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# A Self-contained Proof of the Standard Completeness in HW-algebras

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ABSTRACT. This paper has a survey-character and studies many-valued logic endowed with two different kinds of implication: Łukasiewicz's implication and Gödel's implication. We focus on the class of algebras containing the algebraic counterpart of this new logic: the class of Heyting Wajsberg algebras. We introduce a new direct Chang-style proof of subdirect representation and standard algebraic completeness theorem.

KEYWORDS: Bounded distributive lattice, MV-algebra, HW-algebra, filter, subdirect product, standard algebraic completeness, $l$-group first order theory.

## 1. Introduction

The contribution of this paper is mainly taxonomic and aims to complete the study of Gödel Łukasiewicz Logic in [13]. There is an important connection between any logical calculus S and the class of adequate models for it – i.e. the class of algebraic structures which verify exactly the provable formulae of S. For instance Boolean algebras are the algebraic counterpart of classical propositional logic and Heyting algebras correspond to intuitionistic propositional logic (see pp. 380-3 in [10]).


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Heyting Wajsberg algebras were introduced by Giampiero Cattaneo and Davide Ciucci in [2] and have two different implications as primitive operators: Łukasiewicz's implication and Gödel's implication [11]. By the composition of the two primitive operators with the **0**-element it is possible to define two different negations and the modal operators of necessity and possibility. Moreover the equational theory of the variety of Heyting Wajsberg algebras is capable to contain both the equational theory of Heyting algebras and the one of Wajsberg algebras [13]. Wajsberg algebras are proven to be termwise equivalent to MV-algebras (section 4.2 in [9]). Then the logical calculus whose algebraic counterpart is the class of Heyting Wajsberg algebras (i.e. Gödel Łukasiewicz Logic [13]) results to be an extension of both intuitionistic logic and of Łukasiewicz many-valued logic (i.e. the logical systems arising from Heyting and MV-algebras). Furthermore, in [13] Gödel Łukasiewicz Logic is shown to be decidable, to have the deduction-detachment theorem and to be strongly complete.

All these results hold with the necessary support of the standard completeness theorem. Up to now, the standard completeness of Heyting Wajsberg algebras has been obtained indirectly by the equivalence proven in [4] between Heyting Wajsberg algebras and other algebraic structures, for instance MVΔ-algebras (Theorem 3.2.13 in [12]). The main contribution of this paper is to give a direct proof of the standard completeness theorem for Heyting Wajsberg algebras in a traditional Chang-like style.

In section 2 the basic notions and properties of this algebraic structure are introduced. In section 3 I introduce a suitable extension of the definition of implicative filter and show that any Heyting Wajsberg algebra is isomorphic to a subdirect product of linear Heyting Wajsberg algebras. Finally, in section 4 I prove the whole variety of Heyting Wajsberg algebras to be generated by the real unit interval model. It is worth reminding that from the logical point of view this result entails that in Gödel Łukasiewicz Logic any tautology is provable.

## 2. Basic notions

**Definition 2.1.** Let $\mathscr{A} = \langle A, \rightarrow_L, \rightarrow_G, \mathbf{0} \rangle$ be an algebraic structure of type $\langle 2,2,0 \rangle$. $\mathscr{A}$ is a *Heyting Wajsberg algebra* (briefly HW-algebra) if for any $x, y, z \in A$, once defined

$$\neg x := x \to_L 0$$
$$\sim x := x \to_G 0$$
$$x \wedge y := \neg((\neg x \to_L \neg y) \to_L \neg y)$$
$$x \vee y := (x \to_L y) \to_L y$$
$$\mathbf{1} := \neg \mathbf{0}$$

the following identities are satisfied:

(HW1) $x \to_G x = \mathbf{1}$

(HW2) $x \to_G (y \wedge z) = (x \to_G z) \wedge (x \to_G y)$

(HW3) $x \wedge (x \to_G y) = x \wedge y$

(HW4) $(x \vee y) \to_G z = (x \to_G z) \wedge (y \to_G z)$

(HW5) $\mathbf{1} \to_L x = x$

(HW6) $x \to_L (y \to_L z) = \neg(x \to_L z) \to_L \neg y$

(HW7) $\neg \sim x \to_L \sim\sim x = \mathbf{1}$

(HW8) $(x \to_G y) \to_L (x \to_L y) = \mathbf{1}$

It is useful to define also the following operators:

$$x \oplus y := \neg x \to_L y$$
$$x \odot y := \neg(\neg x \oplus \neg y)$$
$$\flat x := \neg \sim \neg x$$
$$x \oslash y := x \oplus \neg y$$

We assume familiarity with the basic notions of MV-algebra and its main properties. Any of them can be found by the readers in [9]. Moreover any HW-algebra $\mathscr{A} = \langle A, \to_L, \to_G, \mathbf{0} \rangle$ has the MV-algebra $\mathscr{A}^\star = \langle A, \oplus, \neg, \mathbf{0} \rangle$ as term reduct and any HW-algebra $\mathscr{A} = \langle A, \to_L, \to_G, \mathbf{0} \rangle$ has the bounded distributive lattice $\mathscr{A}^{\star\star} = \langle A, \wedge, \vee, \mathbf{0}, \mathbf{1} \rangle$ as term reduct ([4],[3]).

