such an interpretation is impossible, because the theory obtained adding to Peano arithmetic the truth predicate for the extended language L^* with Tarski's biconditionals is inconsistent. Not so for a dialetheist, who loves inconsistent models! We will show that, if one uses the classical rules of conditional in natural deduction (from which ABS is derivable) and Tarski's scheme:

$$T(\ulcorner A \urcorner) \leftrightarrow A,$$

the model N* turns out to be trivial.

For, let A be any sentence of L^*. By diagonalisation, there is a natural number k such that

$$k = \ulcorner T(\underline{k}) \to A \urcorner.$$

We can derive A as follows:

1	(1)	$T(\underline{k}) \leftrightarrow \ulcorner T(\underline{k}) \rightarrow A \urcorner$	Tarski's schema
2	(2)	$T(\underline{k})$	Assumption
1, 2	(3)	$(T(\underline{k}) \rightarrow A)$	1, 2 MPP
1, 2	(4)	A	2, 3 MPP
1	(5)	$(T(\underline{k}) \rightarrow A)$	2, 4 → Introduction
1	(6)	$T(\underline{k})$	1, 5 MPP
1	(7)	A	5, 6 MPP

3. On the general validity of MPP

Priest blocks this derivation in *LP* by rejecting the general validity of MPP. According to him, this rule is not valid but *quasi-valid*, i.e. valid insofar as no dialetheia is involved.

Priest in (Priest 2006), in order to reject MPP, identifies, in the object language, $(A \rightarrow B)$ with $(\neg A \lor B)$. Then, the rejection proceeds as follows. Suppose that A is a dialetheia; $(\neg A \lor B)$ is true even if B is not. In this case, if you infer B from A and $(A \rightarrow B)$, you get from true premises a not true conclusion. This shows that MPP may fail to preserve truth. Thus, the possibility of dialetheias justifies the rejection of MPP.

Nevertheless, that move is inappropriate for a dialetheist: the possibility of A being a dialetheia should lead dialetheists to reject the classical equivalence between $(A \rightarrow B)$ and $(\neg A \lor B)$. This equivalence holds in classical logic because the truth of $(\neg A \lor B)$ guarantees that truth is preserved from A to B for the reason that dialetheiae are excluded.

Of course, nothing prevents a dialetheist from *defining* $(A \rightarrow B)$ as $(\neg A \lor B)$, but in this way she cannot transfer the intuitive meaning of "if ... then ..." from the metalanguage to the object language, against Priest's claim (ii).
On the other hand, if a dialetheist uses the material conditional both in the object language and in the metalanguage, she can recover the validity of MPP. For, the definition of truth-preservation involves a meta-linguistic conditional. An inference rule is truth-preserving if it satisfies the following condition: *if the premises are true, so is the conclusion*. Reading this conditional as a material conditional, the truth preservation of MPP is expressed as follows:

(*) A is false or ($\neg A \lor B$) is false or B is true.

Now, if A is a dialetheia, (*) is a dialetheia too. So, in any case (*) is true, i.e. MPP is truth preserving.

For these reasons we think that the rejection of the general validity of MPP is not justified by the mere presence of dialetheiae. It is an *ad hoc* move for avoiding trivialism at the cost of a severe limitation of the expressive power of the logical language.

Anyway, in the next section we argue how it is possible to obtain Curry's Paradox without MPP.

4. A new argument for trivialism

We will propose a new version of Curry's paradox, without making use of MPP. We use a notion of *naïve deducibility* that, in our opinion, should be perfectly acceptable by Priest.

In his discussion about Goedel's theorem, Priest defends the *naïve* notion of proof:

> Proof, as understood by mathematicians (not logicians), is that process of deductive argumentation by which we establish certain mathematical claims to be true. In other words, suppose we have a mathematical assertion, say a claim of number theory, whose truth or falsity we wish to establish. We look for a proof or a refutation, that is a proof of its negation…I will call the informal deductive arguments from basic statements *naïve proofs*. (Priest 2006b, 40)

Priest holds that *naïve proof* is inconsistent but correct, since it is the official means for recognizing mathematical truths. According to him, inconsistency arises from the presence of dialetheias. Furthermore, he argues for the thesis that the notion of *naïve proof* of an arithmetical sentence is expressible in the arithmetical language.