It is also shown in [4] (proposition 1.1) that the natural partial order $\leq$ defined by $\wedge$ or $\vee$ (*i.e.* $x \leq y := x \wedge y = x$ or $x \leq y := x \vee y = y$) has the following property:

$$x \leq y \Leftrightarrow x \to_L y = \mathbf{1} \Leftrightarrow x \to_G y = \mathbf{1}$$

**Remark 1.** Any linear MV-algebra can be enriched in a natural way with a new unary operator in order to have a HW-algebra term reduct.

*Proof.* By [4] any HW-algebra is termwise equivalent to a Stonean MV-algebra. An MV-algebra is Stonean when there can be defined a Stonean negation (see also [5]). Any linear MV-algebra is trivially Stonean once defined the Stonean negation $\sim$:

$$\sim x = \begin{cases} \mathbf{0} & \text{if } x \neq \mathbf{0} \\ \mathbf{1} & \text{if } x = \mathbf{0} \end{cases}$$

Then any linear MV-algebra enriched in such a way has a HW-algebra term reduct. □

In the sequel we'll adopt the following notation. Given a HW-algebra $\mathscr{A}$, $\forall x \in A$ and $\forall n \in N$:

$$nx = \begin{cases} \mathbf{0} & \text{if } n = 0 \\ x & \text{if } n = 1 \\ \underbrace{x \oplus \ldots \oplus x}_{n-times}, & \text{if } 2 \leq n \in N \end{cases}$$

and

$$x^n = \begin{cases} \mathbf{1} & \text{if } n = 0 \\ x & \text{if } n = 1 \\ \underbrace{x \odot \ldots \odot x}_{n-times}, & \text{if } 2 \leq n \in N \end{cases}$$

A HW-algebra $\mathscr{A}$ is *linear (or totally ordered)* iff for any pair of elements $x, y \in A$, either $x \leq y$ or $y \leq x$.

Now we introduce the most important example of HW-algebra, the model we will prove at the end of this article to generate the whole variety of HW-algebras.

**Example 1** (Standard HW-algebra)**.** $\mathscr{A}_{[0,1]} = \langle [0,1], \rightarrow_L, \rightarrow_G, 0 \rangle$ where:

$$[0,1] \subset R,$$

$$x \rightarrow_L y := \begin{cases} 1 & \text{if } x \leq y \\ 1 - x + y & \text{otherwise} \end{cases},$$

and

$$x \rightarrow_G y := \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}.$$

We recall some important basic results that will be useful in the sequel below.

**Lemma 2.1.** Let $\mathscr{A}$ be a HW-algebra and $x \in A$. Then
(i) $\sim\sim x = \neg \sim x$
(ii) $x \wedge \sim x = \mathbf{0}$ $\qquad x \vee \flat x = \mathbf{1}$
(iii) $x \leq \sim\sim x$ $\qquad \flat\flat x \leq x$
(iv) $x \leq y \quad \Rightarrow \quad \sim y \leq \sim x, \flat y \leq \flat x$

*Proof.* $(i)$ is the interconnection rule ((in)p. 336 in [3]) and can be derived from (HW7) and (HW8) (see Proposition 4.6 in [3]). In $(ii)$ we find B3 (p. 336) and its dual AB4 (p.337) of [3]. $(iii)$ is SBL$_\neg$2 (p. 347, [3]) and AB1 (p. 337, [3]). $(iv)$ is (B2b) p. 335 in [3] and its dual.

□

**Lemma 2.2.** Let $\mathscr{A}$ be a HW-algebra and $x, y \in A$. Then
(i) $\sim (x \wedge y) = \sim x \vee \sim y$ $\qquad \flat(x \vee y) = \flat x \wedge \flat y$
(ii) $\sim (x \vee y) = \sim x \wedge \sim y$ $\qquad \flat(x \wedge y) = \flat x \vee \flat y$

*Proof.* $(i)$ is reported B2a (p. 335) and its dual AB3 (p. 337) in [3]. $(ii)$ is B2 (p. 335) and its dual AB2 (p. 337) in [3].

□

**Lemma 2.3.** Let $\mathscr{A}$ be a HW-algebra and $x, y \in A$. Then
(i) $\sim (x \oplus y) = \sim x \odot \sim y$ $\qquad \flat(x \odot y) = \flat x \oplus \flat y$
(ii) $x \wedge \sim y = x \odot \sim y$ $\qquad x \vee \flat y = x \oplus \flat y$
(iii) $x \vee \sim y = x \oplus \sim y$ $\qquad x \wedge \flat y = x \odot \flat y$
(iv) $\sim x \oplus \sim x = \sim x \odot \sim x = \sim x$

*Proof.* $(i)$, $(ii)$ and $(ii)$ are $(v)$ and $(iii)$ with duals in Lemma 1.1, p. 361 in [4]. $(iv)$ follows directly from $(iii)$.

□

**Corollary 2.1.** Let $\mathscr{A}$ be a HW-algebra and $x, y \in A$. Then
(i) $\sim\sim (x \oplus y) = \sim (\sim x \odot \sim y) = \sim\sim x \oplus \sim\sim y$
(ii) $\flat\flat(x \odot y) = \flat(\flat x \oplus \flat y) = \flat\flat x \odot \flat\flat y$

*Proof.* $(ii)$ By Lemma 2.3 $(i)$ $\flat\flat(x \odot y) = \flat(\flat x \oplus \flat y)$. By Lemma 2.3 $(ii), (iii)$ and Lemma 2.2 $(i)$, $\flat(\flat x \oplus \flat y) = \flat(\flat x \vee \flat y) = \flat\flat x \wedge \flat\flat y = \flat\flat x \odot \flat\flat y$. $(i)$ is dual.

□

## 3. Subdirect representation

**Definition 3.1.** A *filter* $F$ of a HW-algebra $\mathscr{A}$ is a subset of $A$ which satisfies the following conditions:
(F1) $\mathbf{1} \in F$
(F2) if $x \in F$ and $x \leq y$, then $y \in F$
(F3) if $x \in F$ and $y \in F$, then $x \odot y \in F$
(F4) if $x \in F$ then $\flat\flat x \in F$

Notice that the filter defined above is an implicative MV-filter (Definition 4.2.6, p. 86 [9]) plus the additional condition F4. Then any MV-filter $F$ can be extended naturally in a filter $F^\star$ in the following trivial way:

$$F^\star := \{x \in A \mid \exists y \in F : x \geq \flat\flat y\}$$

It has to be also noticed that $F^\star$ is trivially the smallest HW-filter containing $F$.

**Definition 3.2.** A filter $F$ of a HW-algebra $\mathscr{A}$ is *proper* iff $\mathbf{0} \notin F$.

**Definition 3.3.** Let $\mathscr{A}=\langle A, \rightarrow_L, \rightarrow_G, \mathbf{0}\rangle$ be a HW-algebra, let $F$ be a filter of $\mathscr{A}$ and $x \in F$. We introduce the definition of *filter generated by* $F \cup \{x\}$, denoted $Fi(F \cup \{x\}) := \{y \in A \mid y \leq i \odot x^n$, for some $i \in F$ and some $n \in N\}$. Further the *filter generated by* $x$, denoted $Fi(x) :=$ the filter generated by $\{\mathbf{1}\} \cup \{x\}$.

**Definition 3.4.** A filter $J$ of a HW-algebra $\mathscr{A}$ is *maximal* iff it is proper and for any filter $F$ of $\mathscr{A}$ s.t. $J \subseteq F$, either $F = J$ or $F = A$.

**Definition 3.5.** A filter $J$ of a HW-algebra $\mathscr{A}$ is *prime* iff it is proper and if for any pair of elements $x, y \in A$, either $x \oslash y \in J$ or $y \oslash x \in J$.

It can be noticed that in general $\{\mathbf{1}\}$ is a non-prime filter.

**Definition 3.6** (Distance function on a HW-algebra $\mathscr{A}$)**.** Let $x, y \in A$, $q(x, y) := (x \oslash y) \odot (y \oslash x)$.

**Definition 3.7.** Let $F$ be a filter of a HW-algebra $\mathscr{A}$, $\forall x, y \in A$:
$x \equiv_F y \Leftrightarrow q(x, y) \in F$.

In order to prove $\equiv_F$ to be a congruence relation we need to prove the following lemma.

**Lemma 3.1.** Let $F$ be a filter of a HW-algebra $\mathscr{A}$ and let $x,y \in A$, if $x \oplus y \in F$ then $\flat\flat x \oplus \flat\neg y \in F$.

*Proof.* By Lemma 2.1 $(i)$ and $(iii)$ $y \leq \sim\sim y = \neg\sim y = \flat\neg y$ and thus, by monotonicity, $x \oplus y \leq x \oplus \flat\neg y \in F$. By Lemma 2.3 $(ii)$ $x \oplus \flat\neg y = x \vee \flat\neg y \in F$. By F4 $\flat\flat(x \vee \flat\neg y) \in F$. Hence, by Lemma 2.2 $(i)$ and $(ii)$, $\flat\flat x \vee \flat\flat\flat\neg y = \flat\flat x \vee \flat\neg y \in F$. Since in any MV-algebra and then in any HW-algebra $x \vee y \leq x \oplus y$ we obtain $\flat\flat x \oplus \flat\neg y \in F$.

□

**Theorem 3.1.** Let $F$ be a filter of a HW-algebra $\mathscr{A}$, $\forall x,y \in A$: $x \equiv_F y$ is a congruence relation on $\mathscr{A}$.