We do not endorse the latter highly problematic thesis. Instead, consider the extension L' of the language L of first order arithmetic, obtained by introducing a new the predicate $P(x)$. Extend the standard model N of arithmetic to the model N', where P is interpreted as naïve provability for the language L' (with reference to a numerical codification of the syntax of L'). More precisely, $P(\ulcorner A \urcorner)$ means: it is naively provable that A is true in N'.

sely, $P(\ulcorner A \urcorner)$ means: it is naively provable that A is true in N'. According to Priest, this predicate satisfies the following principles:

(a) $P(\ulcorner A \urcorner) \rightarrow A$ is naively provable;
(b)   If A is naively provable, than $P(\ulcorner A \urcorner)$ is naively provable.

He justifies principles (a) and (b) by observing that "for (a) it is analytic that whatever is naively provable is true. Naïve proof is just that sort of mathematical argument that establishes something as true. And since this is analytic, it is itself naively provable... For (b), if something is naively proved then this fact itself constitutes a proof that A is provable". (Priest 2006b, 238).

   Similarly, extend L to a language L^* by introducing a binary predicate $D(x, y)$. Then extend the standard model N to the model N^* of L^*, where D is interpreted as the *naïve deducibility* relation for L^* (with reference to a codification of L^*). $D(x, y)$ means "*y is naively deducible from x*"; more explicitly "*there is a naïve proof that, assuming that x is true in N*, leads to the conclusion that y is true in N* *". In a system of natural deduction, the analogous principles to (a) and (b) are the following *elimination* and *introduction* rules for D:

DE: From premises A and $D(\ulcorner A \urcorner, \ulcorner B \urcorner)$ one can derive B. The conclusion depends on all assumptions the premises depend on.

DI: From premise B, depending on the unique assumption A, one can infer $D(\ulcorner A \urcorner, \ulcorner B \urcorner)$, discharging A.

It is worthy to compare DE with MPP: while the possibility of A being a dialetheia blocks the inference from A and $(\neg A \vee B)$ to B, it does not block the inference from A and $D(\ulcorner A \urcorner, \ulcorner B \urcorner)$ to B. Indeed, according to the intended meaning of D, if $D(\ulcorner A \urcorner, \ulcorner B \urcorner)$ is true, the truth of B is provable under the mere assumption that A is true (the fact that A may also be false is not at issue here). Regarding DI, if B is derived from the unique assumption A, since the soundness of inference rules is recognized by informal reasoning, there is a naïve reasoning that leads from A to B.

Theorem. From DE and DI *trivialism follows*.

Proof. Let A be any L^*-sentence. By diagonalization, we get a natural number k such that $k = \ulcorner D(\underline{k}, \ulcorner A \urcorner) \urcorner$. Using $\underline{k}$ as a name of $D(\underline{k}, \ulcorner A \urcorner)$, suppose that $\underline{k}$

is true. Since $\underline{k}$ says that A is deducible from $\underline{k}$ and deduction is sound, A is true. So we have proved A from the assumption $\underline{k}$. Hence D(k , $\ulcorner A \urcorner$), i.e. $\underline{k}$, is true. And, since deduction is sound, A is true.

A formal proof of A in natural deduction (where $\underline{k}$ is used again as a name of D($\underline{k}$, $\ulcorner$A$\urcorner$)) is as follows.

1	(1)	$\underline{k}$	Assumption
1	(2)	D($\underline{k}$, $\ulcorner A \urcorner$)	1 Identity
1	(3)	A	1, 2 DE
	(4)	D($\underline{k}$, $\ulcorner A \urcorner$)	1,3 DI (discharging (1))
	(5)	$\underline{k}$	4 Identity
	(6)	A	4, 5 DE

 Since A is arbitrary, N^* is trivial. But N^* differs from N only for the introduction of the relation of *naïve deducibility*: the arithmetical sentences of L are interpreted in N^* as in N. Therefore N is trivial as well.

5. Conclusion

We conclude that, according to his perspective, Priest should recognize that the standard model of arithmetic is trivial. After all, why should trivialism not be accepted as a *fact of life*? However, never mind! The triviality of arithmetic does not invalidate the proof by induction that every arithmetical sentence has a unique truth-value. Thus dialetheists should not be afraid of trivialism. Their philosophy allows them to hold that, though arithmetic is trivial, it is also nevertheless not.

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