*Proof.* First we prove that $\equiv_F$ is an equivalence relation. $\equiv_F$ is trivially symmetric and since any HW-algebra defines an MV-algebra $\mathscr{A}^\star$=$\langle A, \oplus, \neg, \mathbf{0}\rangle$ and in any MV-algebra $x \oplus \neg x = \mathbf{1} = \mathbf{1}\odot\mathbf{1} \in F$ we have $\equiv_F$ is reflexive. To prove transitivity we have just to prove $q(x,z) \geq q(x,y) \odot q(y,z)$. We assume familiarity with MV-algebra and lattice properties. $\mathbf{0}= y \odot \neg y \geq (y \wedge z) \odot (\neg y \wedge \neg x) = z \odot (\neg z \oplus y) \odot \neg x \odot (x \oplus \neg y)$. Thus $\neg(x \oslash z) \odot (x \oslash y) \odot (y \oslash z) = \mathbf{0}$. By Lemma 1.1.2 in [9] (p. 9) in any MV-algebra and then in any HW-algebra $\neg y \odot x = \mathbf{0} \Leftrightarrow y \geq x$. It follows $(x \oslash z) \geq (x \oslash y) \odot (y \oslash z)$. Analogously we obtain $(z \oslash x) \geq (y \oslash x) \odot (z \oslash y)$. Then by monotonicity we have $(x \oslash z) \odot (z \oslash x) \geq (x \oslash y) \odot (y \oslash z) \odot (y \oslash x) \odot (z \oslash y)$ that is $q(x,z) \geq q(x,y) \odot q(y,z)$. Since in [4] (Theorem 2.5 and 2.6) it is proven $x \to_G y = \sim (x \odot \neg y) \oplus y$, $x \to_L y = \neg x \oplus y = \neg(x \odot \neg y)$ and since $\sim x = \neg\flat\neg x$, in order to prove $\equiv_F$ preserves $\to_G$ and $\to_L$ we have just to show that $\equiv_F$ preserves $\neg$, $\flat$ and $\odot$. By $\neg\neg x = x$ we have trivially that $q(x,y) = q(\neg x, \neg y)$ and $\equiv_F$ preserves $\neg$. About $\odot$ to prove $x \equiv_F y$ and $s \equiv_F t$ implies $x \odot s \equiv_F y \odot t$, by F2 and F3 we have just to show that $q(x \odot s, y \odot t) \geq q(x,y) \odot q(s,t)$. $\mathbf{0}= x \odot s \odot \neg(x \odot s) \geq \neg(x \odot s) \odot x \odot (\neg x \oplus y) \odot s \odot (\neg s \oplus t) = \neg(x \odot s) \odot (x \wedge y) \odot (s \wedge t) = \neg(x \odot s) \odot y \odot (\neg y \oplus x) \odot t \odot (\neg t \oplus s) = \neg(x \odot s) \odot y \odot t \odot (x \oslash y) \odot (s \oslash t) = \neg((x \odot s) \oslash (y \odot t)) \odot (x \oslash y) \odot (s \oslash t) = \mathbf{0}$. By Lemma 1.1.2 in [9] (p.9) in any MV-algebra and then in any HW-algebra $\neg y \odot x = \mathbf{0} \Leftrightarrow y \geq x$. This means $(x \odot s) \oslash (y \odot t) \geq (x \oslash y) \odot (s \oslash t)$. Analogously we obtain $(y \odot t) \oslash (x \odot s) \geq (y \oslash x) \odot (t \oslash s)$. By monotonicity we have $q(x \odot s, y \odot t) \geq q(x,y) \odot q(s,t)$. Now we show how $\equiv_F$ preserves $\flat$: $q(x,y) \in F \Rightarrow q(\flat x, \flat y) \in F$. If $(x \oslash y) \odot (y \oslash x) \in F$ then $(x \oplus \neg y) \in F$ and $(y \oplus \neg x) \in F$. By Lemma 3.1 we have $(\flat\flat x \oplus \flat\neg\neg y) \in F$ and $(\flat\flat y \oplus \flat\neg\neg x) \in F$.

By F4 and Lemma 2.1 $(i)$, it follows $(\neg\flat x \oplus \flat y) \odot (\neg\flat y \oplus \flat x) \in F$. Thus $\equiv_F$ is a congruence relation and it induces a quotient HW-algebra $\mathscr{A}/F$ homomorphic to the original $\mathscr{A}$ (for general concepts of universal algebra see [1]).

□

Moreover, by duality on Chen Chung Chang's result on MV-algebras [6] with ideals, if $F$ is a prime MV-filter, then the quotient MV-algebra $\mathscr{A}/F$ is linear. It follows that if $F$ is prime, then the quotient HW-algebras $\mathscr{A}/F$ is linear. Let us now define the last main concepts necessary to present the subdirect representation Theorem.

**Definition 3.8.** A *direct product* of a given family of HW-algebras $\{\mathscr{A}_i \mid i \in I\}$ is a HW-algebra $\Pi_{i\in I}\mathscr{A}_i = \langle \Pi_{i\in I}A_i, \rightarrow_L, \rightarrow_G, \mathbf{0}\rangle$ where $\Pi_{i\in I}A_i$ := the cartesian product of $\{A_i \mid i \in I\}$ and the operators are defined componentwise as the operators of each original MV-algebra $\mathscr{A}_i$. The $\mathbf{0}$-element is obviously the sequence of all the $\mathbf{0}$-elements of $\{A_i \mid i \in I\}$.

Every element $x$ of a direct product $\Pi_{i\in I}\mathscr{A}_i$ of HW-algebras $\{\mathscr{A}_i \mid i \in I\}$ is expressed in the following way: $x = \langle x_1, ..., x_n, ..\rangle$ where each $x_i$ belongs to each HW-algebra $\mathscr{A}_i$ of $\Pi_{i\in I}\mathscr{A}_i$.

**Definition 3.9.** Let a HW-algebra $\Pi_{i\in I}\mathscr{A}_i$ be a direct product of a family of HW-algebras $\{\mathscr{A}_i \mid i \in I\}$ and $j \in I$. Let $\pi_j : \Pi_{i\in I}A_i \mapsto A_j$ be the *j-th projection function* s.t. $\forall x = \langle x_1, ..., x_n, ..\rangle \in \Pi_{i\in I}A_i$, $\pi_j(x) := x_j$. A HW-algebra $\mathscr{A}$ is a *subdirect product* of a given family of HW-algebras $\{\mathscr{A}_i \mid i \in I\}$ iff there exists a one-one homomorphism $h : \mathscr{A} \mapsto \Pi_{i\in I}\mathscr{A}_i$ such that for any $j \in I$, the compose map $\pi_j \circ h$ is a homomorphism onto $\mathscr{A}_j$.

Obviously every subdirect product of a family of HW-algebras $\{\mathscr{A}_i \mid i \in I\}$ is a subalgebra of the direct product of the same family of HW-algebras.

**Theorem 3.2.** A HW-algebra $\mathscr{A}$ is isomorphic to a subdirect product of a family of linear HW-algebras if there is a family of prime filters $\{F_i \mid i \in I\}$ of $\mathscr{A}$ such that $\bigcap F_i = \{\mathbf{1}\}$.

*Proof.* By duality to Theorem 1.3.2 in [9].

□

**Remark 2.** Given a HW-algebra $\mathscr{A} = \langle A, \rightarrow_L, \rightarrow_G, \mathbf{0}\rangle$, $\{\mathbf{1}\}$ is trivially a HW-filter of $\mathscr{A}$.

To prove the next theorem we need the following three lemmas.

**Lemma 3.2.** In any HW-algebra $\mathscr{A}$, $\forall a,x,y,z \in A$, $a \geq x \odot y, a \geq x \odot z \Rightarrow a \geq x \odot (y \vee z)$.

*Proof.* By duality to Theorem 1.5 in [6] and axiom 11 of [7] we have $a = a \vee a \geq (x \odot y) \vee (x \odot z) = x \odot (y \vee z)$.

□

**Lemma 3.3.** In any HW-algebra $\mathscr{A}$, $\forall x,y \in A$, $\forall m \in N$, $(x \oslash y)^m \vee (y \oslash y)^m = \mathbf{1}$.

*Proof.* This lemma is Theorem 3.7 in [6].

□

**Lemma 3.4.** In any HW-algebra $\mathscr{A}$, $\forall x,y, \in A$, $x \vee y = \mathbf{1} \Rightarrow \flat\flat x \vee \flat\flat y = \mathbf{1}$.

*Proof.* By Lemma 2.1 $(i)$ and $(iii)$, $\mathbf{0} \geq \flat\flat\mathbf{0} = \neg \sim \mathbf{0} = \sim \mathbf{1}$. Thus if $x \vee y = \mathbf{1}$ then $\flat\flat(x \vee y) = \sim \neg(x \vee y) = \mathbf{1}$. By Lemma 2.2 $(i)$ we have $\flat(\flat x \wedge \flat y) = \mathbf{1}$. Then, by Lemma 2.2 $(ii)$, $\flat\flat x \vee \flat\flat y = \mathbf{1}$.

□

**Theorem 3.3.** Let $\mathscr{A}$ be a HW-algebra. For any $z \in A, z \neq \mathbf{1}$, there is a prime HW-filter $F \subseteq A$ such that $z \notin F$.

*Proof.* $\{\mathbf{1}\}$ is trivially a HW-filter of $\mathscr{A}$ and a MV-filter of its MV-algebra term reduct $\mathscr{A}^\star$. Suppose $z \neq \mathbf{1}$. By the duality between filters and ideals with a routine application of Zorn's lemma $\{\mathbf{1}\}$ can be extended into a HW-filter $F$ which is maximal with respect to the property "$z \notin F$". We show that $F$ is prime: suppose, by ctr., $\exists x,y \in A$ s.t. $x \oslash y \notin F$ and $y \oslash y \notin F$. We define for any $x,y \in A$, $F^\star_{x \oslash y} := (Fi(F \cup \{x \oslash y\}))^\star$. $F^\star_{x \oslash y}$ and $F^\star_{y \oslash x}$ are HW-filters containing $Fi(F \cup \{x \oslash y\})$ and $Fi(F \cup \{y \oslash x\})$. By maximality of $F$ with respect to "$z \notin F$", $z \in F^\star_{x \oslash y}$ and $z \in F^\star_{y \oslash x}$. Thus, $\exists r \in Fi(F \cup \{x \oslash y\})$ and $\exists s \in Fi(F \cup \{y \oslash x\})$ s.t. $z \geq \flat\flat r$ and $z \geq \flat\flat s$. Now $r = i \odot (x \oslash y)^n$ for some $i \in F$ and $n \in N$, $s = j \odot (x \oslash x)^m$ for some $j \in F$ and $m \in N$. Then, by Corollary 2.1, we have that $z \geq \flat\flat(i \odot (x \oslash y)^n) = \flat\flat i \odot \flat\flat((x \oslash y)^n)$ and $z \geq \flat\flat(j \odot (y \oslash x)^m) = \flat\flat j \odot \flat\flat((y \oslash x)^m)$. It is important to remind that by two application of Lemma 2.1 $(iv)$, if $x \leq y$ then $\flat\flat x \leq \flat\flat y$. Hence, let $k = max\{n,m\}$, by monotonicity $z \geq (\flat\flat i \odot \flat\flat j) \odot \flat\flat((x \oslash y)^k)$ and $z \geq (\flat\flat i \odot \flat\flat j) \odot \flat\flat((x \oslash y)^k)$. By Lemma 3.2 we have $z \geq (\flat\flat i \odot \flat\flat j) \odot (\flat\flat((x \oslash y)^k) \vee \flat\flat((y \oslash x)^k))$. By Lemma 3.3 $(x \oslash y)^k \vee (y \oslash x)^k = \mathbf{1}$. By Lemma 3.4

$\flat\flat((x \oslash y)^k) \vee \flat\flat((y \oslash x)^k) = \mathbf{1}$. Hence $z \geq \flat\flat i \odot \flat\flat j$. Since $F$ is a HW-filter and $i, j \in F$, by F4 $\flat\flat i \in F$ and $\flat\flat j \in F$. Thus $\flat\flat i \odot \flat\flat j \in F$. By F2 we have $z \in F$, against our ab absurdo hypothesis. Then $F$ is a prime HW-filter.

□

Now we can state the subdirect representation theorem.

**Theorem 3.4.** Any HW-algebra $\mathscr{A}$ is isomorphic to a subdirect product of a family of linear HW-algebras.

*Proof.* We have already all the ingredients. Since for any $z \in A$, $\{\mathbf{1}\}$ can be extended in a prime HW-filter $F$ such that $z \notin F$, we have that $\{\mathbf{1}\} = \bigcap\{F_i \mid F_i$ is a maximal prime filter of $\mathscr{A}\}$. By Theorem 3.2 and Theorem 3.3 we have the thesis.

□

## 4. Standard algebraic completeness

We will prove that an equation defined on the language of the HW-algebras holds in any HW-algebra if it holds in the standard HW-algebra. We will follow the track of Chang's standard completeness theorem for MV-algebras [7]. Then we assume familiarity with this proof and with all the results utilized to pursue it (see also [8]). Chang's proof exploits the completeness of the first order theory of divisible totally ordered Abelian groups (Chang's references are [14] and [15] but, as reported in footnote at page 79 [7], Tarski's proof has never appeared explicitly, then for a clear presentation of this result we advise the readers to consult appendix at page 91 of [8]). As a fundamental step of his proof, Chang had build a totally ordered abelian group made of infinite copies of an MV-algebra. Since any HW-algebra has an MV-algebra term reduct we can exploit the same argument. We introduce this expedient:

**Definition 4.1.** Let $\mathscr{A}$ be a linear HW-algebra. The algebraic structure $\mathscr{G}_A$ is defined in the following way, $G_A := \{(n,x) \mid n \in Z, x \in \mathrm{A} - \{\mathbf{1}\}\}$. Its operators are defined as:

$$(m,x) + (n,y) := \begin{cases} (n+m, x \oplus y) & \text{if } x \oplus y \neq \mathbf{1} \\ (n+m+1, x \odot y) & \text{if } x \oplus y = \mathbf{1} \end{cases}$$

$$-(n,x) := \begin{cases} (-n,\mathbf{0}) & \text{if } x=\mathbf{0} \\ \\ (-(n+1),\neg x) & \text{if } \mathbf{0}\neq x\neq \mathbf{1} \end{cases}$$

and its related order relation is

$$(n,x) \sqsubseteq (m,y) \quad := \quad n<m \ \text{ or, } \ n=m \ \text{ and } \ x\leq y$$

Chang in [9] proved that $\mathscr{G}_{\mathscr{A}} = \langle G_A, +, -, \sqsubseteq, (0,\mathbf{0})\rangle$ is a totally ordered abelian group. Moreover if we define:

**Definition 4.2.** Let $\mathscr{G}$=$\langle G,+,-,0,\sqsubseteq\rangle$ be a totally ordered abelian group and $u\in G$:

$$\Gamma(\mathrm{G},u) := \{x\in G \mid 0\sqsubseteq x\sqsubseteq u\}$$
$$\neg x := u-x$$
$$x\oplus y := min\{u,x+y\}$$

we can immediately verify that $\Gamma(\mathscr{G},u) = \langle\Gamma(\mathrm{G},u),\oplus,\neg,0\rangle$ is a linear MV-algebra. By Remark 1, once defined

$$\sim x := \begin{cases} \mathbf{0} & \text{if } x\neq\mathbf{0} \\ \mathbf{1} & \text{if } x=\mathbf{0} \end{cases}$$

the arising structure $\Gamma(\mathscr{G},u)^{\star} = \langle\Gamma(\mathrm{G},u),\rightarrow_L,\rightarrow_G,0\rangle$, where for any $x,y\in\Gamma(\mathrm{G},u)$,

$$x\rightarrow_L y := \neg x\oplus y \text{ and}$$
$$x\rightarrow_G y :=\sim\neg(\neg x\oplus y)\oplus y$$

is a linear HW-algebra.

**Remark 3.** The above definition of Gödel implication introduced in [4] (see Theorem 2.5 and 2.6) is given in terms of $\sim$, $\neg$ and $\oplus$. It is worth to be observed that by linearity since in any linear MV-algebra and then in any linear HW-algebra $\neg(\neg x\oplus y) = x\odot\neg y = \mathbf{0}$ if and only if $x\leq y$, we obtain that in any linear HW-algebra

$$x\rightarrow_G y = \begin{cases} \mathbf{1} & \text{if } x\leq y \\ y & \text{otherwise} \end{cases}$$

We recall that $u \in G$ is a *strong unit* iff for any $x \in G$ there exists an $n \in N$ s.t. $x \sqsubseteq nu$. $\mathscr{G}_{\mathscr{A}}$ is composed of infinite copies of $\mathscr{A}$; $\Gamma(\mathrm{G}_A,(1,\mathbf{0}))^\star$ belongs to them, then we have:

**Theorem 4.1.** If $\mathscr{A}$ is a linear HW-algebra, $\Gamma(\mathrm{G}_A,(1,\mathbf{0}))^\star$ is isomorphic to $\mathscr{A}$.

This result can be generalized to:

**Theorem 4.2.** If $u$ is the strong unit of a totally ordered abelian group $\mathscr{G}$, there exists an isomorphism $f$ from $\mathscr{G}$ onto $\mathscr{H} = \mathscr{G}_{\Gamma,(\mathrm{G},u)^\star}$:
i) $f(u) = (1,\mathbf{0})$
ii) $x \sqsubseteq y$ in $\mathscr{G} \Leftrightarrow f(x) \sqsubseteq f(y)$ in $\mathscr{H}$

*Proof.* It follows either Theorem 2.4.10 in [8] or [7]. □

The first order language of totally ordered abelian groups theory $L'$ is composed by the usual logic symbols and 0,+,-,$\sqcap$,$\sqcup$ with their traditional meaning. We have to fix their corresponding definitions:

**Definition 4.3.** A language $L$ of a HW-algebra $\mathscr{A}$ is composed by:
$\mathbf{0}$ : costant
$x_1,...,x_n,..$ : variables
$\to_L$ : binary functor
$\to_G$ : binary functor.
We define inductively a *HW-term*:
1) $\mathbf{0}$, $x_1,...,x_n,..$ are HW-terms.
2) If $x_i$ and $x_j$ are HW-terms, then $x_i \to_G x_j$ is a HW-term.
3) If $x_i$ and $x_j$ are HW-terms, then $x_i \to_L x_j$ is a HW-term.
Let $p$ be a HW-term containing the variables $x_1,...,x_t$ and assume $a_1,...,a_t$ are elements of $\mathscr{A}$. Substituting an element $a_i \in A$ for all occurrences of the variable $x_i$ in $p$, for $i = 1,...,t$, by the above rules 1)-3) and interpreting the symbols $\mathbf{0}$,$\to_L$ and $\to_G$ as the corresponding operations in $\mathscr{A}$, we obtain an element of $A$, denoted $p^{\mathscr{A}}(a_1,...,a_t)$. In more detail, proceeding by induction on the number of operation symbols occurring in $p$, we define $p^{\mathscr{A}}(a_1,...,a_t)$ as follows:
i) $x_i^{\mathscr{A}} = a_i$, for each $i = 1,...,t$;
ii) $(p \to_L q)^{\mathscr{A}} = (p^{\mathscr{A}} \to_L q^{\mathscr{A}})$;
iii) $(p \to_G q)^{\mathscr{A}} = (p^{\mathscr{A}} \to_G q^{\mathscr{A}})$;

By the above definition, given any HW-algebra $\mathscr{A}$ we can associate each HW-term in the variables $x_1,...,x_n$ with a function $p^{\mathscr{A}} : A^n \mapsto A$. These functions are called *term functions on A*.

A *HW-equation on variables* $x_1,...,x_t$ is an expression $p = q$, where $p$ and $q$ are HW-term containing at most the variables $x_1,...,x_t$. We say that a HW-algebra $\mathscr{A}$ *satisfies* a HW-equation $p = q$ (we write $\mathscr{A} \models p = q$) if and only if for any sequence of elements $(a_1,...,a_t) \in A$, $p^{\mathscr{A}}(a_1,...,a_t) = q^{\mathscr{A}}(a_1,...,a_t)$.

**Theorem 4.3.** If a HW-algebra $\mathscr{A}$ is a subdirect product of a family of linear HW-algebras $\{\mathscr{A}_i \mid i \in \mathrm{I}\}$, then $\mathscr{A} \models p = q \Leftrightarrow$ for any $i$ $\mathscr{A}_i \models p = q$.

*Proof.* In the subdirect representation theorem (Theorem 3.4) there is a homomorphism from $\mathscr{A}$ onto any linear HW-algebra of its subdirect product: the Łukasiewicz implication operator $\rightarrow_L$ and the Gödel implication operator $\rightarrow_G$ are preserved into these structures; then every HW-equation continues to hold in any $\mathscr{A}_i$. Vice versa if a HW-equation holds in any $\mathscr{A}_i$, it holds in their direct product $\Pi_{i\in I}\mathscr{A}_i$. Since $\mathscr{A}$ is isomorphic to a subalgebra of $\Pi_{i\in I}\mathscr{A}_i$, it holds in $\mathscr{A}$.

□

**Corollary 4.1.** A HW-equation is satisfied in any HW-algebras if and only if it is satisfied in any linear HW-algebra.

We will report in the following steps Chang's standard completeness proof, as it has been presented in [8], to check its validity with respect to the Heyting Wajsberg algebras case. Every totally ordered abelian group can be embedded into a divisible totally ordered abelian group. From the completeness of the first order theory of these last structures it follows that every universal sentence of the first order theory of totally ordered abelian groups is satisfied in the additive group $Q$ of rational numbers if and only if it is satisfied in any totally ordered abelian group [7]. Then any HW-equation has to be associated to an universal sentence of the first order language of totally ordered abelian groups theory $L'$ to exploit its completeness. We will do it by induction on the degree of complexity of a HW-term.

**Definition 4.4.** The *degree of complexity* of a HW-term $p$: $d(p)$:= the number of times that symbols $\rightarrow_L$ and $\rightarrow_G$ appear in $p$.
We associate to any HW-term $p$ a term $p' \in L'$ by induction on the degree of complexity of $p$:

If $d(p)$=0 ($p$=0 or $p = x_i$) then $p' = p$.
We suppose to have associated HW-terms until degree of complexity $n$; then if $d(p)$=$n+1$, we can have either:
1) $p = q \rightarrow_L r$ with $d(q) \leq n$ and $d(r) \leq n$ or
2) $p = q \rightarrow_G r$ with $d(q) \leq n$ and $d(r) \leq n$.
Let $z$ be a free variable that belongs to $L'$, we define, fir case 1 and 2 respectively:
1) $p' = z \sqcap (z - q' + r')$;
2) $p' = \begin{cases} z & \text{if } q' \sqsubseteq r' \\ r' & \text{otherwise} \end{cases}$
Then we define $\alpha_{pq} := \forall x_1, ..., x_n (0 \sqsubseteq x_i \sqsubseteq z \bigwedge, ..., \bigwedge 0 \sqsubseteq x_n \sqsubseteq z) \rightarrow p' = q'$.

As a routine it can be checked, by the way $\mathscr{G}_A$ has been built, that the following sentence holds:

**Proposition 4.1.** Let $\mathscr{A}$ be a linear HW-algebra, let $p = q$ be a HW-equation; $\mathscr{A} \models p = q \Leftrightarrow \alpha_{pq}(z)$ is true in $\mathscr{G}_A$ when we attribute to $z$ the value $(1, \mathbf{0})$.

At last we can introduce:

**Theorem 4.4** (Standard Completeness Theorem)**.** A HW-equation is satisfied in any HW-algebra if and only if it is satisfied in the standard HW-algebra $\mathscr{A}_{[0,1]}$.

*Proof.* $\Leftarrow$ (not trivial) : By contradiction we suppose there is a HW-algebra $\mathscr{A}$ such that $\mathscr{A} \not\models p = q$. From Corollary 4.1 we infer that there is a linear HW-algebra $\mathscr{B}$ s.t. $\mathscr{B} \not\models p = q$. By Proposition 4.1 above there is an universal sentence $\beta$ of the $1^o$ order theory of the totally ordered Abelian groups, $\beta = \forall z > 0\ \alpha_{pq}(z)$ s.t. $\beta$ is false in $\mathscr{G}_B$, and hence, by the completeness of totally ordered abelian groups, $\beta$ is false in $Q$ (group of rational numbers with usual operations). It means that there is a $c > 0, c \in Q$ s.t. $c$ does not verify $\beta$ in $Q$. Let's consider $f$: $Q \mapsto Q$ defined by $f(x) := c^{-1}x$. $f(c) = 1$. $f$ is an isomorphism from $Q$ onto itself (antiautomorphism), then $f$ preserves falsity of sentences and therefore $\beta$ is false in $Q$ when we attribute to $z$ the value $1 \in Q$. By Theorem 4.2 $Q$ is isomorphic to $\mathscr{G}_{\Gamma(Q,1)^\star}$. Thus $\beta$ is false in $\mathscr{G}_{\Gamma(Q,1)^\star}$ with $z = 1$ and, by Proposition 4.1, $\Gamma(Q, 1)^\star = \mathscr{A}_{[0,1]} \not\models p = q$. □

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